CHAPTER VII

The cohomology of coherent sheaves

1. Basic Čech cohomology

We begin with the general set-up.

(i) X any topological space

$$\begin{gathered}\mathcal{U} = \{U_\alpha\}_{\alpha \in S} \text{ an open covering of } X \\ \mathcal{F} \text{ a presheaf of abelian groups on } X.\end{gathered}$$

Define:

(ii)

$$\begin{aligned} C^i(\mathcal{U}, \mathcal{F}) &= \text{group of } i\text{-cochains with values in } \mathcal{F} \\ &= \prod_{\alpha_0,\dots,\alpha_i \in S} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_i}). \end{aligned}$$

We will write an i-cochain $s = s(\alpha_0, \dots, \alpha_i)$, i.e.,

$$s(\alpha_0, \dots, \alpha_i) = \text{the component of } s \text{ in } \mathcal{F}(U_{\alpha_0} \cap \cdots U_{\alpha_i}).$$

(iii) $\delta\colon C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$ by

$$\delta s(\alpha_0, \dots, \alpha_{i+1}) = \sum_{j=0}^{i+1} (-1)^j \operatorname{res} s(\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_{i+1}),$$

where res is the restriction map

$$\mathcal{F}(U_\alpha \cap \cdots \cap \widehat{\mathcal{U}}_{\alpha_j} \cap \cdots \cap U_{\alpha_{i+1}}) \longrightarrow \mathcal{F}(U_{\alpha_0} \cap \cdots U_{\alpha_{i+1}})$$

and $\widehat{\quad}$ means “omit”. For $i = 0, 1, 2$, this comes out as

$$\begin{aligned} \delta s(\alpha_0, \alpha_1) &= s(\alpha_1) - s(\alpha_0) \text{ if } s \in C^0 \\ \delta s(\alpha_0, \alpha_1, \alpha_2) &= s(\alpha_1, \alpha_2) - s(\alpha_0, \alpha_2) + s(\alpha_0, \alpha_1) \text{ if } s \in C^1 \\ \delta s(\alpha_0, \alpha_1, \alpha_2, \alpha_3) &= s(\alpha_1, \alpha_2, \alpha_3) - s(\alpha_0, \alpha_2, \alpha_3) + s(\alpha_0, \alpha_1, \alpha_3) - s(\alpha_0, \alpha_1, \alpha_2) \text{ if } s \in C^2. \end{aligned}$$

One checks very easily that the composition δ^2:

$$C^i(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^{i+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^{i+2}(\mathcal{U}, \mathcal{F})$$

is 0. Hence we define:

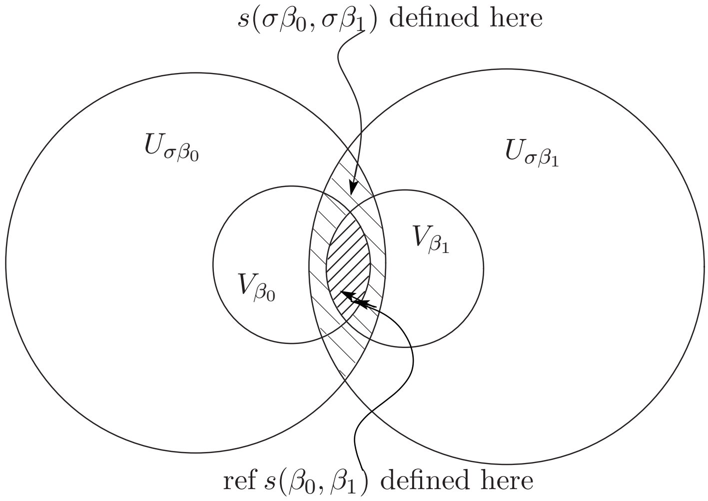

FIGURE VII.1

(iv)

$$\begin{aligned} Z^i(\mathcal{U}, \mathcal{F}) &= \mathrm{Ker}\left[\delta\colon C^i(\mathcal{U}, \mathcal{F}) \longrightarrow C^{i+1}(\mathcal{U}, \mathcal{F})\right] \\ &= \text{group of } i\text{-cocycles}, \\ B^i(\mathcal{U}, \mathcal{F}) &= \mathrm{Image}\left[\delta\colon C^{i-1}(\mathcal{U}, \mathcal{F}) \longrightarrow C^i(\mathcal{U}, \mathcal{F})\right] \\ &= \text{group of } i\text{-coboundaries} \\ H^i(\mathcal{U}, \mathcal{F}) &= Z^i(\mathcal{U}, \mathcal{F})/B^i(\mathcal{U}, \mathcal{F}) \\ &= i\text{-th Čech-cohomology group with respect to } \mathcal{U}. \end{aligned}$$

For $i = 0, 1$, this comes out:

$$\begin{aligned} H^0(\mathcal{U}, \mathcal{F}) =& \text{group of maps } \alpha \mapsto s(\alpha) \in \mathcal{F}(U_\alpha) \text{ such that} \\ & \quad s(\alpha_1) = s(\alpha_0) \text{ in } \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) \\ \cong& \Gamma(X, \mathcal{F}) \textit{ if } \mathcal{F} \textit{ is a sheaf}. \\ H^1(\mathcal{U}, \mathcal{F}) =& \text{group of cochains } s(\alpha_0, \alpha_1) \text{ such that} \\ & \quad s(\alpha_0, \alpha_2) = s(\alpha_0, \alpha_1) + s(\alpha_1, \alpha_2) \\ & \quad \text{modulo the cochains of the form} \\ & \quad s(\alpha_0, \alpha_1) = t(\alpha_0) - t(\alpha_1). \end{aligned}$$

Next suppose $\mathcal{U} = \{U_\alpha\}_\alpha$ and $\mathcal{V} = \{V_\beta\}_{\beta \in T}$ are two open coverings and that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$, i.e., for all $V_\beta \in \mathcal{V}$, $V_\beta \subset U_\alpha$ for some $\alpha \in S$. Fixing a map $\sigma\colon T \to S$ such that $V_\beta \subset U_{\sigma(\beta)}$, define

(v) the refinement homomorphism

$$\mathrm{ref}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F})$$

by the homomorphism on i-cochains:

$$\mathrm{ref}^\sigma_{\mathcal{U},\mathcal{V}}(s)(\beta_0, \dots, \beta_i) = \mathrm{res}\, s(\sigma\beta_0, \dots, \sigma\beta_i)$$

(using res: $\mathcal{F}(U_{\sigma\beta_0} \cap \cdots \cap U_{\sigma\beta_i}) \to \mathcal{F}(V_{\beta_0} \cap \cdots \cap V_{\beta_i})$ and checking that $\delta \circ \mathrm{ref}^\sigma_{\mathcal{U},\mathcal{V}} = \mathrm{ref}^\sigma_{\mathcal{U},\mathcal{V}} \circ \delta$, so that ref on cochains induces a map ref on cohomology groups.) (cf. Figure VII.1)

Now one might fear that the refinement map depends on the choice of $\sigma\colon T \to S$, but here we encounter the first of a series of nice identities that make cohomology so elegant — although "ref" on cochains depends on σ, "ref" on cohomology does not.

(vi) Suppose $\sigma, \tau\colon T \to S$ satisfy $V_\beta \subset U_{\sigma\beta} \cap U_{\tau\beta}$. Then

a) for all 1-cocycles s for the covering $\mathcal{U}$,

$$\begin{aligned}
\operatorname{ref}^{\sigma}_{\mathcal{U},\mathcal{V}} s(\alpha_0,\alpha_1) &= s(\sigma\alpha_0,\sigma\alpha_1)\\
&= s(\sigma\alpha_0,\tau\alpha_1) - s(\sigma\alpha_1,\tau\alpha_1)\\
&= \{s(\sigma\alpha_0,\tau\alpha_0) + s(\tau\alpha_0,\tau\alpha_1)\} - s(\sigma\alpha_1,\tau\alpha_1)\\
&= s(\tau\alpha_0,\tau\alpha_1) + s(\sigma\alpha_0,\tau\alpha_0) - s(\sigma\alpha_1,\tau\alpha_1)\\
&= \operatorname{ref}^{\tau}_{\mathcal{U},\mathcal{V}} s(\alpha_0,\alpha_1) + s(\sigma\alpha_0,\tau\alpha_0) - s(\sigma\alpha_1,\tau\alpha_1)
\end{aligned}$$

i.e., the two ref's differ by the coboundary δt, where

$$t(\alpha) = \operatorname{res} s(\sigma\alpha,\tau\alpha) \in \mathcal{F}(V_\alpha).$$

More generally, one checks easily that

b) if $s \in Z^i(\mathcal{U},\mathcal{F})$, then

$$\operatorname{ref}^{\sigma}_{\mathcal{U},\mathcal{V}} s - \operatorname{ref}^{\tau}_{\mathcal{U},\mathcal{V}} s = \delta t$$

where

$$t(\alpha_0,\dots,\alpha_{i-1}) = \sum_{j=0}^{i-1}(-1)^j s(\sigma\alpha_0,\dots,\sigma\alpha_j,\tau\alpha_j,\dots,\tau\alpha_{i-1}).$$

For general presheaves $\mathcal{F}$ and topological spaces X, one finally passes to the limit via ref over finer and finer coverings and defines:

(vii) [1]

$$H^i(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} H^i(\mathcal{U},\mathcal{F}).$$

Here are three important variants of the standard Čech complex. The first is called the *alternating cochains*:

$C^i_{\mathrm{alt}}(\mathcal{U},\mathcal{F})$ = group of i-cochains s as above such that:

a) $s(\alpha_0,\dots,\alpha_n) = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$

b) $s(\alpha_{\pi 0},\dots,\alpha_{\pi n}) = \operatorname{sgn}(\pi)\cdot s(\alpha_0,\dots,\alpha_n)$ for all permutations π.

For $i = 1$, one sees that every 1-cocycle is automatically alternating; but for $i > 1$, this is no longer so. One checks immediately that $\delta(C^i_{\mathrm{alt}}) \subset C^{i+1}_{\mathrm{alt}}$, hence we can form the cohomology of the complex $(C^\bullet_{\mathrm{alt}},\delta)$. By another beautiful identity, it turns out that the cohomology of the subcomplex $C^\bullet_{\mathrm{alt}}$ and the full complex $C^\bullet$ are exactly the same! This can be proved as follows:

a) Order the set S of open sets U_α.

b) Define

$$D^n_{(k)} \subset C^n \text{ to be the group of cochains } s$$

such that

$$s(\alpha_0,\dots,\alpha_n) = 0 \text{ if } \alpha_0 < \cdots < \alpha_{n-k}.$$

[1] This group, the Čech cohomology, is often written $\check{H}^i(X,\mathcal{F})$ to distinguish it from the "derived functor" cohomology. In most cases they are however canonically isomorphic and as we will not define the latter, we will not use the ˇ .

c) Note that $\delta(D^n_{(k)}) \subset D^{n+1}_{(k)}$, that

$$(0) = D^n_{(n)} \subset D^n_{(n-1)} \subset \cdots \subset D^n_{(0)}$$

and that

$$C^n \cong C^n_{\mathrm{alt}} \oplus D^n_{(0)}.$$

d) Therefore

$$H^n(\text{complex } C^\bullet, \delta) \cong H^n(\text{complex } C^\bullet_{\mathrm{alt}}, \delta) \oplus H^n(\text{complex } D^\bullet_{(0)}, \delta)$$

so it suffices to prove that the last is 0. We use the elementary fact: if $(G^\bullet, d)$ is a complex of abelian groups and $H^n \subset G^n$ is a subcomplex, then (H^n, d) exact and $(G^n/H^n, d)$ exact imply (G^n, d) exact. Thus we need only check that $(D^n_{(k)}/D^n_{(k+1)}, \delta)$ is exact for each k.

e) We may interpret:

$$D^n_{(k)}/D^n_{(k+1)} = \left\{ \text{cochains } s \;\middle|\; \begin{array}{l} s(\alpha_0, \dots, \alpha_n) \text{ defined only if} \\ \alpha_0 < \cdots < \alpha_{n-k-1} \text{ and} \\ = 0 \text{ if } \alpha_0 < \cdots < \alpha_{n-k} \end{array} \right\}.$$

Define

$$h\colon D^n_{(k)/D^n_{(k+1)}} \longrightarrow D^{n-1}_{(k)}/D^{n-1}_{(k+1)}$$

as follows: Assume $\alpha_0 < \cdots < \alpha_{n-k-2}$; set

$$\begin{aligned} hs(\alpha_0, \dots, \alpha_{n-1}) =& 0 \text{ if } \alpha_{n-k-1} = \alpha_i, \text{ some } 0 \le i \le n-k-2 \\ =& (-1)^{i+1} s(\alpha_0, \dots, \alpha_i, \alpha_{n-k-1}, \alpha_{i+1}, \dots, \alpha_n) \\ & \quad \text{if } \alpha_i < \alpha_{n-k-1} < \alpha_{i+1}, \text{ some } 0 \le i \le n-k-3 \\ =& 0 \text{ if } \alpha_{n-k-1} < \alpha_i. \end{aligned}$$

One checks with some patience that

$$h\delta + \delta h = \text{identity}$$

hence $(D^\bullet_{(k)}/D^\bullet_{(k+1)}, \delta)$ is exact.

The second variant is *local cohomology.* Suppose $Y \subset X$ is a closed subset and that the covering $\mathcal{U}$ has the property:

$$X \setminus Y = \bigcup_{\alpha \in S_0} U_\alpha.$$

Consider the subgroups:

$$C^i_{S_0}(\mathcal{U}, \mathcal{F}) = \{s \in C^i(\mathcal{U}, \mathcal{F}) \mid s(\alpha_0, \dots, \alpha_i) = 0 \text{ if } \alpha_0, \dots, \alpha_i \in S_0\}.$$

One checks that $\delta(C^i_{S_0}) \subset C^{i+1}_{S_0}$, hence one can define $H^i_{S_0}(\mathcal{U}, \mathcal{F})$ = cohomology of complex $(C^\bullet_{S_0}, \delta)$. Passing to a limit with refinements ($\mathcal{V}$, T_0 refines $\mathcal{U}$, S_0 if $\exists \rho\colon T \to S$ such that $V_\beta \subset U_{\rho\beta}$ and $\rho(T_0) \subset S_0$), one gets $H^i_Y(X, \mathcal{F})$ much as above.

The third variation on the same theme is the hypercohomology of a complex of presheaves:

$$\mathcal{F}^\bullet\colon 0 \longrightarrow \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{m-1}} \mathcal{F}^m \longrightarrow 0$$

(i.e., $d_{i+1} \circ d_i = 0$, for all i). If $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ is an open covering, we get a double complex

$$C^{ij} = C^i(\mathcal{U}, \mathcal{F}^j)$$

where

$\delta_1\colon C^{ij} \longrightarrow C^{i+1,j}$ is the Čech coboundary

$\delta_2\colon C^{ij} \longrightarrow C^{i,j+1}$ is given by applying d_j to the cochain.

Then $\delta_1\delta_2 = \delta_2\delta_1$ and if we set

$$C^{(n)} = \sum_{i+j=n} C^{i,j}$$

and use $d = \delta_1 + (-1)^i\delta_2 \colon C^{(n)} \to C^{(n+1)}$ as differential, then $d^2 = 0$. This is called the associated "total complex". Define

$$\mathbb{H}^n(\mathcal{U}, \mathcal{F}^\bullet) = n\text{-th cohomology group of complex } (C^{(\bullet)}, d).$$

Passing to a limit with refinements, one gets $\mathbb{H}^n(X, \mathcal{F}^\bullet)$. This variant is very important in the De Rham theory (cf. §VIII.3 below).

The most important property of Čech cohomology is the *long exact cohomology sequence.* Suppose

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of *presheaves* (which means that

$$0 \longrightarrow \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U) \longrightarrow \mathcal{F}_3(U) \longrightarrow 0$$

is exact for every open U). Then for every covering $\mathcal{U}$, we get a big diagram relating the cochain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^{i-1}(\mathcal{U},\mathcal{F}_1) \arrow[r] \arrow[d, "\delta"] & C^{i-1}(\mathcal{U},\mathcal{F}_2) \arrow[r] \arrow[d, "\delta"] & C^{i-1}(\mathcal{U},\mathcal{F}_3) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^{i}(\mathcal{U},\mathcal{F}_1) \arrow[r] \arrow[d, "\delta"] & C^{i}(\mathcal{U},\mathcal{F}_2) \arrow[r] \arrow[d, "\delta"] & C^{i}(\mathcal{U},\mathcal{F}_3) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^{i+1}(\mathcal{U},\mathcal{F}_1) \arrow[r] \arrow[d] & C^{i+1}(\mathcal{U},\mathcal{F}_2) \arrow[r] \arrow[d] & C^{i+1}(\mathcal{U},\mathcal{F}_3) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows, i.e., a short exact sequence of complexes of abelian groups. By a standard fact in homological algebra, this always leads to a long exact sequence relating the cohomology groups of the three complexes. In this case, this gives:

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{U}, \mathcal{F}_1) \longrightarrow H^0(\mathcal{U}, \mathcal{F}_2) \longrightarrow H^0(\mathcal{U}, \mathcal{F}_3) \xrightarrow{\delta} H^1(\mathcal{U}, \mathcal{F}_1) \\ \longrightarrow H^1(\mathcal{U}, \mathcal{F}_2) \longrightarrow H^1(\mathcal{U}, \mathcal{F}_3) \xrightarrow{\delta} H^2(\mathcal{U}, \mathcal{F}_1) \longrightarrow \cdots . \end{aligned}$$

Moreover, we may pass to the limit over refinements, getting:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{F}_1) \longrightarrow H^0(X, \mathcal{F}_2) \longrightarrow H^0(X, \mathcal{F}_3) \xrightarrow{\delta} H^1(X, \mathcal{F}_1) \\ \longrightarrow H^1(X, \mathcal{F}_2) \longrightarrow H^1(X, \mathcal{F}_3) \xrightarrow{\delta} H^2(X, \mathcal{F}_1) \longrightarrow \cdots . \end{aligned}$$

In almost all applications, we are only interested in the cohomology of *sheaves* and unfortunately short exact sequences of sheaves are seldom exact as sequences of presheaves. Still, in reasonable cases the long exact cohomology sequence continues to hold. The problem can be analyzed as follows: let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be a short exact sequence of *sheaves*. If we define a subpresheaf $\mathcal{F}_3^* \subset \mathcal{F}_3$ by

$$\mathcal{F}_3^*(U) = \text{Image}\left[\mathcal{F}_2(U) \longrightarrow \mathcal{F}_3(U)\right],$$

then

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3^* \longrightarrow 0$$

is an exact sequence of presheaves, hence we get a long exact sequence:

$$\cdots \longrightarrow H^i(X,\mathcal{F}_1) \longrightarrow H^i(X,\mathcal{F}_2) \longrightarrow H^i(X,\mathcal{F}_3^*) \xrightarrow{\delta} H^{i+1}(X,\mathcal{F}_1) \longrightarrow \cdots$$

Now $\mathcal{F}_3$ is the sheafification of $\mathcal{F}_3^*$ so a long exact sequence for the cohomology of the sheaves $\mathcal{F}_i$ follows if we can prove the more general assertion:

$$(*) \qquad \begin{array}{c}\text{for all presheaves } \mathcal{F}, \text{ the canonical maps}\\ H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathrm{sh}(\mathcal{F}))\\ \text{are isomorphisms.}\end{array}$$

Breaking up $\mathcal{F} \to \mathrm{sh}(\mathcal{F})$ into a diagram of presheaves:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{F} \arrow[rr] \arrow[dr] & & \mathrm{sh}\,\mathcal{F} \arrow[r] & \mathcal{C} \arrow[r] & 0 \\ & & & \mathcal{F}' \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

($\mathcal{K}$ = kernel, $\mathcal{C}$ = cokernel, $\mathcal{F}'$ = image) and applying twice the long exact sequence for presheaves, $(*)$ follows from:

$$(**) \qquad \text{If } \mathcal{F} \text{ is a presheaf such that } \mathrm{sh}(\mathcal{F}) = (0), \text{ then } H^i(X,\mathcal{F}) = (0).$$

The standard case where $(**)$ and hence $(*)$ is satisfied is for paracompact Hausdorff spaces[2] X: we will use this fact once in (3.11) below and §VIII.3 in comparing classical and algebraic De Rham cohomology for complex varieties. Schemes however are far from Hausdorff so we need to take a different tack. In fact, suppose X is a scheme (separated as usual) and

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of *quasi-coherent* sheaves. Then in the above notations:

$$\mathcal{F}_3^*(U) \xrightarrow{\approx} \mathcal{F}_3(U), \quad \text{all } \textit{affine } U$$

so

$$\mathcal{K}(U) = \mathcal{C}(U) = (0), \quad \text{all affine } U.$$

Now if $\mathcal{U}$ is any affine open covering of X, then X separated implies $U_{\alpha_0} \cap \cdots \cap U_{\alpha_i}$ affine for all $\alpha_0, \ldots, \alpha_i$, hence $C^i(\mathcal{U},\mathcal{K}) = C^i(\mathcal{U},\mathcal{C}) = (0)$, hence $H^i(\mathcal{U},\mathcal{K}) = H^i(\mathcal{U},\mathcal{C}) = (0)$. Since affine coverings are cofinal among all coverings, $H^i(X,\mathcal{K}) = H^i(X,\mathcal{C}) = (0)$, hence $H^i(X,\mathcal{F}_3^*) \xrightarrow{\approx} H^i(X,\mathcal{F}_3)$ and we get a long exact sequence for the cohomology of the $\mathcal{F}_i$'s for much more elementary reasons!

What are the functorial properties of cohomology groups? Here are three important kinds:

[2]The proof is as follows: We may compute $H^i(X,\mathcal{F})$ by locally finite coverings $\mathcal{U}$ so let $\mathcal{U}$ be one and let $s \in C^i(\mathcal{U},\mathcal{F})$. A paracompact space is normal so one easily constructs a covering $\mathcal{V}$ with the same index set I such that $\overline{V}_\alpha \subset U_\alpha$, $\forall \alpha \in I$. Now for all $x \in X$, the local finiteness of $\mathcal{U}$ shows that $\exists$ neighborhood N_x of x such that

a) $x \in U_{\alpha_0} \cap \cdots \cap U_{\alpha_i} \Longrightarrow N_x \subset U_{\alpha_0} \cap \cdots \cap U_{\alpha_i}$ and $\mathrm{res}_{N_x}\, s(\alpha_0, \ldots, \alpha_i) = 0$. Shrinking N_x, we can also assume that N_x meets only a finite set of U_α's hence there is a smaller neighborhood $M_x \subset N_x$ of x such that:

b) $M_x \subset$ some V_α and if $M_x \cap V_\beta \neq \emptyset$, then $M_x \subset V_\beta$. Let $\mathcal{W} = \{M_x\}_{x \in X}$. Then $\mathcal{W}$ refines $\mathcal{V}$ and it follows immediately that $\mathrm{ref}_{\mathcal{V},\mathcal{W}}(s) \equiv 0$ as a cochain.

a) If $f\colon X \to Y$ is a continuous map of topological spaces, $\mathcal{F}$ (resp. $\mathcal{G}$) a presheaf on X (resp. Y), and $\alpha\colon \mathcal{G} \to \mathcal{F}$ a homomorphism covering f, (i.e., a set of homomorphisms:

$$\alpha(U)\colon \mathcal{G}(U) \longrightarrow \mathcal{F}(f^{-1}(U)), \qquad \text{all open } U \subset Y$$

commuting with restriction), then we get canonical maps:

$$(f,\alpha)^*\colon H^i(Y,\mathcal{G}) \longrightarrow H^i(X,\mathcal{F}), \qquad \text{all } i.$$

b) If we have two short exact sequences of presheaves and a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d, "\alpha_1"'] & \mathcal{F}_2 \arrow[r] \arrow[d, "\alpha_2"'] & \mathcal{F}_3 \arrow[r] \arrow[d, "\alpha_3"'] & 0 \\ 0 \arrow[r] & \mathcal{G}_1 \arrow[r] & \mathcal{G}_2 \arrow[r] & \mathcal{G}_3 \arrow[r] & 0, \end{tikzcd}$$

then the δ's in the long exact cohomology sequences give a commutative diagram:

$$\begin{tikzcd} H^i(X,\mathcal{F}_3) \arrow[r, "\delta"] \arrow[d, "\alpha_3"'] & H^{i+1}(X,\mathcal{F}_1) \arrow[d, "\alpha_1"] \\ H^i(X,\mathcal{G}_3) \arrow[r, "\delta"] & H^{i+1}(X,\mathcal{G}_1), \end{tikzcd}$$

(i.e., the $H^i(X,\mathcal{F})$'s together are a "cohomological δ-functor").

c) If $\mathcal{F}$ and $\mathcal{G}$ are two presheaves of abelian groups, define a presheaf $\mathcal{F}\otimes\mathcal{G}$ by $(\mathcal{F}\otimes\mathcal{G})(U) = \mathcal{F}(U)\otimes\mathcal{G}(U)$. Then there is a bilinear map:

$$H^i(X,\mathcal{F}) \times H^j(X,\mathcal{G}) \longrightarrow H^{i+j}(X,\mathcal{F}\otimes\mathcal{G})$$

called cup product, and written $\cup$.

To construct the map in (a), take the obvious map of cocycles and check that it commutes with δ; (b) is a straightforward computation; as for (c), define $\cup$ on couples by:

$$(s\cup t)(\alpha_0,\dots,\alpha_{i+j}) = \operatorname{res} s(\alpha_0,\dots,\alpha_i) \otimes \operatorname{res} t(\alpha_i,\dots,\alpha_{i+j})$$

and check that $\delta(s\cup t) = \delta s \cup t + (-1)^i s \cup \delta t$. It is not hard to check that $\cup$ is associative and has a certain skew-commutativity property:

c′) If $s_i \in H^{k_i}(X,\mathcal{F}_i)$, $i=1,2,3$, then in the group $H^{k_1+k_2+k_3}(X,\mathcal{F}_1\otimes\mathcal{F}_2\otimes\mathcal{F}_3)$ we have

$$(s_1\cup s_2)\cup s_3 = s_1 \cup (s_2 \cup s_3).$$

c″) If $\operatorname{Symm}^2\mathcal{F}$ is the quotient presheaf of $\mathcal{F}\otimes\mathcal{F}$ by the subsheaf of elements $a\otimes b - b\otimes a$, and $s_i \in H^{k_i}(X,\mathcal{F})$, $i=1,2$, then in the group $H^{k_1+k_2}(X,\operatorname{Symm}^2\mathcal{F})$ we have

$$s_1\cup s_2 = (-1)^{k_1k_2} s_2 \cup s_1.$$

The proofs are left to the reader.

The cohomology exact sequence leads to the method of computing cohomology by *acyclic resolutions*: suppose a sheaf $\mathcal{F}$ is given and we construct a long exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \cdots,$$

such that:

a) $H^i(X,\mathcal{G}_k) = (0)$, $i \geq 1$, $k \geq 0$.

b) If $\mathcal{K}_k = \operatorname{Ker}(\mathcal{G}_{k+1} \to \mathcal{G}_{k+2})$ and $\mathcal{C}_k = \text{cokernel as presheaf } (\mathcal{G}_{k-1} \to \mathcal{G}_k)$ so that $\mathcal{K}_k = \operatorname{sh}(\mathcal{C}_k)$, then assume

$$H^i(X,\mathcal{C}_k) \xrightarrow{\approx} H^i(X,\mathcal{K}_k), \quad i \geq 0,\ k \geq 0.$$

Then $H^i(X,\mathcal{F})$ is isomorphic to the i-th cohomology group of the complex:

$$0 \longrightarrow \mathcal{G}_0(X) \longrightarrow \mathcal{G}_1(X) \longrightarrow \mathcal{G}_2(X) \longrightarrow \cdots .$$

To see this, use induction on i. We may split off the first part of our resolution like this:

i) $0 \to \mathcal{F} \to \mathcal{G}_0 \to \mathcal{C}_0 \to 0$, exact as *presheaves*.
ii) $0 \to \mathcal{K}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to \cdots$, exact as *sheaves*.

So by the cohomology sequence of (i) and induction applied to the resolution (ii):

a)
$$\begin{aligned} H^0(X,\mathcal{F}) &\cong \operatorname{Ker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{C}_0)\right] \\ &\cong \operatorname{Ker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{K}_0)\right] \\ &\cong \operatorname{Ker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{G}_1)\right] \end{aligned}$$

b)
$$\begin{aligned} H^1(X,\mathcal{F}) &\cong \operatorname{Coker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{C}_0)\right] \\ &\cong \operatorname{Coker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{K}_0)\right] \\ &\cong \operatorname{Coker}\left[H^0(\mathcal{G}_0) \longrightarrow \operatorname{Ker}\left[H^0(\mathcal{G}_1) \longrightarrow H^0(\mathcal{G}_2)\right]\right] \\ &\cong H^1(\text{the complex } H^0(\mathcal{G}_\bullet)). \end{aligned}$$

c)
$$\begin{aligned} H^i(X,\mathcal{F}) &\cong H^{i-1}(X,\mathcal{C}_0) \\ &\cong H^{i-1}(X,\mathcal{K}_0) \\ &\cong H^i(\text{the complex } H^0(\mathcal{G}_\bullet)), \quad i \geq 2. \end{aligned}$$

If $\mathcal{F}$ is a sheaf, we have seen that $H^0(X,\mathcal{F})$ is just $\Gamma(X,\mathcal{F})$ or $\mathcal{F}(X)$. $H^1(X,\mathcal{F})$ also has a simple interpretation in terms of "twisted structures" over X. Define

A principal $\mathcal{F}$-sheaf
= a sheaf of sets $\mathcal{G}$, plus an action of $\mathcal{F}$ on $\mathcal{G}$
(i.e., $\mathcal{F}(U)$ acts on $\mathcal{G}(U)$ commuting with restriction)
such that $\exists$ a covering $\{U_\alpha\}$ of X where:
$\operatorname{res}_{U_\alpha}$ ($\mathcal{G}$, as sheaf with $\mathcal{F}$-action)
$\cong \operatorname{res}_{U_\alpha}$ ($\mathcal{F}$, with $\mathcal{F}$-action on itself by translation).

Then if $\mathcal{F}$ is a sheaf:

$(*)$ $\quad H^1(X,\mathcal{F}) \cong \{\text{set of principal } \mathcal{F}\text{-sheaves, modulo isomorphism}\}$.

$$H^1(\mathcal{U},\mathcal{F}) \cong \left\{ \begin{array}{l} \text{subset of those principal } \mathcal{F}\text{-sheaves which are trivial} \\ \text{on the open sets } U_\alpha \text{ of the covering } \mathcal{U} \end{array} \right\}.$$

In fact,

a) Given $\mathcal{G}$, let $\phi_\alpha \colon \mathcal{G}|_{U_\alpha} \xrightarrow{\approx} \mathcal{F}|_{U_\alpha}$ be an $\mathcal{F}$-isomorphism. Then on $U_\alpha \cap U_\beta$, $\phi_\alpha \circ \phi_\beta^{-1} \colon \mathcal{F}|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}|_{U_\alpha \cap U_\beta}$ is an $\mathcal{F}$-automorphism. If it carries the 0-section to $s(\alpha,\beta) \in \mathcal{F}(U_\alpha \cap U_\beta)$, it will be the map $x \mapsto x + s(\alpha,\beta)$. One checks that s is a 1-cocycle, hence it defines a cohomology class in $H^1(\mathcal{U},\mathcal{F})$, and by refinement in $H^1(X,\mathcal{F})$.

b) Conversely, given $\sigma \in H^1(X, \mathcal{F})$, represent σ by a 1-cocycle $s(\alpha, \beta)$ for a covering $\{U_\alpha\}$. Define a sheaf $\mathcal{G}_\sigma$ by

$$\mathcal{G}_\sigma(V) = \left\{ \begin{array}{l} \text{collections of elements } t_\alpha \in \mathcal{F}(V \cap U_\alpha) \text{ such that} \\ \operatorname{res} t_\alpha + s(\alpha, \beta) = \operatorname{res} t_\beta \text{ in } \mathcal{F}(V \cap U_\alpha \cap U_\beta) \end{array} \right\}.$$

Intuitively, $\mathcal{G}_\sigma$ is obtained by "glueing" the sheaves $\mathcal{F}|_{U_\alpha}$ together by translation by $s(\alpha, \beta)$ on $U_\alpha \cap U_\beta$.

We leave it to the reader to check that $\mathcal{G}_\sigma$ is independent of the choice of s and that the constructions (a) and (b) are inverse to each other. The same ideas exactly allow you to prove:

If $\mathcal{O}_X$ is a sheaf of *rings* on X and $\mathcal{O}_X^* =$ subsheaf of units in $\mathcal{O}_X$, then

$$H^1(X, \mathcal{O}_X^*) \cong \left\{ \begin{array}{l} \text{set of sheaves of } \mathcal{O}_X\text{-modules, locally isomorphic} \\ \text{to } \mathcal{O}_X \text{ itself, modulo isomorphism} \end{array} \right\}$$

(cf. §III.6)

and

If X is locally connected and $(\mathbb{Z}/n\mathbb{Z})_X =$ sheafification of the constant presheaf $\mathbb{Z}/n\mathbb{Z}$, then

$$H^1(X, (\mathbb{Z}/n\mathbb{Z})_X) \cong \left\{ \begin{array}{l} \text{set of covering spaces } \pi\colon Y \to X \text{ with } \mathbb{Z}/n\mathbb{Z} \\ \text{acting on } Y, \text{ permuting freely and transitively} \\ \text{the points of each set } \pi^{-1}(x),\ x \in X \end{array} \right\}.$$

2. The case of schemes: Serre's theorem

From now on, we assume that X is a *scheme*[3] and that $\mathcal{F}$ is a *quasi-coherent sheaf.* The main result is this:

THEOREM 2.1 (Serre). *Let $\mathcal{U}$ and $\mathcal{V}$ be two affine open coverings of X, with $\mathcal{V}$ refining $\mathcal{U}$. Then*

$$\operatorname{res}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F})$$

is an isomorphism.

The proof consists in two steps. The first is a general criterion for ref to be an isomorphism. The second is an explicit computation for modules and distinguished affine coverings. The general criterion is this:

PROPOSITION 2.2. *Let X be any topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathcal{U}$ and $\mathcal{V}$ two open coverings of X. Suppose $\mathcal{V}$ refines $\mathcal{U}$. For every finite subset $S_0 = \{\alpha_0, \dots, \alpha_p\} \subset S$, let*

$$U_{S_0} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$$

and let $\mathcal{V}|_{U_{S_0}}$ denote the covering of U_{S_0} induced by $\mathcal{V}$. Assume:

$$H^i(\mathcal{V}|_{U_{S_0}}, \mathcal{F}|_{U_{S_0}}) = (0), \quad \textit{all } S_0,\ i > 0.$$

Then $\operatorname{ref}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U}, \mathcal{F}) \to H^i(\mathcal{V}, \mathcal{F})$ is an isomorphism for all i.

---

[3] Our approach works only because all our schemes are separated. In the general case, Čech cohomology is not good and either derived functors (via Grothendieck) or a modified Čech complex (via Lubkin or Verdier) must be used.

Proof. The technique is to compare the two Čech cohomologies via a big double complexes:

$$C^{p,q} = \prod_{\alpha_0,\ldots,\alpha_p \in S} \prod_{\beta_0,\ldots,\beta_q \in T} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \cap V_{\beta_0} \cap \cdots \cap V_{\beta_q}).$$

By ignoring either the α's or the β's and taking δ in the β's or α's, we get two coboundary maps:

$$\delta_1 \colon C^{p,q} \longrightarrow C^{p+1,q}$$

$$\delta_1 s(\alpha_0,\ldots,\alpha_{p+1},\beta_0,\ldots,\beta_q) = \sum_{j=0}^{p+1} (-1)^j s(\alpha_0,\ldots,\widehat{\alpha}_j,\ldots,\alpha_{p+1},\beta_0,\ldots,\beta_q)$$

and

$$\delta_2 \colon C^{p,q} \longrightarrow C^{p,q+1}$$

$$\delta_2 s(\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\beta_{q+1}) = \sum_{j=0}^{q+1} (-1)^j s(\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\widehat{\beta}_j,\ldots,\beta_{q+1}).$$

One checks immediately that these satisfy $\delta_1^2 = \delta_2^2 = 0$ and $\delta_1\delta_2 = \delta_2\delta_1$. As in §1, we get a "total complex" by setting:

$$C^{(n)} = \sum_{\substack{p+q=n \\ p,q \geq 0}} C^{p,q}$$

and with $d = \delta_1 + (-1)^p \delta_2 \colon C^{(n)} \to C^{(n+1)}$ as differential. Here is the picture:

$$\begin{tikzcd}
{} & & \\
C^{0,2} \arrow[u] \arrow[r] & {} & \\
C^{0,1} \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & C^{1,1} \arrow[u] \arrow[r] & {} \\
C^{0,0} \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & C^{1,0} \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & C^{2,0} \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

where

$$C^{0,2} = \prod_{\substack{\alpha\in S\\ \beta_0,\beta_1,\beta_2\in T}} \mathcal{F}(U_\alpha \cap V_{\beta_0} \cap V_{\beta_1} \cap V_{\beta_2})$$
$$C^{0,1} = \prod_{\substack{\alpha\in S\\ \beta_0,\beta_1\in T}} \mathcal{F}(U_\alpha \cap V_{\beta_0} \cap V_{\beta_1})$$
$$C^{1,1} = \prod_{\substack{\alpha_0,\alpha_1\in S\\ \beta_0,\beta_1\in T}} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap V_{\beta_0} \cap V_{\beta_1})$$
$$C^{0,0} = \prod_{\substack{\alpha\in S\\ \beta\in T}} \mathcal{F}(U_\alpha \cap V_\beta)$$
$$C^{1,0} = \prod_{\substack{\alpha_0,\alpha_1\in S\\ \beta\in T}} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap V_\beta)$$
$$C^{2,0} = \prod_{\substack{\alpha_0,\alpha_1,\alpha_2\in S\\ \beta\in T}} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap V_\beta).$$

We need to observe four things about this situation:

(A) The columns of this double complex are just products of the Čech complexes for the coverings $\mathcal{V}|_{U_{S_0}}$ for various $S_0 \subset S$: in fact the p-th column $C^{p,0} \to C^{p,1} \to \cdots$ is the product of these complexes for all S_0 with $\#S_0 = p+1$. By assumption these complexes have no cohomology beyond the first place, hence

the δ_2-cohomology of the columns

$$\operatorname{Ker}\left[\delta_2\colon C^{p,q} \longrightarrow C^{p,q+1}\right] / \operatorname{Image}\left[\delta_2\colon C^{p,q-1} \longrightarrow C^{p,q}\right]$$

is (0) for all $p \geq 0$, all $q > 0$.

(B) The rows of this double complex are similarly products of the Čech complexes for the coverings $\mathcal{U}|_{V_{T_0}}$ for various $T_0 \subset T$. Now $V_{T_0} \subset$ some $V_\beta \subset$ some U_α, hence the covering $\mathcal{U}|_{V_{T_0}}$ of V_{T_0} includes among its open sets the whole space V_{T_0}. For such silly coverings, Čech cohomology always vanishes —

Lemma 2.3. *X a topological space, $\mathcal{F}$ a sheaf, and $\mathcal{U}$ an open covering of X such that $X \in \mathcal{U}$. Then $H^i(\mathcal{U}, \mathcal{F}) = (0)$, $i > 0$.*

Proof of Lemma 2.3. Let $X = U_\zeta \in \mathcal{U}$. For all $s \in Z^i(\mathcal{U}, \mathcal{F})$, define an $(i-1)$-cochain by:

$$t(\alpha_0, \ldots, \alpha_{i-1}) = s(\zeta, \alpha_0, \ldots, \alpha_{i-1})$$

[OK since $U_\zeta \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_{i-1}} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_{i-1}}$!] An easy calculation shows that $s = \delta t$. □

Hence

the δ_1-cohomology of the rows is (0) at the (p,q)-th spot, for all $p > 0$, $q > 0$.

(C) Next there is a big diagram-chase —

Lemma 2.4 (The easy lemma of the double complex). *Let $\{C^{p,q}, \delta_1, \delta_2\}_{p,q\geq 0}$ be any double complex (meaning $\delta_1^2 = \delta_2^2 = 0$ and $\delta_1\delta_2 = \delta_2\delta_1$). Assume that the δ_2-cohomology:*

$$H^{p,q}_{\delta_2} = \operatorname{Ker}\left[\delta_2\colon C^{p,q} \longrightarrow C^{p,q+1}\right] / \operatorname{Image}\left[\delta_2\colon C^{p,q-1} \longrightarrow C^{p,q}\right]$$

is (0) *for all* $p \geq 0$, $q > 0$. *Then there is an isomorphism:*

$\left(\delta_1\text{-}\textit{cohomology of } H^{p,0}_{\delta_2}\right) = ((d = \delta_1 + (-1)^p\delta_2)\text{-}\textit{cohomology of total complex})$

i.e.,

$$\frac{\{x \in C^{p,0} \mid \delta_1 x = \delta_2 x = 0\}}{\{\delta_1 x \mid x \in C^{p-1,0} \textit{ with } \delta_2 x = 0\}} \cong \frac{\left\{x \in \sum_{i+j=p} C^{i,j} \mid dx = 0\right\}}{\left\{dx \mid x \in \sum_{i+j=p-1} C^{i,j}\right\}}.$$

Proof of Lemma 2.4. We give the proof in detail for $p = 2$ in such a way that it is clear how to set up the proof in general. Start with $x = (x_{2,0}, x_{1,1}, x_{0,2}) \in \sum_{i+j=2} C^{i,j}$ such that $dx = 0$, i.e.,

$$\delta_1 x_{2,0} = 0;\ \delta_1 x_{1,1} + \delta_2 x_{2,0} = 0;\ \delta_1 x_{0,2} - \delta_2 x_{1,1} = 0;\ \delta_2 x_{0,2} = 0.$$

$$\begin{tikzcd}
0 & & \\
x_{0,2} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & \pm y & \\
& x_{1,1} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & \pm z \\
& & x_{2,0} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & 0
\end{tikzcd}$$

Now $\delta_2 x_{0,2} = 0 \implies x_{0,2} = \delta_2 x_{0,1}$ for some $x_{0,1}$. Alter x by the coboundary $d(0, -x_{0,1})$: we find

$$x \sim x' = (x'_{2,0}, x'_{1,1}, 0) \qquad (\sim \text{ means cohomologous}).$$

But then $dx' = 0 \implies \delta_2 x'_{1,1} = 0 \implies x'_{1,1} = \delta_2 x_{1,0}$ for some $x_{1,0}$. Alter x' by the coboundary $d(x_{1,0}, 0)$: we find

$$x' \sim x'' = (x''_{2,0}, 0, 0)$$

and $dx'' = 0 \implies \delta_1 x''_{2,0}$ and $\delta_2 x''_{2,0}$ are 0. Thus $x''_{2,0}$ defines an element of $H^2_{\delta_1}\left(\text{the complex } H^{p,0}_{\delta_2}\right)$. This argument (generalized in the obvious way) shows that the map:

$$\Phi\colon \left(\delta_1\text{-cohomology of } H^{p,0}_{\delta_2}\right) \longrightarrow (d\text{-cohomology of total complex})$$

is *surjective.* Now say $x_{2,0} \in C^{2,0}$ satisfies $\delta_1 x_{2,0} = \delta_2 x_{2,0} = 0$. Say $(x_{2,0}, 0, 0) = dx$, $x = (x_{1,0}, x_{0,1}) \in \sum_{i+j=1} C^{i,j}$, i.e.,

$$x_{2,0} = \delta_1 x_{1,0};\ -\delta_2 x_{1,0} + \delta_1 x_{0,1} = 0; \delta_2 x_{0,1} = 0.$$

$$\begin{tikzcd}
0 & \\
x_{0,1} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & \pm y \\
& x_{1,0} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & x_{2,0}
\end{tikzcd}$$

Then $\delta_2 x_{0,1} = 0 \implies x_{0,1} = \delta_2 x_{0,0}$, for some $x_{0,0}$. Alter x by the coboundary $-dx_{0,0}$: we find

$$x \sim x' = (x'_{0,0}, 0)$$

and $dx' = (x_{2,0}, 0, 0)$. Then $\delta_2 x'_{1,0} = 0$ and $\delta_1 x'_{1,0} = x_{2,0}$, i.e., $x_{2,0}$ goes to 0 in the δ_1-cohomology of $H^{p,0}_{\delta_2}$. This gives injectivity of Φ. $\square$

If we combine (A), (B) and (C), applied both to the rows and columns of our double complex, we find isomorphisms:

$$\begin{aligned} H^n_d(\text{total complex } C^{(\cdot)}) &\cong H^n_{\delta_1}\left(\text{the complex } \operatorname{Ker}\delta_2 \text{ in } C^{\bullet,0}\right) \\ &\cong H^n_{\delta_2}\left(\text{the complex } \operatorname{Ker}\delta_1 \text{ in } C^{0,\bullet}\right). \end{aligned}$$

But

$$\begin{aligned} \operatorname{Ker}\left(\delta_2 \colon C^{n,0} \longrightarrow C^{n,1}\right) &\cong \prod_{\alpha_0,\dots,\alpha_n \in S} \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_n}) = C^n(\mathcal{U}, \mathcal{F}) \\ \operatorname{Ker}\left(\delta_1 \colon C^{0,n} \longrightarrow C^{1,n}\right) &\cong \prod_{\beta_0,\dots,\beta_n \in T} \mathcal{F}(V_{\beta_0} \cap \dots \cap V_{\beta_n}) = C^n(\mathcal{V}, \mathcal{F}), \end{aligned}$$

so in fact

$$\begin{aligned} H^n_d\left(\text{total complex } C^{(\cdot)}\right) &\cong H^n(\mathcal{U}, \mathcal{F}) \\ &\cong H^n(\mathcal{V}, \mathcal{F}). \end{aligned}$$

It remains to check:

(D) The above isomorphism is the refinement map, i.e., if $s(\alpha_0, \dots, \alpha_n)$ is an n-cocycle for $\mathcal{U}$, then $s \in C^{n,0}$ and $\operatorname{ref}^{\sigma}_{\mathcal{U},\mathcal{V}} s \in C^{0,n}$ are cohomologous in the total complex. In fact, define $t \in C^{(n-1)}$ by setting its $(l, n-1-l)$-th component equal to:

$$t_l(\alpha_0, \dots, \alpha_l, \beta_0, \dots, \beta_{n-1-l}) = (-1)^l \operatorname{res} s(\alpha_0, \dots, \alpha_l, \sigma\beta_0, \dots, \sigma\beta_{n-1-l}).$$

Thus a straightforward calculation shows that $dt = (\operatorname{ref}^{\sigma}_{\mathcal{U},\mathcal{V}} s) - s$.

This completes the proof of Proposition 2.2. $\square$

Now return to the proof of Theorem 2.1 for quasi-coherent sheaves on schemes! The second step in its proof is the following explicit calculation:

PROPOSITION 2.5. *Let* $\operatorname{Spec} R$ *be an affine scheme,* $\mathcal{U} = \{\operatorname{Spec} R_{f_i}\}_{i \in I}$ *a finite distinguished affine covering and* $\widetilde{M}$ *a quasi-coherent sheaf on* X. *Then* $H^i(\mathcal{U}, \widetilde{M}) = (0)$, *all* $i > 0$.

PROOF. Since $\widetilde{M}(\operatorname{Spec} R_f) \cong M_f$ and $\bigcap_{i \in I_0} \operatorname{Spec} R_{f_i} = \operatorname{Spec} R_{(\prod_{i \in I_0} f_i)}$, the complex of Čech cochains reduces to:

$$\prod_{i \in I} M_{f_i} \longrightarrow \prod_{i_0, i_1 \in I} M_{(f_{i_0} \cdot f_{i_1})} \longrightarrow \prod_{i_0, i_1, i_2 \in I} M_{(f_{i_0} \cdot f_{i_1} \cdot f_{i_2})} \longrightarrow \cdots.$$

Using the fact that the covering is finite, we can write a k-cochain:

$$m(i_0, \dots, i_k) = \frac{m_{i_0,\dots,i_k}}{(f_{i_0} \cdots f_{i_k})^N}, \quad m_{i_0,\dots,i_k} \in M$$

with fixed denominator. Then

$$\begin{aligned} (\delta m)(i_0, \dots, i_{k+1}) &= \frac{m_{i_1,\dots,i_{k+1}}}{(f_{i_1} \cdots f_{i_k})^N} - \frac{m_{i_0,i_2,\dots,i_{k+1}}}{(f_{i_0} f_{i_2} \cdots f_{i_{k+1}})^N} + \dots + (-1)^{k+1} \frac{m_{i_0,\dots,i_k}}{(f_{i_0} \cdots f_{i_k})^N} \\ &= \frac{f^N_{i_0} m_{i_1,\dots,i_{k+1}} - f^N_{i_1} m_{i_0,i_2,\dots,i_{k+1}} + \dots + (-1)^{k+1} f^N_{i_{k+1}} m_{i_0,\dots,i_k}}{(f_{i_0} \cdots f_{i_{k+1}})^N}. \end{aligned}$$

If $\delta m = 0$, then this expression is 0 in $M_{(f_{i_0} \cdots f_{i_{k+1}})}$, hence

$$(f_{i_0} \cdots f_{i_{k+1}})^{N'} \left[f^N_{i_0} m_{i_1,\dots,i_{k+1}} - f^N_{i_1} m_{i_0,i_2,\dots,i_{k+1}} + \dots + (-1)^{k+1} f^N_{i_{k+1}} m_{i_0,\dots,i_k} \right] = 0$$

in M if N' is sufficiently large. But rewriting the original cochain m with N replaced by $N+N'$, we have

$$m(i_0,\dots,i_k) = \frac{m'_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_k})^{N+N'}}, \quad m'_{i_0,\dots,i_k} = (f_{i_0}\cdots f_{i_k})^{N'} m_{i_0,\dots,i_k}$$

so that

$$(*) \quad f_{i_0}^{N+N'} m'_{i_1,\dots,i_{k+1}} - f_{i_1}^{N+N'} m'_{i_0,i_2,\dots,i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^{N+N'} m'_{i_0,\dots,i_k} = 0 \quad \text{in } M.$$

Now since

$$\operatorname{Spec} R = \bigcup \operatorname{Spec} R_{f_i} = \bigcup \operatorname{Spec} R_{(f_i^{N+N'})},$$

it follows that $1 \in (\dots, f_i^{N+N'}, \dots)$, i.e., we can write

$$1 = \sum_{i\in I} g_i \cdot f_i^{N+N'}$$

for some $g_i \in R$. Now define a $(k-1)$-cochain n by the formula:

$$n(i_0,\dots,i_{k-1}) = \frac{n_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_{k-1}})^{N+N'}}$$
$$n_{i_0,\dots,i_{k-1}} = \sum_{l\in I} g_l \cdot m'_{l,i_0,\dots,i_{k=1}}.$$

Then $m = \delta n$! In fact

$$\begin{aligned}(\delta n)(i_0,\dots,i_k) &= \sum_{j=0}^{k} (-1)^j \cdot \frac{n_{i_0},\dots,\widehat{i}_j,\dots,i_k}{(f_{i_0}\cdots \widehat{f}_{i_j}\cdots f_{i_k})^{N+N'}} \\
&= \frac{1}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{j=0}^{k} (-1)^j f_{i_j}^{N+N'} \cdot \sum_{l\in I} g_l m'_{l,i_0,\dots,\widehat{i}_j,\dots,i_k} \\
&= \frac{1}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{l\in I} g_l \sum_{j=0}^{k} (-1)^j f_{i_j}^{N+N'} m'_{l,i_0,\dots,\widehat{i}_j,\dots,i_k} \\
&= \frac{1}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{l\in I} g_l f_i^{N+N'} m'_{i_0,\dots,i_k} \quad \text{(by } (*)) \\
&= \frac{m'_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{l\in I} g_l f_i^{N+N'} \\
&= m(i_0,\dots,i_k).\end{aligned}$$

□

Corollary 2.6. *Let X be an affine scheme, $\mathcal{U}$ any affine covering of X and $\widetilde{M}$ a quasi-coherent sheaf on X. Then $H^i(\mathcal{U},\widetilde{M}) = (0)$, $i > 0$.*

Proof. Since the distinguished affines form a basis for the topology of X, and X is quasi-compact, we can find a finite distinguished affine covering $\mathcal{V}$ of X refining $\mathcal{U}$. Consider the map

$$\mathrm{ref}_{\mathcal{U},\mathcal{V}} \colon H^i(\mathcal{U},\widetilde{M}) \longrightarrow H^i(\mathcal{V},\widetilde{M}).$$

By Proposition 2.5, $H^i(\mathcal{V},\widetilde{M}) = (0)$ all $i > 0$, *and* $H^i(\mathcal{V}|_{U_{S_0}}, \widetilde{M}|_{U_{S_0}}) = (0)$ for all $i > 0$ and for all finite intersections $U_{S_0} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ (since each $V_\beta \cap U_{S_0}$ is a distinguished affine in U_{S_0} too). Therefore by Proposition 2.2, $\mathrm{ref}_{\mathcal{U},\mathcal{V}}$ is an isomorphism, hence $H^i(\mathcal{U},\widetilde{M}) = (0)$ for all $i > 0$. □

Theorem 2.1 now follows immediately from Proposition 2.2 and Corollary 2.6, in view of the fact that since X is separated, each U_{S_0} as a finite intersection of affines, is also affine as are the open sets $V_\beta \cap U_{S_0}$ that cover it.

Theorem 2.1 implies:

COROLLARY 2.7. *For all schemes X, quasi-coherent $\mathcal{F}$ and affine covering $\mathcal{U}$, the natural map:*

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$$

is an isomorphism.

The "easy lemma of the double complex" (Lemma 2.4) has lots of other applications in homological algebra. We sketch one that we can use later on.

a) Let R be any commutative ring, let $M^{(1)}$, $M^{(2)}$ be R-modules, choose free resolutions $F_\bullet^{(1)} \to M^{(1)}$ and $F_\bullet^{(2)} \to M^{(2)}$, i.e., exact sequences

$$\begin{aligned}
\longrightarrow F_n^{(1)} \longrightarrow F_{n-1}^{(1)} \longrightarrow \cdots \longrightarrow F_1^{(1)} \longrightarrow F_0^{(1)} \longrightarrow M^{(1)} \longrightarrow 0\\
\longrightarrow F_n^{(2)} \longrightarrow F_{n-1}^{(2)} \longrightarrow \cdots \longrightarrow F_1^{(2)} \longrightarrow F_0^{(2)} \longrightarrow M^{(2)} \longrightarrow 0
\end{aligned}$$

where all $F_j^{(i)}$ are free R-modules. Look at the double complex $C_{i,j} = F_i^{(1)} \otimes_R F_j^{(2)}$, $0 \le i, j$ with boundary maps

$$\begin{aligned}
d^{(1)} &: C_{i,j} \longrightarrow C_{i-1,j}\\
d^{(2)} &: C_{i,j} \longrightarrow C_{i,j-1}
\end{aligned}$$

induced by the d's in the two resolutions. Then Lemma 2.4 shows that

$$\begin{aligned}
H_n(\text{total complex } C_{\bullet,\bullet}) &\cong H_n(\text{complex } F_\bullet^{(1)} \otimes_R M^{(2)})\\
&\cong H_n(\text{complex } M^{(1)} \otimes_R F_\bullet^{(2)}).
\end{aligned}$$

Note that the arrows here are reversed compared to the situation in the text. For complexes in which d decreases the index, we take of homology on H_n instead of cohomology on H^n. It is not hard to check that the above R-modules are *independent* of the resolutions $F_\bullet^{(1)}$, $F_\bullet^{(2)}$. They are called $\mathrm{Tor}_n^R(M^{(1)}, M^{(2)})$. The construction could be globalized: if X is a scheme, $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ are quasi-coherent sheaves, then there are canonical quasi-coherent sheaves $\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})$ such that for all affine open $U \subset X$, if

$$\begin{aligned}
U &= \operatorname{Spec} R\\
\mathcal{F}^{(i)} &= \widetilde{M^{(i)}},
\end{aligned}$$

then

$$\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})|_U = \mathrm{Tor}_n^R(M^{(1)}, M^{(2)})^{\sim}.$$

I want to conclude this section with the classical explanation of the "meaning" of $H^1(X, \mathcal{O}_X)$, via so-called "Cousin data". Let me digress to give a little history: in the 19th century Mittag-Leffler proved that for any discrete set of points $\alpha_i \in \mathbb{C}$ and any positive integer n_i, there is a meromorphic function $f(z)$ with poles of order n_i at α_i and no others. Cousin generalized this to meromorphic functions $f(z_1, \dots, z_n)$ on $\mathbb{C}^n$ in the following form: say $\{U_i\}$ is an open covering of $\mathbb{C}^n$ and f_i is a meromorphic function on U_i such that $f_i - f_j$ is holomorphic on $U_i \cap U_j$. Then there exists a meromorphic function f such that $f - f_i$ is holomorphic on U_i. We can easily pose an algebraic analog of this —

a) Let X be a reduced and irreducible scheme.

b) Let $\mathbb{R}(X)$ = function field of X.

c) Cousin data consists in an open covering $\{U_\alpha\}_{\alpha\in S}$ of X plus $f_\alpha \in \mathbb{R}(X)$ for each α such that

$$f_\alpha - f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X), \quad \text{all } \alpha, \beta.$$

d) The Cousin problem for this data is to find $f \in \mathbb{R}(X)$ such that

$$f - f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X), \quad \text{all } \alpha,$$

i.e., f and f_α have the same "polar part" in U_α.

For all Cousin data $\{f_\alpha\}$, let $g_{\alpha\beta} = f_\alpha - f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$. Then $\{g_{\alpha\beta}\}$ is a 1-cocycle in $\mathcal{O}_X$ for the covering $\{U_\alpha\}$ and by refinement, it defines an element of $H^1(X, \mathcal{O}_X)$, which we call $\mathrm{ob}(\{f_\alpha\})$(= the "obstruction").

Proposition 2.8. $\mathrm{ob}(\{f_\alpha\}) = 0$ *iff the Cousin problem has a solution.*

Proof. If $\mathrm{ob}(\{f_\alpha\}) = 0$, then there is a finer covering $\{V_\alpha\}_{\alpha\in T}$ and $h_\alpha \in \Gamma(V_\alpha, \mathcal{O}_X)$ such that if $\sigma\colon T \to S$ is a refinement map, then

$$h_\alpha - h_\beta = \mathrm{res}\, g_{\sigma\alpha,\sigma\beta} = \mathrm{res}(f_{\sigma\alpha} - f_{\sigma\beta})$$

(equality here being in the ring $\Gamma(V_\alpha \cap V_\beta, \mathcal{O}_X)$). But then in $\mathbb{R}(X)$,

$$h_\alpha - f_{\sigma\alpha} = h_\beta - f_{\sigma\beta},$$

i.e., $f_{\sigma\alpha} - h_\alpha = F$ is independent of α. Then F has the same polar part as $f_{\sigma\alpha}$ in V_α. And for any $x \in U_\alpha$, take β so that $x \in V_\beta$ too; then since $f_\alpha - f_{\sigma\beta} \in \mathcal{O}_{x,X}$, it follows that $F - f_\alpha = (F - f_{\sigma\beta}) + (f_{\sigma\beta} - f_\alpha) \in \mathcal{O}_{x,X}$, i.e., F has the same polar part as f_α throughout U_α, so F solves the Cousin problem. Conversely, if such F exists, let $h_\alpha = f_\alpha - F$; then $h_\alpha - h_\beta = g_{\alpha\beta}$ and $h_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X)$, i.e., $\{g_{\alpha\beta}\} = \delta(\{h_\alpha\})$ is a 1-coboundary. □

3. Higher direct images and Leray's spectral sequence

One of the main tools that is used over and over again in computing cohomology is the higher direct image sheaf and the Leray spectral sequence. Let $f\colon X \to Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups on X. For all $i \geq 0$, consider the presheaf on Y:

a) $U \longmapsto H^i(f^{-1}(U), \mathcal{F}), \quad \forall U \subset Y$ open

b) if $U_1 \subset U_2$, then

$$\mathrm{res}\colon H^i(f^{-1}(U_2), \mathcal{F}) \longrightarrow H^i(f^{-1}(U_1), \mathcal{F})$$

is the canonical map.

Definition 3.1. $R^i f_*(\mathcal{F})$ = the sheafification of this presheaf, i.e., the universal sheaf which receives homomorphisms:

$$H^i(f^{-1}(U), \mathcal{F}) \longrightarrow R^i f_* \mathcal{F}(U), \quad \text{all } U.$$

Proposition 3.2. *If X and Y are schemes, $f\colon X \to Y$ is quasi-compact and $\mathcal{F}$ is quasi-coherent $\mathcal{O}_X$-module, then $R^i f_*(\mathcal{F})$ is quasi-coherent $\mathcal{O}_Y$-module. Moreover, if U is affine or if $i = 0$, then*

$$H^i(f^{-1}(U), \mathcal{F}) \longrightarrow R^i f_*(\mathcal{F})(U)$$

is an isomorphism.

PROOF. In fact, by the sheaf axiom for $\mathcal{F}$, it follows immediately that the presheaf $U \mapsto H^0(f^{-1}(U), \mathcal{F}) = \mathcal{F}(f^{-1}(U))$ is a sheaf on Y. Therefore $H^0(f^{-1}(U), \mathcal{F}) \to R^0 f_* \mathcal{F}(U)$ is an isomorphism for all U. The rest of the proposition falls into the set-up of (I.5.9). As stated there, it suffices to verify that if U is affine, $R = \Gamma(U, \mathcal{O}_X)$ and $g \in R$, then we get an isomorphism:

$$H^i(f^{-1}(U), \mathcal{F}) \otimes_R R_g \xrightarrow{\approx} H^i(f^{-1}(U_g), \mathcal{F}).$$

But since f is quasi-compact, we may cover $f^{-1}(U)$ by a finite set of affines $\{V_1, \ldots, V_N\} = \mathcal{V}$. Then $f^{-1}(U_g)$ is covered by

$$\big\{(V_1)_{f^*(g)}, \ldots, (V_N)_{f^*(g)}\big\} = \mathcal{V}|_{f^{-1}(U_g)}$$

which is again an affine covering. Therefore

$$\begin{aligned} H^i(f^{-1}(U), \mathcal{F}) &= H^i(C^\bullet(\mathcal{V}, \mathcal{F})) \\ H^i(f^{-1}(U_g), \mathcal{F}) &= H^i(C^\bullet(\mathcal{V}|_{f^{-1}(U_g)}, \mathcal{F})). \end{aligned}$$

The cochain complexes are:

$$\begin{aligned} C^i(\mathcal{V}, \mathcal{F}) &= \prod_{1 \le \alpha_0, \ldots, \alpha_i \le N} \mathcal{F}(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}) \\ C^i(\mathcal{V}|_{f^{-1}(U_g)}, \mathcal{F}) &= \prod_{1 \le \alpha_0, \ldots, \alpha_i \le N} \mathcal{F}((V_{\alpha_0})_{f^*g} \cap \cdots \cap (V_{\alpha_i})_{f^*g}). \end{aligned}$$

Since if $S = \Gamma(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}, \mathcal{O}_X)$:

$$\begin{aligned} \mathcal{F}((V_{\alpha_0})_{f^*g} \cap \cdots \cap (V_{\alpha_i})_{f^*g}) &\cong \mathcal{F}(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}) \otimes_S S_{f^*g} \\ &\cong \mathcal{F}(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}) \otimes_R R_g \end{aligned}$$

it follows that

$$C^i(\mathcal{V}|_{f^{-1}(U_g)}, \mathcal{F}) \cong C^i(\mathcal{V}, \mathcal{F}) \otimes_R R_g$$

(since $\otimes$ commutes with finite products). But now localizing commutes with kernels and cokernels, so for any complex $A^\bullet$ of R-modules, $H^i(A^\bullet) \otimes_R R_g \cong H^i(A^\bullet \otimes_R R_g)$. Thus

$$H^i(f^{-1}(U_g), \mathcal{F}) \cong H^i(f^{-1}(U), \mathcal{F}) \otimes_R R_g$$

as required. □

COROLLARY 3.3. *If $f\colon X \to Y$ is an affine morphism (cf. Proposition-Definition* I.7.3*) and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, then*

$$R^i f_* \mathcal{F} = 0, \qquad \forall i > 0.$$

A natural question to ask now is whether the cohomology of $\mathcal{F}$ on X can be reconstructed by taking the cohomology on Y of the higher direct images $R^i f_* \mathcal{F}$. The answer is: almost. The relationship between them is a *spectral sequence*. These are the biggest monsters that occur in homological algebra and have a tendency to strike terror into the heart of all eager students. I want to try to debunk their reputation of being so difficult[4].

DEFINITION 3.4. A spectral sequence $E_2^{pq} \Longrightarrow E^n$ consists in two pieces of data[5]:

[4](Added in publication) Fancier notions of "derived categories and derived functors" have since become indispensable not only in algebraic geometry but also in analysis, mathematical physics, etc. Among accessible references are: Hartshorne [**49**], Kashiwara-Schapira [**59**], [**60**] and Gelfand-Manin [**38**].

[5]Sometimes one also has a spectral sequence that "begins" with an E_1^{pq}. Then the first differential is

$$d_1^{pq}\colon E_1^{pq} \longrightarrow E_1^{p+1,q}$$

and if you set $E_2^{pq} = (\operatorname{Ker} d_1^{p,q})/(\operatorname{Image} d_1^{p-1,q})$, you get a spectral sequence as above.

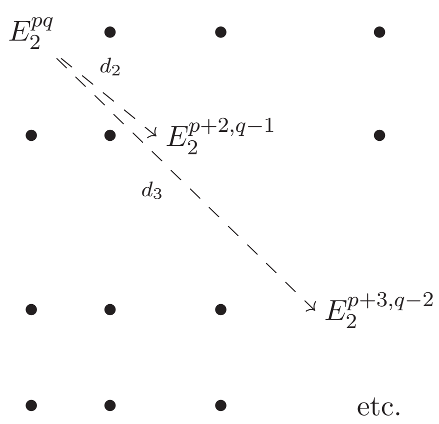

Figure VII.2

(A) A doubly infinite collection of abelian groups E_2^{pq}, $(p, q \in \mathbb{Z},\ p, q \geq 0)$ called the initial terms plus filtrations on each E_2^{pq}, which we write like this:

$$E_2^{pq} = Z_2^{pq} \supset Z_3^{pq} \supset Z_4^{pq} \supset \cdots \supset B_4^{pq} \supset B_3^{pq} \supset B_2^{pq} = (0),$$

also, let

$$Z_\infty^{pq} = \bigcap_r Z_r^{pq}$$
$$B_\infty^{pq} = \bigcup_r B_r^{pq},$$

plus a set of homomorphisms d_r^{pq} that allow us to determine inductively Z_{r+1}^{pq}, B_{r+1}^{pq} from the previous ones Z_r^{pq}, B_r^{pq}:

$$d_r^{pq}\colon Z_r^{pq} \longrightarrow E_2^{p+r,q-r+1}/B_r^{p+r,q-r+1}$$

(cf. Figure VII.2).

The d's should have the properties

i) $B_r^{pq} \subset \mathrm{Ker}(d_r^{pq})$, $Z_r^{p+r,q-r+1} \supset \mathrm{Image}(d_r^{pq})$ so that d_r^{pq} induces a map

$$Z_r^{pq}/B_r^{pq} \longrightarrow Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

This sub-quotient of E_2^{pq} is called E_r^{pq}.

ii) $d^2 = 0$; more precisely, the composite

$$Z_r^{pq}/B_r^{pq} \longrightarrow Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1} \longrightarrow Z_r^{p+2r,q-2r+2}/B_r^{p+2r,q-2r+2}$$

is 0.

iii) $Z_{r+1}^{pq} = \mathrm{Ker}(d_r^{pq})$; $B_{r+1}^{p+r,q-r+1} = \mathrm{Image}(d_r^{pq})$. This implies that E_{r+1}^{pq} is the cohomology of the complex formed by the E_r^{pq}'s and the d_r's!

(B) The so-called "abutment": a simply infinite collection of abelian groups E^n plus a filtration on each E^n whose successive quotients are precisely the groups $E_\infty^{p,n-p}$:

$$E^n = \underbrace{F^0(E^n) \supset F^1}_{\cong E_\infty^{0,n}}\underbrace{(E^n) \supset \cdots}_{\cong E_\infty^{1,n-1}}\underbrace{\cdots\cdots \supset F^n}_{\cdots}\underbrace{(E^n) \supset F^{n+1}(E^n)}_{\cong E_\infty^{n,0}} = (0)$$

To illustrate what is going on here, look at the terms of lowest total degree. One sees easily that one gets the following exact sequences:

a) $E_2^{00} \cong E^0$.

b) $0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E^2$.

c) For all n, one gets "edge homomorphisms"

$$E_2^{n,0} \longrightarrow E_\infty^{n,0} \longrightarrow E^n$$

and

$$E^n \longrightarrow E_\infty^{0,n} \longrightarrow E_2^{0,n} :$$

i.e.,

$$\begin{tikzcd} & & E^n \arrow[dll] \\ E_2^{0,n} \arrow[dr, dash, dashed] & & \\ & E_2^{n,0} \arrow[uur] & \end{tikzcd}$$

Theorem 3.5. [6] *Given any quasi-compact morphism $f\colon X \to Y$ and quasi-coherent sheaf $\mathcal{F}$ on X, there is a canonical spectral sequence, called* Leray's spectral sequence*, with initial terms*

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F})$$

and abutment $E^n = H^n(X, \mathcal{F})$.

Proof. Choose open affine coverings $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ of Y and $\mathcal{V} = \{V_\beta\}_{\beta \in T}$ of X and consider the double complex introduced in §2 for the two coverings $f^{-1}(\mathcal{U})$ and $\mathcal{V}$ of X:

$$C^{pq} = \prod_{\alpha_0,\dots,\alpha_p \in S} \prod_{\beta_0,\dots,\beta_q \in T} \mathcal{F}(f^{-1}U_{\alpha_0} \cap \cdots \cap f^{-1}U_{\alpha_p} \cap V_{\beta_0} \cap \cdots \cap V_{\beta_q}).$$

Note that all the open sets here are affine because of Proposition II.4.5.

Now the q-th row of our double complex is the product over all $\beta_0, \dots, \beta_q \in T$ of the Čech complex $C^\bullet(f^{-1}(\mathcal{U}) \cap V_{\beta_0} \cap \cdots \cap V_{\beta_q}, \mathcal{F})$, i.e., the Čech complex for an affine open covering of an affine $V_{\beta_0} \cap \cdots \cap V_{\beta_q}$. Therefore all the rows are exact except at their first terms where their cohomology is $\prod_{\beta_0,\dots,\beta_q} \mathcal{F}(V_{\beta_0} \cap \cdots \cap V_{\beta_q})$, i.e., $C^q(\mathcal{V}, \mathcal{F})$. Hence by the easy lemma of the double complex (Lemma 2.4),

1)
$$\begin{aligned} H^n(\text{total complex}) &\cong H^n(C^\bullet(\mathcal{V}, \mathcal{F})) \\ &\cong H^n(X, \mathcal{F}). \end{aligned}$$

But on the other hand, the p-th column of our double complex is the product over all $\alpha_0, \dots, \alpha_p \in S$ of the Čech complex $C^\bullet(\mathcal{V} \cap f^{-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}), \mathcal{F})$. The cohomology of this complex at the q-th spot is $H^q(f^{-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}), \mathcal{F})$ which is also the same as $R^q f_* \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$. Therefore:

2) [vertical δ_2-cohomology of p-th column at (p,q)] $\cong \prod_{\alpha_0,\dots,\alpha_p \in S} R^q f_* \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$.
But now the horizontal maps $\delta_1\colon C^{p,q} \to C^{p+1,q}$ induce maps from the [δ_2-cohomology at (p,q)-th spot] $\to$ [δ_2-cohomology at $(p+1,q)$-th spot] and we see easily that

3) [q-th row of vertical cohomology groups] $\underset{\text{as complex}}{\cong}$ Čech complex $C^\bullet(\mathcal{U}, R^q f_* \mathcal{F})$. Therefore finally:

4) [horizontal δ_1-cohomology at (p,q) of vertical δ_2-cohomology group] $\cong H^p(Y, R^q f_* \mathcal{F})$!

Theorem 3.5 is now reduced to:

---

[6] Theorem 3.5 also holds for continuous maps of paracompact Hausdorff spaces and arbitrary sheaves $\mathcal{F}$, but we will not use this.

LEMMA 3.6 (The hard lemma of the double complex). *Let $(C^{p,q}, \delta_1, \delta_2)$ be any double complex. Make no assumption on the δ_2-cohomology, but consider instead its δ_1-cohomology:*

$$E_2^{p,q} = H^p_{\delta_1}(H^q_{\delta_2}(C^{\bullet,\bullet})).$$

Then there is a spectral sequence starting at $E_2^{p,q}$ and abutting at the cohomology of the total complex. Alternatively, one can "start" this spectral sequence at

$$E_1^{p,q} = H^q_{\delta_2}(C^{p,\bullet}) = (\textit{cohomology in vertical direction})$$

with d_1 being the maps induced by δ_1 on δ_2-cohomology[7]*. Also, since the rows and columns of a double complex play symmetric roles, one gets as a consequence a second spectral sequence with*

$$E_2^{p,q} = H^p_{\delta_2}(H^q_{\delta_1}(C^{\bullet,\bullet}))$$

or

$$E_1^{p,q} = H^q_{\delta_1}(C^{\bullet,p}) = (\textit{cohomology in horizontal direction}),$$

abutting also to the cohomology of the total complex.

A hard-nosed detailed proof of this is not very long but quite unreadable. I think the reader will find it easier if I sketch the idea of the proof far enough so that he/she can work out for himself/herself as many details as he/she wants. To begin with, we may describe E_2^{pq} rather more explicitly as:

$$E_2^{pq} = \frac{\{x \in C^{p,q} \mid \delta_2 x = 0 \text{ and } \delta_1 x = \delta_2 y, \text{ some } y \in C^{p+1,q-1}\}}{\delta_2(C^{p,q-1}) + \delta_1\{x \in C^{p-1,q} \mid \delta_2 x = 0\}}.$$

The idea is — how hard is it to "extend" the δ_2-cocycle x to a whole d-cocycle in the total complex: more precisely, to a set of elements

$$\begin{array}{ll} x \in C^{p,q} & \delta_2 x = 0 \\ y_1 \in C^{p+1,q-1} & \delta_2 y_1 = \delta_1 x \\ y_2 \in C^{p+2,q-2} & \delta_2 y_2 = \delta_1 y_1 \\ \text{etc.} & \text{etc.} \end{array}$$

so that $d(x \pm y_1 \pm y_2 \pm \cdots) = 0$ (the signs being mechanically chosen here taking into account that $d = \delta_1 + (-1)^p \delta_2$). See Figure VII.3.

Define Z_∞^{pq} to be the subgroup of E_2^{pq} for which such a sequence of y_i's exist; define Z_3^{pq} to be the set of x's such that such y_1 and y_2 exist; define Z_4^{pq} to be the set of x's such that such y_1, y_2 and y_3 exist; etc.

On the other hand, a δ_2-cocycle x may be a d-coboundary in various ways — let

$$B_3^{pq} = \text{image in } E_2^{pq} \text{ of } \left\{ x \in C^{p,q} \;\middle|\; \begin{array}{l} w_1 \in C^{p-1,q},\ w_2 \in C^{p-2,q-1} \\ \delta_1 w_1 = x,\ \delta_2 w_1 = \delta_1 w_2,\ \delta_2 w_2 = 0 \end{array} \right\}$$

$$B_4^{pq} = \text{image in } E_2^{pq} \text{ of } \left\{ x \in C^{p,q} \;\middle|\; \begin{array}{l} w_1, w_2 \text{ as above},\ w_3 \in C^{p-3,q-2} \\ \delta_1 w_1 = x,\ \delta_2 w_1 = \delta_1 w_2,\ \delta_2 w_2 = \delta_1 w_3,\ \delta_2 w_3 = 0 \end{array} \right\}$$

etc.

(cf. Figure VII.4)

[7] More precisely, to construct the spectral sequence, one doesn't need both gradings on $\bigoplus C^{p,q}$ and both differentials; it is enough to have one grading (the grading by total degree), one filtration ($F_k = \bigoplus_{p \geq k} C^{p,q}$) and the total differential: for details cf. MacLane [**68**, Chapter 11, §§3 and 6].

$$
\begin{tikzcd}
0 \\
x \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & z_1 \\
& y_1 \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & z_2 \\
& & y_2 \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & {} \\
& & & \ddots & \ddots & & d_r x! \arrow[dd, dashed, bend right] \\
& & & & y_{r-1} \arrow[r, mapsto, "\delta_1"] & z_{r-1} \\
& & & & & y_{r-1} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & ?
\end{tikzcd}
$$

FIGURE VII.3

$$
\begin{tikzcd}
0 \\
w_r \arrow[u, mapsto, "\delta_2"'] \arrow[r, mapsto, "\delta_1"'] & u_{r-1} \\
& w_{r-1} \arrow[u, mapsto, "\delta_2"'] & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & u_2 \\
& & & & w_2 \arrow[u, mapsto, "\delta_2"'] \arrow[r, mapsto, "\delta_1"'] & u_1 \\
& & & & & w_1 \arrow[u, mapsto, "\delta_2"'] \arrow[r, mapsto, "\delta_1"'] & x
\end{tikzcd}
$$

FIGURE VII.4

As for $d_r^{pq}\colon Z_r^{pq} \to E_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}$, suppose $x \in C^{p,q}$ defines an element of Z_r^{pq}, i.e., $\exists y_1 \in C^{p+1,q-1}, \ldots, y_{r-1} \in C^{p+r-1,q-r+1}$ such that $\delta_2 y_{i+1} = \delta_1 y_i$, $i < r-1$; $\delta_2 y_1 = \delta_1 x$. Define

$$d_r^{pq}(x) = \delta_1 y_{r-1}.$$

This is an element of $C^{p+r,q-r+1}$ killed by δ_1 and δ_2, hence it defines an element of $E_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}$. At this point there are quite a few points to verify — that d_r is well-defined so long as the image is taken modulo B_r and that d_r has the three properties of the definition. These are all mechanical and we omit them.

$$
\begin{array}{cccc|cccc}
0 & & & & F^k & & & \\
 & 0 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 0 & & & & \\
 & & & & x_{k,n-k} & & & \\
 & & & & & \ddots & & \\
 & & & & & & x_{n-1,1} & \\
 & & & & & & & x_{n,0}
\end{array}
$$

FIGURE VII.5

Finally, define the filtration on the cohomology of the total complex:

$$
\begin{aligned}
F^k(E^n) = &\text{those elements of } \left[\frac{\operatorname{Ker} d \text{ in } \sum_{p+q=n} C^{p,q}}{d\left(\sum_{p+q=n-1} C^{p,q}\right)}\right] \\
&\text{which can be represented by a } d\text{-cocycle} \\
&\text{with components } x_{pq} \in C^{p,q},\ x_{pq} = 0 \text{ if } p < k
\end{aligned}
$$

(cf. Figure VII.5). The whole point of these definitions, which is now reasonable I hope, is the isomorphism:

$$F^pE^n/F^{p+1}E^n \cong Z_\infty^{p,n-p}/B_\infty^{p,n-p}.$$

The details are again omitted. □

An important remark is that the edge homomorphisms in the Leray spectral sequence:

a) $H^n(Y, f_*\mathcal{F}) \cong E_2^{n,0} \to E^n \cong H^n(X, \mathcal{F})$
b) $H^n(X, \mathcal{F}) \cong E^n \to E_2^{0,n} \cong H^0(Y, R^n f_*\mathcal{F})$

are just the maps induced by the functorial properties of cohomology (i.e., the set of maps $f_*\mathcal{F}(U) \to \mathcal{F}(f^{-1}(U))$ means that there is a map of sheaves $f_*\mathcal{F} \to \mathcal{F}$ with respect to f and this gives (a); and the maps $H^n(X, \mathcal{F}) \to H^n(f^{-1}U, \mathcal{F}) \to R^n f_*\mathcal{F}(U)$ for all U give (b)). This comes out if $\mathcal{V}$ is a refinement of $f^{-1}(\mathcal{U})$ by the calculation used in the proof of Theorem 3.5.

PROPOSITION 3.7. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $f\colon X \to Y$ is an affine morphism (cf. Proposition-Definition I.7.3), then*

$$H^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(Y, f_*\mathcal{F}), \qquad \forall p.$$

PROOF. Leray's spectral sequence (Theorem 3.5) and Corollary 3.3. □

COROLLARY 3.8. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $i\colon X \to Y$ is a closed immersion of schemes (cf. Definition 3.1), then*

$$H^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(Y, i_*\mathcal{F}), \qquad \forall p.$$

REMARK. If X is identified with its image $i(X)$ in Y, $i_*\mathcal{F}$ is nothing but the quasi-coherent $\mathcal{O}_Y$-module obtained as the extension of the $\mathcal{O}_X$-module $\mathcal{F}$ by (0) outside X.

A second important application of the hard lemma (Lemma 3.6) is to hypercohomology and in particular to De Rham cohomology (cf. §VIII.3 below). Let $\mathcal{F}^\bullet$ be any complex of sheaves on a topological space X. Then if $\mathcal{U}$ is an open covering, $\mathbb{H}^n(\mathcal{U}, \mathcal{F}^\bullet)$ is by definition the cohomology of the total complex of the double complex $C^q(\mathcal{U}, \mathcal{F}^p)$, hence we get two spectral sequences abutting to it. The first is gotten by taking vertical cohomology (with respect to the superscript q):

$$E_1^{pq} = H^q(\mathcal{U}, \mathcal{F}^p) \Longrightarrow E^n = \mathbb{H}^n(\mathcal{U}, \mathcal{F}^\bullet)$$
$$\text{(with } d_1^{pq} \text{ the map induced on cohomology by } d\colon \mathcal{F}^p \to \mathcal{F}^{p+1}\text{).}$$

Passing to the limit over finer coverings, we get:

$$E_1^{pq} = H^q(X, \mathcal{F}^p) \Longrightarrow E^n = \mathbb{H}^n(X, \mathcal{F}^\bullet). \tag{3.9}$$

The second is gotten by taking horizontal cohomology (with respect to p) and *then* vertical cohomology. To express this conveniently, define *presheaves* $\mathcal{H}^p_{\text{pre}}(\mathcal{F}^\bullet)$ by

$$\mathcal{H}^p_{\text{pre}}(\mathcal{F}^\bullet)(U) = \frac{\operatorname{Ker}(\mathcal{F}^p(U) \to \mathcal{F}^{p+1}(U))}{\operatorname{Image}(\mathcal{F}^{p-1}(U) \to \mathcal{F}^p(U))}.$$

The sheafification of these presheaves are just:

$$\mathcal{H}^p(\mathcal{F}^\bullet) = \frac{\operatorname{Ker}(\mathcal{F}^p \to \mathcal{F}^{p+1})}{\operatorname{Image}(\mathcal{F}^{p-1} \to \mathcal{F}^p)}$$

but $\mathcal{H}^p_{\text{pre}}$ will not generally be a sheaf already. The horizontal cohomology of the double complex $C^q(\mathcal{U}, \mathcal{F}^p)$ is just $C^q(\mathcal{U}, \mathcal{H}^p_{\text{pre}})$ and the vertical cohomology of this is $H^q(\mathcal{U}, \mathcal{H}^p_{\text{pre}})$, hence we get the second spectral sequence:

$$E_2^{pq} = H^p(\mathcal{U}, \mathcal{H}^q_{\text{pre}}(\mathcal{F}^\bullet)) \Longrightarrow E^n = \mathbb{H}^n(\mathcal{U}, \mathcal{F}^\bullet).$$

Passing to the limit over $\mathcal{U}$, this gives:

$$E_2^{pq} = H^p(X, \mathcal{H}^q_{\text{pre}}(\mathcal{F}^\bullet)) \Longrightarrow E^n = \mathbb{H}^n(X, \mathcal{F}^\bullet). \tag{3.10}$$

In good cases, e.g., X paracompact Hausdorff (cf. §1), the cohomology of a presheaf is the cohomology of its sheafification, so we get finally:

$$E_2^{pq} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow E^n = \mathbb{H}^n(X, \mathcal{F}^\bullet). \tag{3.11}$$

4. Computing cohomology (1): Push $\mathcal{F}$ into a huge acyclic sheaf

Although the apparatus of cohomology of quasi-coherent sheaves may seem at first acquaintace rather formidable, it should always be remembered that it is really only fancy linear algebra. In many specific cases, it is no great problem to compute it. To stress the flexibility of the tools available for computing cohomology, we present in a fugal style four calculations each using a different method.

A standard approach for cohomology is via a resolution of the type:

$$0 \longrightarrow \mathcal{F} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots\cdots$$

where the I_k's are injective, or "flasque" or "mou" or at least are acyclic. (See Godement [**39**] or Swan [**99**].) Sheaves of this type tend to be huge monsters, but there has been quite a bit of work done on injectives in the category of sheaves of $\mathcal{O}_X$-modules on a noetherian X (see Hartshorne [**49**, p. 120]). We use the method as follows:

LEMMA 4.1. *If $U \subset X$ is affine and $i\colon U \to X$ the inclusion map, then for all quasi-coherent $\mathcal{F}$ on U, $i_*\mathcal{F}$ is acyclic, i.e., $H^p(X, i_*\mathcal{F}) = (0)$, all $p \geq 1$.*

PROOF. In fact, for $V \subset X$ affine, $i^{-1}(V) = U \cap V$ is affine, so the presheaf $V \mapsto H^p(i^{-1}V, \mathcal{F})$ is (0) on affines ($p \geq 1$). Thus $R^p i_*\mathcal{F} = (0)$ if $p \geq 1$. Then Leray's spectal sequence (Theorem 3.5) degenerates since

$$E_2^{pq} = H^p(X, R^q i_*\mathcal{F}) = (0), \quad q \geq 1.$$

Thus $E_2^{pq} \cong E_\infty^{pq} \cong E^{p+q}$, and the edge homomorphism

$$H^p(X, i_*\mathcal{F}) \longrightarrow H^p(U, \mathcal{F})$$

is an isomorphism. Since $H^p(U, \mathcal{F}) = (0)$, $p \geq 1$, the lemma is proven. □

If $\mathcal{F}$ is quasi-coherent on X, and $i\colon U \to X$ is the inclusion of an affine, there is a canonical map:

$$\phi\colon \mathcal{F} \longrightarrow i_*(\mathcal{F}|_U)$$

via

$$\mathcal{F}(V) \xrightarrow{\text{res}} \mathcal{F}(U \cap V) \cong i_*(\mathcal{F}|_U)(V), \quad \forall \text{open } V,$$

which is an isomorphism on U. We can apply this to prove:

PROPOSITION 4.2. *Let X be a noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Let $n = \dim(\operatorname{Supp}\mathcal{F})$, i.e., n is the maximum length of chains:*

$$Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_r \subset \operatorname{Supp}(\mathcal{F}), \qquad Z_i \textit{ closed irreducible.}$$

Then $H^i(X, \mathcal{F}) = (0)$ if $i > n$.

PROOF. Use induction on n. If $n = 0$, then $\operatorname{Supp}\mathcal{F}$ is a finite set of closed points $\{x_1, \ldots, x_N\}$. For all i, let $U_i \subset X$ be an affine neighborhood of X_i such that $x_j \notin U_i$, all $j \neq i$; let $\{U_\beta\}_{\beta \in T}$ be an affine covering of $X \setminus \{x_1, \ldots, x_N\}$. Then $\{U_1, \ldots, U_N\} \cup \{U_\beta\}$ is an affine covering of X such that for any two disjoint open sets U_α, $U_{\alpha'}$ in it, $U_\alpha \cap U_{\alpha'} \cap \operatorname{Supp}\mathcal{F} = \emptyset$. Thus $C^i(\mathcal{U}, \mathcal{F}) = (0)$, $i \geq 1$, and hence $H^i(X, \mathcal{F}) = (0)$, $i \geq 1$.

In general, decompose $\operatorname{Supp}\mathcal{F}$ into irreducible sets:

$$\operatorname{Supp}\mathcal{F} = S_1 \cup \cdots \cup S_N.$$

Let $U_i \subset X$ be an affine open set such that

$$\begin{aligned} &U_i \cap S_i \neq \emptyset \\ &U_i \cap S_j = \emptyset, \quad \text{all } j \neq i. \end{aligned}$$

Let $i_k\colon U_k \to X$ be the inclusion map, and let

$$\mathcal{F}_k = i_{k,*}(\mathcal{F}|_{U_k}).$$

As above we have a canonical map:

$$\mathcal{F} \xrightarrow{\phi} \bigoplus_{k=1}^{N} \mathcal{F}_k$$

given by:

$$\mathcal{F}(V) \xrightarrow{\text{res}} \bigoplus_{k=1}^{N} \mathcal{F}(U_k \cap V) \cong \left[\bigoplus_{k=1}^{N} i_{k,*}(\mathcal{F}|_{U_k})\right](V).$$

Concerning ϕ, we have the following facts:

a) If $i \neq j$, $U_i \cap U_j \cap \operatorname{Supp} \mathcal{F} = \emptyset$, hence $\mathcal{F}(U_i \cap U_j) = (0)$. Therefore if $V \subset U_{k_0}$,

$$\bigoplus_{k=1}^{N} \mathcal{F}(U_k \cap V) = \mathcal{F}(U_{k_0} \cap V) = \mathcal{F}(V).$$

Therefore ϕ is an *isomorphism* of sheaves on each of the open sets U_k.

b) If $V \cap S_k = \emptyset$, then $V \cap U_k \cap \operatorname{Supp} \mathcal{F} = \emptyset$ so $\mathcal{F}_k(V) = \mathcal{F}(U_k \cap V) = (0)$. Thus $\operatorname{Supp} \mathcal{F}_k \subset S_k$.

c) Each $\mathcal{F}_k$ is quasi-coherent by Proposition 3.2, hence $\mathcal{K}_1 = \operatorname{Ker} \phi$ and $\mathcal{K}_2 = \operatorname{Coker} \phi$ are quasi-coherent.

Putting all this together, if $i = 1, 2$

$$\begin{aligned} \operatorname{Supp} \mathcal{K}_i &\subset (S_1 \cup \cdots \cup S_N) \setminus (\text{open set where } \phi \text{ is an isomorphism}) \\ &\subset \bigcup_{k=1}^{N} (S_k \setminus S_k \cap U_k). \end{aligned}$$

Therefore $\dim \operatorname{Supp} \mathcal{K}_i < n$, and we can apply induction. If we set $\mathcal{K}_3 = \mathcal{F}/\mathcal{K}_1$, we get two short exact sequences:

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}_3 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K}_3 \longrightarrow \bigoplus_{k=1}^{N} \mathcal{F}_k \longrightarrow \mathcal{K}_2 \longrightarrow 0,$$

hence if $p > n$:

$$(*) \qquad \begin{array}{ccccc} H^p(X, \mathcal{K}_1) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{K}_3) \\ \| \text{ by induction} & & & & \\ (0) & & & & (0) \\ \| \text{ by induction} & & & & \| \text{ by Lemma 4.1} \\ H^{p-1}(X, \mathcal{K}_2) & \longrightarrow & H^p(X, \mathcal{K}_3) & \longrightarrow & \bigoplus_{k=1}^{N} H^p(X, \mathcal{F}_k). \end{array}$$

This proves that $H^p(X, \mathcal{F}) = (0)$ if $p > n$. □

5. Computing cohomology (2): Directly via the Čech complex

We illustrate this approach by calculating $H^i(\mathbb{P}^n_R, \mathcal{O}(m))$ for any ring R. We need some more definitions first:

a) Let R be a ring, $f_1, \ldots, f_n \in R$. Let M be an R-module. Introduce formal symbols $\omega_1, \ldots, \omega_n$ such that

$$\omega_i \wedge \omega_j = -\omega_j \wedge \omega_i, \quad \omega_i \wedge \omega_i = 0.$$

Define an R-module:

$$K^p(f_1, \ldots, f_n; M) = \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} M \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

Define

$$d \colon K^p(f_1, \ldots, f_n; M) \longrightarrow K^{p+1}(f_1, \ldots, f_n; M)$$

by

$$dm = \left(\sum_{i=1}^{n} f_i \omega_i \right) \wedge m.$$

Note that $d^2 = 0$. This gives us the *Koszul complex* $K^{\bullet}(f_1, \ldots, f_n; M)$:

$$\begin{array}{ccccccccc} 0 \longrightarrow & K^0((f);M) & \longrightarrow & K^1((f);M) & \longrightarrow \cdots \longrightarrow & K^n((f);M) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & M & & \bigoplus_{i=1}^n M \cdot \omega_i & & M \cdot \omega_1 \wedge \cdots \wedge \omega_n & \end{array}$$

b) Now say R is a graded ring, $f_i \in R_{d_i}$ is homogeneous and M is a graded module. Then we assign ω_i the degree $-d_i$, so that $\sum f_i \omega_i$ is homogeneous of degree 0. Then $K^{\bullet}(f_1, \ldots, f_n; M)$ is a complex of graded modules with degree preserving maps d. We let $K^{\bullet}(f_1, \ldots, f_n; M)^0$ denote the degree 0 subcomplex, i.e.,

$$K^p(f_1, \ldots, f_n; M)^0 = \bigoplus_{i_1 < \cdots < i_p} M_{d_{i_1} + \cdots + d_{i_p}} \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

c) Next compare the Koszul complexes $K^{\bullet}(f_1^{\nu}, \ldots, f_n^{\nu}; M)$ for various $\nu \geq 1$. If we write

$$K^p(f_1^{\nu}, \ldots, f_n^{\nu}; M) = \bigoplus_{i_1 < \cdots < i_p} M \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)}$$

and set

$$\omega_i^{(\nu)} = f_i^{(\nu')} \cdot \omega_i^{(\nu + \nu')}$$

then we get a natural homomorphism

$$K^p(f_1^{\nu}, \ldots, f_n^{\nu}; M) \longrightarrow K^p(f_1^{\nu+\nu'}, \ldots, f_n^{\nu+\nu'}; M)$$

which commutes with d.

The point of all this is:

Proposition 5.1. *If R is a graded ring, $f_i \in R_{d_i}$, $1 \leq i \leq n$, M a graded R-module, $U_i = (\operatorname{Proj} R)_{f_i}$, $\mathcal{U} = \{U_1, \ldots, U_n\}$, then there is a natural isomorphism:*

$$C_{\text{alt}}^{p-1}(\mathcal{U}, \widetilde{M}) \cong \varinjlim_{\nu} K^p(f_1^{\nu}, \ldots, f_n^{\nu}; M)^0, \quad p \geq 1,$$

under which the Čech coboundary δ and the Koszul d correspond.

Proof. We have

$$\begin{aligned} C_{\text{alt}}^{p-1}(\mathcal{U}, \widetilde{M}) &= \bigoplus_{i_1 < \ldots < i_p} \widetilde{M}(U_{i_1} \cap \cdots \cap U_{i_p}) \\ &= \bigoplus_{i_1 < \cdots < i_p} \left(M_{f_{i_1} \cdots f_{i_p}}\right)^0 \end{aligned}$$

and

$$\begin{aligned} \varinjlim_{\nu} K^p((f^{\nu}); M)^0 &= \bigoplus_{i_1 < \ldots < i_p} \varinjlim_{\nu} \left[M \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)}\right]^0 \\ &= \bigoplus_{i_1 < \cdots < i_p} \varinjlim_{\nu} M_{\nu(d_{i_1} + \cdots + d_{i_p})} \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)}. \end{aligned}$$

Define a homomorphism:

$$M_{\nu(d_{i_1} + \cdots + d_{i_p})} \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)} \longrightarrow (M_{f_{i_1} \cdots f_{i_p}})^0$$

by taking $\omega_j^{(\nu)}$ to $1/f_j^{\nu}$. This clearly commutes with the limit operation in ν and since for any ring S, any S-module N and $g \in S$, $N_g \cong$ direct limit of the system

$$N \xrightarrow{\text{multiplication by } g} N \xrightarrow{\text{multiplication by } g} N \xrightarrow{\text{multiplication by } g} \cdots,$$

it follows that

$$\varinjlim_{\nu} M_{\nu(d_{i_1}+\cdots d_{i_p})}\omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)} \cong (M_{f_{i_1}\cdots f_{i_p}})^0.$$

We leave it to the reader to check that δ and d correspond. $\square$

The complex $\{K^p\}$ goes down to $p = 0$ while $\{C^{p-1}\}$ only goes down to $p = 1$. We can extend $\{C^{p-1}\}$ one further step so that it matches up with $\{K^p\}$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \xrightarrow{\varepsilon} & C^0_{\text{alt}}(\mathcal{U}, \widetilde{M}) & \longrightarrow & C^1_{\text{alt}}(\mathcal{U}, \widetilde{M}) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \varinjlim_\nu (K^0)^0 & \longrightarrow & \varinjlim_\nu (K^1)^0 & \longrightarrow & \varinjlim_\nu (K^2)^0 & \longrightarrow & \cdots \end{array}$$

where ε is the composite of the canonical maps:

$$M_0 \longrightarrow \Gamma(\operatorname{Proj} R, \widetilde{M}) \longrightarrow C^0_{\text{alt}}(\mathcal{U}, \widetilde{M}).$$

What we need next is a criterion for a Koszul complex to be exact:

PROPOSITION 5.2 (Koszul). *Let R be a ring, $f_1, \ldots, f_n \in R$ and M an R-module. If f_s is a non-zero-divisor in $M/(f_1, \ldots, f_{s-1}) \cdot M$ for $1 \le s \le t$, then the complex $K^\bullet(f_1, \ldots, f_n; M)$ is exact at $K^s(f_1, \ldots, f_n; M)$ for $0 \le s \le t-1$.*

PROOF. To see how simple this is, it's better to take the first non-trivial case and check it, rather than getting confused in a general inductive proof. Take $t = 3$ and check that

$$\bigoplus_i M\omega_i \xrightarrow{d} \bigoplus_{i_1<i_2} M\omega_{i_1} \wedge \omega_{i_2} \xrightarrow{d} \bigoplus_{i_1<i_2<i_3} M\omega_{i_1} \wedge \omega_{i_2} \wedge \omega_{i_3}$$

is exact. Write an element η of the middle module as

$$\eta = m\omega_1 \wedge \omega_2 + \omega_1 \wedge \left(\sum_{i=3}^n n_i\omega_i\right) + \sum_{2\le i<j} p_{ij}\omega_i \wedge \omega_j.$$

Assume $d\eta = 0$. Looking at the coefficient of $\omega_1 \wedge \omega_2 \wedge \omega_3$, it follows

$$f_3 m = f_2 n_3 - f_1 p_{23} \in (f_1, f_2)M.$$

Therefore by hypothesis $m = f_1 q_1 + f_2 q_2$, $q_i \in M$. Replace η by $\eta - d(q_1\omega_2 - q_2\omega_1)$ and the coefficient m becomes 0. Assuming we have any η with $m = 0$, look at the coefficient of $\omega_1 \wedge \omega_2 \wedge \omega_i$, $i \ge 3$. It follows

$$f_2 n_i = f_1 p_{23} \in f_1 M.$$

Therefore $n_i = f_1 q_i$, $q_i \in M$. Replace η by $\eta - d(\sum_{i=3}^n q_i\omega_i)$ and now all the coefficients m, n_i are 0. Assuming we have any η with $m = n_i = 0$, look at the coefficient of $\omega_1 \wedge \omega_i \wedge \omega_j$, $2 \le i < j$. It follows that

$$f_1 p_{ij} = 0$$

whence by hypothesis $p_{ij} = 0$, hence $\eta = 0$. This idea works for any t. $\square$

Combining the two propositions, we get:

PROPOSITION 5.3. *Let R be a graded ring generated by homogeneous elements $f_i \in R_{d_i}$, $1 \le i \le n$. Let M be a graded R-module. Fix an integer t and assume[8] that, for every ν, and every s, $1 \le s \le t$, f_s^ν is a non-zero-divisor in $M/(f_1^\nu, \ldots, f_{s-1}^\nu) \cdot M$, then*

a) *If $t \ge 1$, $M_d \to \Gamma(\operatorname{Proj} R, \widetilde{M}(d))$ is injective for all d.*
b) *If $t \ge 2$, $M_d \to \Gamma(\operatorname{Proj} R, \widetilde{M}(d))$ is an isomorphism for all d.*

[8] A closer analysis shows that if this condition holds for $\nu = 1$, it automatically holds for larger ν.

c) *If* $t \geq 3$, $H^i(\operatorname{Proj} R, \widetilde{M}(d)) = (0)$, $1 \leq i \leq t-2$, *for all* d.

This follows by combining Propositions 5.1 and 5.2, taking note that we must augment the Čech complex $C^{\bullet}(\mathcal{U}, \widetilde{M}(d))$ by "$0 \to M_d \to$" to get the $\varinjlim$ of Koszul complexes (and also using the fact that a direct limit of exact sequences is exact).

Corollary 5.4. *Let S be a ring. Then*

a)

$$\begin{bmatrix} S\text{-module of homogeneous polynomials} \\ f(X_0, \ldots, X_l) \text{ of degree } d \end{bmatrix} \longrightarrow \Gamma(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d))$$

is an isomorphism for all $d \in \mathbb{Z}$,

b) $H^i(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d)) = (0)$, *for all* $d \in \mathbb{Z}$, $1 \leq i \leq l-1$.

Proof. Apply Proposition 5.3 to $R = S[X_0, \ldots, X_l]$, $n = l+1$, $f_i = X_{i-1}$, $1 \leq i \leq l+1$ and $M = R$. Then in fact multiplication by X_j^ν is injective in the ring of truncated polynomials:

$$R/(X_0^\nu, \ldots, X_{j-1}^\nu) \cdot R$$

so Proposition 5.3 applies with $t = n$. □

On the other hand, for any quasi-coherent $\mathcal{F}$ on $\mathbb{P}^l_S$, using the affine covering $U_i = (\mathbb{P}^l_S)_{X_i}$, $0 \leq i \leq l$, we get non-zero alternating cochains $C^i_{\text{alt}}(\mathcal{U}, \mathcal{F})$ only for $0 \leq i \leq l$. Therefore:

(5.5)
$$H^i(\mathbb{P}^l_S, \mathcal{F}) = (0), \quad i > l, \text{ all quasi-coherent } \mathcal{F}.$$

If we look more closely, we can describe the groups $H^l(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d))$ too. Look first at the general situation R, $(f_1, \ldots, f_n)$, M:

$$\begin{aligned} H^{n-1}(\mathcal{U}, \widetilde{M}) &= H^{n-1}(C^{\bullet}_{\text{alt}}(\mathcal{U}, \widetilde{M})) \\ &= C^{n-1}_{\text{alt}}(\mathcal{U}, \widetilde{M})/\delta(C^{n-2}_{\text{alt}}) \\ &= \widetilde{M}(U_1 \cap \cdots \cap U_n) \Big/ \sum_{i=1}^n \operatorname{res} \widetilde{M}(U_1 \cap \cdots \widehat{U}_i, \ldots, \cap U_n) \\ &= \left(M_{(\prod f_j)}\right)^0 \Big/ \sum_{i=1}^n \left(M_{(\prod_{j \neq i} f_j)}\right)^0. \end{aligned}$$

Thus in the special case:

(5.6)
$$\begin{aligned} H^l(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d)) \cong &\text{elements of degree } d \text{ in the } S[X_0, \ldots, X_l]\text{-module} \\ &S[X_0, \ldots, X_l]_{(\prod X_j)} \Big/ \sum_{i=0}^l S[X_0, \ldots, X_l]_{(\prod_{j \neq i} X_j)} \\ \cong &S\text{-module of rational functions} \\ &\sum_{\substack{\alpha_i \leq -1 \\ \sum \alpha_i = d}} c_{\alpha_0, \ldots, \alpha_l} X_0^{\alpha_0} \cdots X_l^{\alpha_l}. \end{aligned}$$

In particular $H^l(\mathcal{O}_{\mathbb{P}^l}(d)) = (0)$ if $d > -l-1$. It is natural to ask to what extent this is a canonical description of H^l — for instance, if you change coordinates, how do you change the description of an element of H^l by a rational function. The theory of this goes back to Macaulay and his "inverse systems", cf. Hartshorne [**52**, Chapter III].

Koszul complexes have many applications to the local theory too. For instance in Chapter V, we presented smooth morphisms locally as:

$$\begin{array}{c} X = \operatorname{Spec} R[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \\ {\scriptstyle f}\big\downarrow \\ Y = \operatorname{Spec} R \end{array}$$

and in Proposition V.3.19, we described the syzygies among the equations f_i locally. We can strengthen Proposition V.3.19 as follows: let $x \in X$, $y = f(x)$ so that

$$\mathcal{O}_{x,X} = \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}/(f_1, \ldots, f_r)$$

for some prime ideal $\mathfrak{p}$. Then I claim:

$$K^{\bullet}((f), \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}) \longrightarrow \mathcal{O}_{x,X} \longrightarrow 0$$

is a resolution of $\mathcal{O}_{x,X}$ as module over $\mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}$. This follows from the general fact:

PROPOSITION 5.7. *Let R be a regular local ring, M its maximal ideal and let $f_1, \ldots, f_r \in M$ be independent in M/M^2. Then*

$$0 \longrightarrow K^0((f), R) \longrightarrow \cdots \longrightarrow K^r((f), R) \longrightarrow R/(f_1, \ldots, f_r) \longrightarrow 0$$

is exact.

PROOF. Use Proposition 5.2. □

Proposition 5.7 may also be applied to prove that if R is regular, $f_1, \ldots, f_n \in M$ are independent in M/M^2, then:

$$\operatorname{Tor}_i^R(R/(f_1, \ldots, f_k), R/(f_{k+1}, \ldots, f_n)) = (0), \quad i > 0.$$

(cf. discussion of Serre's theory of intersection multiplicity, §V.1.)

6. Computing cohomology (3): Generate $\mathcal{F}$ by "known" sheaves

There are usually no projective objects in categories of sheaves, but it is nontheless quite useful to examine resolutions of the type:

$$\cdots \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where, for instance, the $\mathcal{E}_i$ are locally free sheaves of $\mathcal{O}_X$-modules (on *affine* schemes, such $\mathcal{E}_i$ are projective in the category of quasi-coherent sheaves).

Let S be a noetherian ring. We proved in Theorem III.4.3 due to Serre that for every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^l_S$ there is an integer n_0 such that $\mathcal{F}(n_0)$ is generated by global sections. This means that for some m_0, equivalently,

a) there is a surjection

$$\mathcal{O}^{m_0}_{\mathbb{P}^l_S} \longrightarrow \mathcal{F}(n_0) \longrightarrow 0$$

or

b) there is a surjection

$$\mathcal{O}_{\mathbb{P}^l_S}(-n_0)^{m_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Iterating, we get a resolution of $\mathcal{F}$ by "known" sheaves:

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(-n_1)^{m_1} \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(-n_0)^{m_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We are now in a position to prove Serre's Main Theorem in his classic paper [**87**]:

THEOREM 6.1 (Fundamental theorem of F.A.C.). *Let S be a noetherian ring, and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^l_S$. Then*

1) *$H^i(\mathbb{P}^l_S, \mathcal{F}(n))$ is a finitely generated S-module for all $i \geq 0$, $n \in \mathbb{Z}$.*
2) *$\exists n_0$ such that $H^i(\mathbb{P}^l_S, \mathcal{F}(n)) = (0)$ if $i \geq 1$, $n \geq n_0$.*
3) *Every $\mathcal{F}$ is of the form $\widetilde{M}$ for some finitely generated graded $S[X_0, \ldots, X_l]$-module M; and if $\mathcal{F} = \widetilde{M}$ where M is finitely generated, then $\exists n_1$ such that $M_n \to H^0(\mathbb{P}^l_S, \mathcal{F})$ is an isomorphism if $n \geq n_1$.*

PROOF. We prove (1) and (2) by descending induction on i. If $i > l$, then as we have seen $H^i(\mathcal{F}(n)) = (0)$, all n (cf. Proposition 4.2). Suppose we know (1) and (2) for all $\mathcal{F}$ and $i > i_0 \geq 1$. Given $\mathcal{F}$, put it in an exact sequence as before:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(-n_1)^{n_2} \longrightarrow \mathcal{F} \longrightarrow 0.$$

For every $n \in \mathbb{Z}$, this gives us:

$$0 \longrightarrow \mathcal{G}(n) \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(n - n_1)^{n_2} \longrightarrow \mathcal{F}(n) \longrightarrow 0,$$

hence

$$H^{i_0}(\mathcal{O}_{\mathbb{P}^l_S}(n - n_1))^{n_2} \longrightarrow H^{i_0}(\mathcal{F}(n)) \longrightarrow H^{i_0+1}(\mathcal{G}(n)).$$

By induction $H^{i_0+1}(\mathcal{G}(n))$ is finitely generated for all n and (0) for $n \gg 0$ and by §5, $H^{i_0}(\mathcal{O}_{\mathbb{P}^l_S}(n-n_1))$ is finitely generated for all n and (0) for $n \gg 0$: therefore the same holds for $\mathcal{F}(n)$.

The first half of (3) has been proven in Proposition III.4.4. Suppose $\mathcal{F} = \widetilde{M}$. Let $R = S[X_0, \ldots, X_l]$ and let

$$\bigoplus_\beta R(-n_\beta) \longrightarrow \bigoplus_\alpha R(-m_\alpha) \longrightarrow M \longrightarrow 0$$

be a presentation of M by twists of the free rank one module R. Taking $\widetilde{\ }$, this gives a presentation of $\mathcal{F}$:

$$\bigoplus_\beta \mathcal{O}_{\mathbb{P}^l_S}(-n_\beta) \longrightarrow \bigoplus_\alpha \mathcal{O}_{\mathbb{P}^l_S}(-m_\alpha) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{6.2}$$

Twisting by n and taking sections, we get a diagram:

$$\begin{array}{ccccccc} \bigoplus_\beta R_{n-n_\beta} & \longrightarrow & R_{n-m_\alpha} & \longrightarrow & M_n & \longrightarrow & 0 \\ \downarrow \approx & & \downarrow \approx & & \downarrow & & \\ \bigoplus_\beta \Gamma(\mathcal{O}_{\mathbb{P}^l_S}(n - n_\beta)) & \longrightarrow & \bigoplus_\alpha \Gamma(\mathcal{O}_{\mathbb{P}^l_S}(n - m_\alpha)) & \longrightarrow & H^0(\mathcal{F}(n)) & \longrightarrow & 0 \end{array} \tag{6.3}$$

with top row exact, but the bottom row need not be so. But break up (6.2) into short exact sequences

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_\alpha \mathcal{O}_{\mathbb{P}^l_S}(-m_\alpha) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{H} \longrightarrow \bigoplus_\beta \mathcal{O}_{\mathbb{P}^l_S}(-n_\beta) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Choose n_1 so that

$$H^1(\mathcal{G}(n)) = H^1(\mathcal{H}(n)) = (0), \quad n \geq n_1.$$

Then if $n \geq n_1$

$$0 \longrightarrow H^0(\mathcal{G}(n)) \longrightarrow \bigoplus_\alpha H^0(\mathcal{O}_{\mathbb{P}^l_S}(n - m_\alpha)) \longrightarrow H^0(\mathcal{F}(n)) \longrightarrow 0$$

$$0 \longrightarrow H^0(\mathcal{H}(n)) \longrightarrow \bigoplus_\alpha H^0(\mathcal{O}_{\mathbb{P}^l_S}(n - n_\beta)) \longrightarrow H^0(\mathcal{G}(n)) \longrightarrow 0$$

are exact, hence so is the bottom row of (6.3). This proves (3). $\square$

Corollary 6.4. *Let $f\colon X \to Y$ be a projective morphism (cf. Definition* II.5.8*) with Y a noetherian scheme. Let $\mathcal{L}$ be a relatively ample invertible sheaf on X. Then for all coherent $\mathcal{F}$ on X:*

1) *$R^i f_*(\mathcal{F})$ is coherent on Y.*
2) *$\exists n_0$ such that $R^i f_*(\mathcal{F} \otimes \mathcal{L}^n) = (0)$ if $i \geq 1$, $n \geq n_0$.*
3) *$\exists n_1$ such that all the natural map*

$$f^* f_*(\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow \mathcal{F} \otimes \mathcal{L}^n$$

is surjective if $n \geq n_1$.

Proof. Since Y can be covered by a *finite* set of affines, to prove all of these it suffices to prove them over some fixed affine $U = \operatorname{Spec} R \subset Y$. Then choose $n \geq 1$ and $(s_0, \ldots, s_k) \in \Gamma(f^{-1}(U), \mathcal{L}^n)$ defining a closed immersion $i\colon f^{-1}(U) \hookrightarrow \mathbb{P}^k_R$. Let $X' \subset \mathbb{P}^k_R$ be the image of i, and let $\mathcal{F}'$, $\mathcal{L}'$ be coherent sheaves on $\mathbb{P}^k_R$, (0) outside X' and isomorphic on X' to $\mathcal{F}|_{f^{-1}(U)}$ and $\mathcal{L}|_{f^{-1}(U)}$. By construction $\mathcal{O}_{X'}(1) \cong (\mathcal{L}')^n$. Then applying Serre's theorem (Theorem 6.1):

1) $R^i f_*(\mathcal{F})|_U \cong (H^i(X', \mathcal{F}'))\widetilde{\ }$ is coherent.
2) For any fixed m,

$$\begin{aligned} R^i f_*(\mathcal{F} \otimes \mathcal{L}^{m+\nu n})|_U &\cong (H^i(X', \mathcal{F}' \otimes (\mathcal{L}')^{m+\nu n}))\widetilde{\ } \\ &\cong (H^i(X', (\mathcal{F}' \otimes (\mathcal{L}')^m)(\nu)))\widetilde{\ } \\ &= (0), \qquad \text{if } \nu \geq \nu_0. \end{aligned}$$

Apply this for $m = 0, 1, \ldots, n-1$ to get (2) of Corollary 6.4.

3) For any fixed m,

$$\begin{aligned} f^* f_*(\mathcal{F} \otimes \mathcal{L}^{m+\nu n})|_{f^{-1}(U)} &\cong H^0(X', \mathcal{F}' \otimes (\mathcal{L}')^{m+\nu n}) \otimes_R \mathcal{O}_{X'} \\ &\cong H^0(X', \mathcal{F}' \otimes (\mathcal{L}')^m(\nu)) \otimes_R \mathcal{O}_{X'} \end{aligned}$$

and this maps *onto* $\mathcal{F}' \otimes (\mathcal{L}')^m$ if $\nu \geq \nu_1$.

Apply this for $m = 0, 1, \ldots, n-1$ to get (3) of Corollary 6.4. □

Combining this with Chow's lemma (Theorem II.6.3) and the Leray spectral sequence (Theorem 3.5), we get:

Theorem 6.5 (Grothendieck's coherency theorem). *Let $f\colon X \to Y$ be a proper morphism with Y a noetherian scheme. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $R^i f_*(\mathcal{F})$ is a coherent $\mathcal{O}_Y$-module for all i.*

Proof. The result being local on Y, we need to prove that if $Y = \operatorname{Spec} S$, then $H^i(X, \mathcal{F})$ is a finitely generated S-module. Since X is also a noetherian scheme, its closed subsets satisfy the descending chain condition and we may make a “noetherian induction”, i.e., assume the theorem holds for all coherent $\mathcal{G}$ with $\operatorname{Supp} \mathcal{G} \subsetneqq \operatorname{Supp} \mathcal{F}$. Also, if $\mathcal{I} \subset \mathcal{O}_X$ is the ideal of functions f such that multiplication by f is 0 in $\mathcal{F}$, we may replace X by the closed subscheme X', $\mathcal{O}_{X'} = \mathcal{O}_X/\mathcal{I}$. This has the effect that $\operatorname{Supp} \mathcal{F} = X$. Now apply Chow's lemma to construct

$$\begin{array}{ccc} & & X' \\ & \overset{\pi}{\swarrow} & \\ X & & \\ {\scriptstyle f}\downarrow & & \\ Y & & \end{array} \qquad \text{with } \pi \text{ and } f \circ \pi \text{ projective}$$

where $\operatorname{res}\pi\colon \pi^{-1}(U_0) \to U_0$ is an isomorphism for an open dense $U_0 \subset X$. Now consider the canonical map of sheaves $\alpha\colon \mathcal{F} \to \pi_*(\pi^*\mathcal{F})$ defined by the collection of maps:

$$\alpha(U)\colon \mathcal{F}(U) \longrightarrow \pi^*\mathcal{F}(\pi^{-1}U) = \pi_*(\pi^*)(U).$$

$\mathcal{F}$ coherent implies $\pi^*\mathcal{F}$ coherent and since π is projective, $\pi_*(\pi^*\mathcal{F})$ is coherent by Corollary 6.4. Look at the kernel, cokernel, and image:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \mathcal{F} \arrow[rr, "\alpha"] \arrow[dr] & & \pi_*(\pi^*\mathcal{F}) \arrow[r] & \mathcal{K}_2 \arrow[r] & 0. \\ & & & \mathcal{F}/\mathcal{K}_1 \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Since α is an isomorphism on U_0, $\operatorname{Supp}\mathcal{K}_i \subset X \setminus U_0 \subsetneqq X$. Thus $H^j(\mathcal{K}_i)$ are finitely generated S-modules by induction. But now using the long exact sequences:

$$\begin{tikzcd} H^{i-1}(\mathcal{K}_2) \arrow[r] & H^i(\mathcal{F}/\mathcal{K}_1) \arrow[r] & H^i(\pi_*\pi^*\mathcal{F}) \\ \text{finitely generated} \arrow[u, dashed] \arrow[d, dashed] & & \\ H^i(\mathcal{K}_1) \arrow[r] & H^i(\mathcal{F}) \arrow[r] & H^i(\mathcal{F}/\mathcal{K}_1) \end{tikzcd}$$

it follows readily that if $H^i(\pi_*\pi^*\mathcal{F})$ is finitely generated, so is $H^i(\mathcal{F})$. But now consider the Leray spectral sequence:

$$H^p(R^q\pi_*(\pi^*\mathcal{F})) = E_2^{pq} \Longrightarrow E^n = \underbrace{H^n(\pi^*\mathcal{F})}_{\substack{\text{finitely generated } S\text{-module} \\ \text{because } X' \text{ is projective} \\ \text{over } \operatorname{Spec} S.}}$$

If $q \geq 1$, then $R^q\pi_*(\pi^*\mathcal{F})|_{U_0} = (0)$; and since π is projective, $R^q\pi_*(\pi^*\mathcal{F})$ is coherent by Corollary 6.4. Therefore by noetherian induction, $H^p(R^q\pi_*(\pi^*\mathcal{F}))$ is finitely generated if $q \geq 1$. In other words, we have a spectral sequence of S-modules with E^n (all n) and E_2^{pq} ($q \geq 1$) finitely generated. It is a simple lemma that in such a case E_2^{p0} must be finitely generated too. □

7. Computing cohomology (4): Push $\mathcal{F}$ into a coherent acyclic one

This is a variant on Method (1) taking advantage of what we have learned already — that at least on $\mathbb{P}^l_S$ there are plenty of coherent acyclic sheaves obtained by twists. It is the closest in spirit to the original Italian methods out of which cohomology grew. For simplicity we work only on $\mathbb{P}^n_k$ (and its closed subschemes) for k an infinite field for the rest of §7.

Let $\mathcal{F}$ be coherent on $\mathbb{P}^n_k$. Then if $F(X_0, \ldots, X_n)$ is a homogeneous polynomial of degree d, multiplication by F defines a homomorphism:

$$\mathcal{F} \xrightarrow{F} \mathcal{F}(d).$$

If d is sufficiently large, $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = (0)$, $i > 0$, and the cohomology of $\mathcal{F}$ can be deduced from the kernel $\mathcal{K}_1$ and cokernel $\mathcal{K}_2$ of F as follows:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{F}(d) \arrow[r] & \mathcal{K}_2 \arrow[r] & 0 \\ & & & \mathcal{F}/\mathcal{K}_1 \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

$$\begin{array}{llll} (7.1) & \to H^i(\mathcal{K}_1) \to H^i(\mathcal{F}) \to H^i(\mathcal{F}/\mathcal{K}_1) \longrightarrow & H^{i+1}(\mathcal{K}_1) \longrightarrow \\ & & \approx\uparrow \\ & & H^{i-1}(\mathcal{K}_2),\ \text{if } i \geq 2 \\ & & \text{or} \\ & & H^0(\mathcal{K}_2)/H^0(\mathcal{F}(d)),\ \text{if } i = 1. \end{array}$$

This reduces properties of the cohomology of $\mathcal{F}$ to those of $\mathcal{K}_1$ and $\mathcal{K}_2$ which have, in general, lower dimensional support. In fact, one can easily arrange that F is injective, hence $\mathcal{K}_1 = (0)$ too. In terms of $\mathrm{Ass}(\mathcal{F})$, defined in §II.3, we can give the following criterion:

PROPOSITION 7.2. *Given a coherent $\mathcal{F}$ on $\mathbb{P}^n_k$, let $\mathrm{Ass}(\mathcal{F}) = \{x_1, \ldots, x_t\}$. Then $F\colon \mathcal{F} \to \mathcal{F}(d)$ is injective if and only if $F(x_a) \neq 0$, $1 \leq a \leq t$ (more precisely, if $x_a \notin V(X_{n_a})$, then the function $F/X^d_{n_a}$ is not 0 at x_a).*

PROOF. Let $U_a = \mathbb{P}^n_k \setminus V(X_{n_a})$. If $(F/X^d_{n_a})(x_a) = 0$, then $F/X^d_{n_a} = 0$ on $\overline{\{x_a\}} \cap U_a$. But $\exists s \in \mathcal{F}(U_a)$ with $\mathrm{Supp}(s) = \overline{\{x_a\}} \cap U_a$, so $(F/X^d_{n_a})^N \cdot s = 0$ if $N \gg 0$. Choose N_a so that $(F/X^d_{n_a})^{N_a} \cdot s \neq 0$ but $(F/X^d_{n_a})^{N_a+1} \cdot s = 0$. Then

$$F \cdot \left(\frac{F}{X^d_{n_a}}\right)^{N_a} \cdot s = 0 \quad \text{in } \mathcal{F}(d)(U_a)$$

so F is not injective. Conversely if $F(x_a) \neq 0$ for all a and $s \in \mathcal{F}(U)$ is not 0, then for some a, $s_{x_a} \in \mathcal{F}_{x_a}$ is not 0. But $F/X^d_{n_a}$ is a unit in $\mathcal{O}_{x_a}$, so $(F/X^d_{n_a}) \cdot s_{x_a} \neq 0$, so $F \cdot s_{x_a} \neq 0$. $\square$

Assuming then that F is *injective*, we get

$$(7.1^*) \qquad \begin{aligned} H^i(\mathcal{F}) &\xrightarrow{\ \approx\ } H^{i-1}(\mathcal{K}_2) \text{ if } i \geq 2 \\[2ex] H^1(\mathcal{F}) &\xrightarrow{\ \approx\ } H^0(\mathcal{K}_2)/\,\mathrm{Image}\, H^0(\mathcal{F}(d)). \end{aligned}$$

It is at this point that we make contact with the Italian methods. Let $X \subset \mathbb{P}^n_k$ be a projective variety, i.e., a reduced and irreducible closed subscheme. Let D be a Cartier divisor on X and $\mathcal{O}_X(D)$ the invertible sheaf of functions "with poles on D" (cf. §III.6). Then $\mathcal{O}_X(D)$, extended by (0) outside X, is a coherent sheaf on $\mathbb{P}^n_k$ of $\mathcal{O}_{\mathbb{P}^n}$-modules (cf. Remark after Corollary 3.8) and its cohomology may be computed by (7.1*).

In fact, we may do even better and describe its cohomology by induction *using only sheaves of the same type $\mathcal{O}_X(D)$*! First, some notation —

DEFINITION 7.3. If X is an irreducible reduced scheme, $Y \subset X$ an irreducible reduced subscheme and D is a Cartier divisor on X, then if $Y \not\subseteq \mathrm{Supp}\, D$, define $\mathrm{Tr}_Y\, D$ to be the Cartier divisor on Y whose local equations at $y \in Y$ are just the restrictions to Y of its local equations at $y \in X$. Note that:

$$\mathcal{O}_Y(\mathrm{Tr}_Y\, D) \cong \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_Y.$$

Now take a homogeneous polynomial F endowed with the following properties:

a) $X \not\subseteq V(F)$ and the effective Cartier divisor $H = \mathrm{Tr}_X(V(F))$ is reduced and irreducible,
b) no component D_j of $\mathrm{Supp}\, D$ is contained in $V(F)$.

It can be shown that such an F exists (in fact, in the affine space of all F's, any F outside a proper union of subvarieties will have these properties). Take a second F' with the property

c) $H \not\subseteq V(F')$

and let $H' = \mathrm{Tr}_X(V(F'))$. Start with the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

and tensor with $\mathcal{O}_X(D+H')$. We find

$$0 \longrightarrow \mathcal{O}_X(D+H'-H) \longrightarrow \mathcal{O}_X(D+H') \longrightarrow \mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H') \longrightarrow 0.$$

But the first sheaf is just $\mathcal{O}_X(D)$ via:

$$\mathcal{O}_X(D) \xrightarrow[\approx]{\text{multiply } F/F'} \mathcal{O}_X(D+H'-H)$$

and the second sheaf is just $\mathcal{O}_X(D)(d)$ and the whole sequence is the same exact sequence as before:

(7.4)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(D) \arrow[r, "\substack{\text{multiplication}\\ \text{by } F}"] \arrow[d, "\approx", "\substack{\text{multiplication}\\ \text{by } F/F'}"'] & \mathcal{O}_X(D)(d) \arrow[r] \arrow[d, "\substack{\text{multiplication}\\ \text{by } F/F'}", "\approx"'] & \mathcal{K}_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D+H'-H) \arrow[r, "\substack{\text{natural}\\ \text{inclusion}}"'] & \mathcal{O}_X(D+H') \arrow[r] & \mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H') \arrow[r] & 0
\end{tikzcd}$$

Thus $\mathcal{K}_2 \approx \mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H')$. This inductive precedure allowed the Italian School to discuss the cohomology in another language without leaving the circle of ideas of linear systems. For instance

$$\begin{aligned} H^1(\mathcal{O}_X(D)) &\cong \mathrm{Coker}\left[H^0(\mathcal{O}_X(D+H')) \longrightarrow H^0(\mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H'))\right] \\ &\cong \begin{bmatrix} \text{space of linear conditions that must be imposed} \\ \text{on an } f \in \mathbb{R}(H) \text{ with poles on } \mathrm{Tr}_H D + \mathrm{Tr}_H H' \text{ before} \\ \text{it can be extended to an } f \in \mathbb{R}(X) \text{ with poles in } D+H' \end{bmatrix}. \end{aligned}$$

Classically one dealt with the projective space $|D+H'|_X$ of divisors $V(s)$, $s \in H^0(\mathcal{O}_X(D+H'))$, (which is just the set of 1-dimensional subspaces of $H^0(\mathcal{O}_X(D+H'))$), and provided $\dim X \geq 2$, we can look instead at:

$$\begin{aligned} \left\{ \begin{array}{l} \text{subset of } |D+H'|_X \text{ of divisors} \\ E \text{ with } H \not\subseteq \mathrm{Supp}\, E \end{array} \right\} &\xrightarrow{\mathrm{Tr}_H} |\,\mathrm{Tr}_H D + \mathrm{Tr}_H H'|_H \\ E &\longmapsto \mathrm{Tr}_H E. \end{aligned}$$

Then

$$\begin{aligned} \dim H^1(\mathcal{O}_X(D)) = &\text{codimension of Image of } \mathrm{Tr}_H, \text{ called} \\ &\text{the ``deficiency'' of } \mathrm{Tr}_H |D+H'|_X. \end{aligned}$$

We go on now to discuss another application of method (4) — to the Hilbert polynomial. First of all, suppose X is any scheme proper over k and $\mathcal{F}$ is a coherent sheaf on X. Then one defines:

(7.5)
$$\begin{aligned} \chi(\mathcal{F}) &= \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, \mathcal{F}) \\ &= \text{the Euler characteristic of } \mathcal{F}, \end{aligned}$$

which makes sense because of the H^i are finite-dimensional by Grothendieck's coherency theorem (Theorem 6.5). The importance of this particular combination of the $\dim H^i$'s is that if

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of coherent sheaves, then it follows from the associated long exact cohomology sequence by a simple calculation that:

$$\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3). \tag{7.6}$$

This makes χ particularly easy to compute. In particular, we get:

THEOREM 7.7. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Then there exists a polynomial $P(t)$ with $\deg P = \dim \operatorname{Supp} \mathcal{F}$ such that*

$$\chi(\mathcal{F}(\nu)) = P(\nu), \quad \textit{all } \nu \in \mathbb{Z}.$$

In particular, by Theorem 6.1, *there exists an ν_0 such that*

$$\dim H^0(\mathcal{F}(\nu)) = P(\nu), \quad \textit{if } \nu \in \mathbb{Z},\ \nu \geq \nu_0.$$

$P(t)$ is called the Hilbert polynomial of $\mathcal{F}$.

PROOF. This is a geometric form of Part I [**76**, (6.21)] and the proof is parallel: Let $L(X)$ be a linear form such that $L(x_a) \neq 0$ for any of the associated points x_a of $\mathcal{F}$. Then as above we get an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{L} \mathcal{F}(1) \longrightarrow \mathcal{G} \longrightarrow 0$$

for some coherent $\mathcal{G}$, with

$$\operatorname{Supp} \mathcal{G} = \operatorname{Supp} \mathcal{F} \cap V(L)$$

hence

$$\dim \operatorname{Supp} \mathcal{G} = \dim \operatorname{Supp} \mathcal{F} - 1.$$

Tensoring by $\mathcal{O}_{\mathbb{P}^n}(l)$ we get exact sequences:

$$0 \longrightarrow \mathcal{F}(l) \longrightarrow \mathcal{F}(l+1) \longrightarrow \mathcal{G}(l) \longrightarrow 0 \tag{7.8}$$

for every $l \in \mathbb{Z}$, hence

$$\chi(\mathcal{F}(l+1)) = \chi(\mathcal{F}(l)) + \chi(\mathcal{G}(l)).$$

Now we prove the theorem by induction: if $\dim \operatorname{Supp} \mathcal{F} = 0$, $\operatorname{Supp} \mathcal{F}$ is a finite set, so $\operatorname{Supp} \mathcal{G} = \emptyset$ and $\mathcal{F}(l) \xrightarrow{\approx} \mathcal{F}(l+1)$ for all l by (7.8). Therefore $\chi(\mathcal{F}(l)) = \chi(\mathcal{F}) =$ constant, a polynomial of degree 0! In general, if $s = \dim \operatorname{Supp} \mathcal{F}$, then by induction $\chi(\mathcal{G}(l)) = Q(l)$, Q a polynomial of degree $s-1$. Then

$$\chi(\mathcal{F}(l+1)) - \chi(\mathcal{F}(l)) = Q(l)$$

hence as in Part I [**76**, (6.21)], $\chi(\mathcal{F}(l)) = P(l)$ for some polynomial P of degree s. □

This leads to the following point of view. Given $\mathcal{F}$, one often would like to compute $\dim_k \Gamma(\mathcal{F})$: for $\mathcal{F} = \mathcal{O}_X(D)$, this is the typical problem of the *additive theory of rational functions on X*. But because of the formula (7.6), it is often easier to compute either $\chi(\mathcal{F})$ directly, or $\dim_k \Gamma(\mathcal{F}(\nu))$ for $\nu \gg 0$, hence the Hilbert polynomial, hence $\chi(\mathcal{F})$ again. The Italians called $\chi(\mathcal{F})$ the *virtual dimension* of $\Gamma(\mathcal{F})$ and viewed it as $\dim \Gamma(\mathcal{F})$ (the main term) followed by an alternating sum of "error terms" $\dim H^i(\mathcal{F})$, $i \geq 1$. Thus one of the main reasons for computing the higher cohomology groups is to find how far $\dim \Gamma(\mathcal{F})$ has diverged from $\chi(\mathcal{F})$.

Recall that in Part I [**76**, (6.28)], we defined the arithmetic genus $p_a(X)$ of a projective variety $X \subset \mathbb{P}^n_k$ with a given projective embedding to be

$$p_a(X) = (-1)^r (P(0) - 1)$$
$$\text{where } P(x) = \text{Hilbert polynomial of } X,\ r = \dim X.$$

It now follows:

Corollary 7.9 (Zariski-Muhly).

$$p_a(X) = \dim H^r(\mathcal{O}_X) - \dim H^{r-1}(\mathcal{O}_X) + \cdots + (-1)^{r-1} \dim H^1(\mathcal{O}_X)$$

hence $p_a(X)$ is independent of the projective embedding of X.

Proof. By Theorem 7.7, $P(0) = \chi(\mathcal{O}_X)$ so the formula follows using $\dim H^0(\mathcal{O}_X) = 1$ (Corollary II.6.10). $\square$

I'd like to give one somewhat deeper result analyzing the "point" vis-a-vis tensoring with $\mathcal{O}(\nu)$ at which the higher cohomology vanishes; and which shows how the vanishing of higher cohomology groups alone can imply the existence of sections:

Theorem 7.10 (Generalized lemma of Castelnuovo and syzygy theorem of Hilbert). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Then the following are equivalent:*

i) $H^i(\mathcal{F}(-i)) = (0)$, $1 \le i \le n$,
ii) $H^i(\mathcal{F}(m)) = (0)$, *if* $m + i \ge 0$, $i \ge 1$,
iii) *there exists a "Spencer resolution":*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n)^{r_n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n+1)^{r_{n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{r_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{r_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

If these hold, then the canonical map

$$H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l)) \longrightarrow H^0(\mathcal{F}(l))$$

is surjective, $l \ge 0$.

Proof. We use induction on n: for $n = 0$, $\mathbb{P}^n_k = \operatorname{Spec} k$, $\mathcal{F} = \widetilde{k^n}$ and the result is clear. So we may suppose we know the result on $\mathbb{P}^{n-1}_k$. The implication (ii) $\Longrightarrow$ (i) is obvious and (iii) $\Longrightarrow$ (ii) follows easily from what we know of the cohomology of $\mathcal{O}_{\mathbb{P}^n}(l)$, by splitting the resolution up into a set of short exact sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n)^{r_n} \to \mathcal{O}_{\mathbb{P}^n}(-n+1)^{r_{n-1}} \to \mathcal{F}_{n-1} \to 0$$

$$- - - - - - - - - - - - - - - - -$$

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{r_1} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{r_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

So assume (i). Choose a linear form $L(X)$ such that $L(x_a) \ne 0$ for any associated points x_a of $\mathcal{F}$, getting sequences

$$0 \longrightarrow \mathcal{F}(l-1) \xrightarrow{\otimes L} \mathcal{F}(l) \longrightarrow \mathcal{G}(l) \longrightarrow 0, \quad \text{all } l \in \mathbb{Z}$$

where $\mathcal{G}$ is a coherent sheaf on the hyperplane $H = V(L)$. In fact $\mathcal{G}$ is not only supported on H but is annihilated by the local equations L/X_j of H: hence $\mathcal{G}$ is a sheaf of $\mathcal{O}_H$-modules. Since $H \cong \mathbb{P}^{n-1}_k$, we are in a position to apply our induction hypothesis. The cohomology sequences give:

$$\longrightarrow H^i(\mathcal{F}(-i)) \longrightarrow H^i(\mathcal{G}(-i)) \longrightarrow H^{i+1}(\mathcal{F}(-i-1)) \longrightarrow .$$

Applying this for $i \ge 1$, we find that $\mathcal{G}$ satisfies (i) also; applying it for $i = 0$, we find that $H^0(\mathcal{F}) \to H^0(\mathcal{G})$ is surjective. Therefore by the theorem for $\mathcal{G}$,

$$H^0(\mathcal{G}) \otimes H^0(\mathcal{O}_H(l)) \longrightarrow H^0(\mathcal{G}(l))$$

is surjective. Consider the maps:

$$\begin{array}{ccc} H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l)) & \xrightarrow{\gamma} & H^0(\mathcal{F}(l)) \\ \alpha\big\downarrow & & \beta\big\downarrow \\ H^0(\mathcal{G}) \otimes H^0(\mathcal{O}_H(l)) & \longrightarrow & H^0(\mathcal{G}(l)). \end{array}$$

We prove next that γ is surjective for all $l \geq 0$. By Proposition III.1.8, $H^0(\mathcal{O}_H(l))$ is the space of homogeneous polynomials of degree l in the homogeneous coordinates on H: therefore each is obtained by restricting to H a polynomial $P(X_0, \ldots, X_n)$ of degree l and $H^0(\mathcal{O}_{\mathbb{P}^n}(l)) \to H^0(\mathcal{O}_H(l))$ is surjective. Therefore α is surjective. It follows that if $s \in H^0(\mathcal{F}(l))$, then $\beta(s) = \sum \overline{u}_q \otimes \overline{v}_q$, $\overline{u}_q \in H^0(\mathcal{G})$, $\overline{v}_q \in H^0(\mathcal{O}_H(l))$; hence lifgint $\overline{u}_q$ to $u_q \in H^0(\mathcal{F})$, $\overline{v}_q$ to $v_q \in H^0(\mathcal{O}_{\mathbb{P}^n}(l))$, $s - \sum u_q \otimes v_q$ lies in $\operatorname{Ker}\beta$. But $\operatorname{Ker}\beta =$ Image of $H^0(\mathcal{F}(l-1))$ under the map $\otimes L\colon \mathcal{F}(l-1) \to \mathcal{F}(l)$ and by induction on l, anything in $H^0(\mathcal{F}(l-1))$ is in $H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l-1))$. Thus

$$s - \sum u_q \otimes v_q = \Big(\sum u'_q \otimes v'_q\Big) \otimes L, \quad u'_q \in H^0(\mathcal{F}),\ v'_q \in H^0(\mathcal{O}_{\mathbb{P}^n}(l-1)).$$

Thus

$$s = \sum u_q \otimes v_q + \sum u'_q \otimes (v'_q \otimes L), \quad \text{where } v'_q \otimes L \in H^0(\mathcal{O}_{\mathbb{P}^n}(l))$$

as required.

Next, note that this implies that $\mathcal{F}$ is generated by $H^0(\mathcal{F})$. In fact, if $x \in \mathbb{P}^n_k$, $x \notin V(X_j)$, and $s \in \mathcal{F}_x$, then $X_j^l \cdot s \in \mathcal{F}(l)_x$. For $l \gg 0$, $\mathcal{F}(l)$ is generated by $H^0(\mathcal{F}(l))$. So

$$\begin{array}{c} X_j^l \cdot s \in H^0(\mathcal{F}(l)) \cdot (\mathcal{O}_{\mathbb{P}^n})_x \\ \| \\ \big(H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l))\big) \cdot (\mathcal{O}_{\mathbb{P}^n})_x \end{array}$$

i.e.,

$$X_j^l \cdot s = \sum u_q \otimes v_q \cdot a_q, \quad u_q \in H^0(\mathcal{F}),\ v_q \in H^0(\mathcal{O}_{\mathbb{P}^n}(l)),\ a_q \in (\mathcal{O}_{\mathbb{P}^n})_x$$

hence

$$s = \sum u_q \otimes \underbrace{\frac{v_q}{X_j^l} \cdot a_q}_{\in (\mathcal{O}_{\mathbb{P}^n})_x}.$$

We can now begin to construct a Spencer resolution: let $s_1, \ldots, s_{r_0}$ be a basis of $H^0(\mathcal{F})$ and define

$$\mathcal{O}_{\mathbb{P}^n}^{r_0} \longrightarrow \mathcal{F} \longrightarrow 0$$

by

$$(a_1, \ldots, a_{r_0}) \longmapsto \sum_{q=0}^{r_0} a_q s_q.$$

If $\mathcal{F}_1$ is the kernel, then from the cohomology sequence it follows immediately that $\mathcal{F}_1(1)$ satisfies Condition (i) of the theorem. Hence $\mathcal{F}_1(1)$ is also generated by its sections and choosing a basis $t_1, \ldots, t_{r_1}$ of $H^0(\mathcal{F}_1(1))$, we get the next step:

$$\mathcal{O}_{\mathbb{P}^n}^{r_1} \longrightarrow \mathcal{F}_1(1) \longrightarrow 0$$

$$(a_1, \ldots, a_{r_1}) \longmapsto \sum_{q=1}^{r_1} a_q t_q$$

hence

$$\begin{tikzcd}[column sep=small] \mathcal{O}_{\mathbb{P}^n}(-1)^{r_1} \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n}^{r_0} \arrow[r] & \mathcal{F} \arrow[r] & 0. \\ & \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & \\ 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Continuing in this way, we derive the whole Spencer resolution. It remains to check that after the last step:

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n)^{r_n} \longrightarrow \mathcal{F}_n \longrightarrow 0,$$

the sheaf $\mathcal{F}_{n+1}$ is actually (0)! To prove this, we compute $H^i(\mathcal{F}_{n+1}(l))$ for $0 \le i \le n$, $0 \le l \le n$, using all the cohomology sequences $(*)_m$ associated to:

$$0 \longrightarrow \mathcal{F}_{m+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-m)^{r_m} \longrightarrow \mathcal{F}_m \longrightarrow 0.$$

We get:

a) $H^0(\mathcal{F}_{n+1}(l)) \underset{\text{by } (*)_n}{=} (0)$ (using injectivity of $H^0(\mathcal{F}^{r_n}) \to H^0(\mathcal{F}_n(n))$ when $l = n$)

b)

$$H^1(\mathcal{F}_{n+1}) \begin{cases} \underset{\text{by } (*)_n}{=} (0) & \text{if } l = n \quad \text{(using surjectivity of } H^0(\mathcal{O}^{r_n}) \to H^0(\mathcal{F}_n(n))) \\ \underset{\text{by } (*)_n}{\cong} H^0(\mathcal{F}_n(l)) \underset{\text{by } (*)_{n-1}}{=} (0) & \text{if } l < n \quad \text{(using injectivity of } H^0(\mathcal{O}^{r_{n-1}}) \to H^0(\mathcal{F}_{n-1}(n-1)) \text{ when } l = n-1) \end{cases}$$

. . . .

x)

$$H^n(\mathcal{F}_{n+1}(l)) \underset{\text{by } (*)_n}{\cong} H^{n-1}(\mathcal{F}_n(l)) \underset{\text{by } (*)_{n-1}}{\cong} \cdots$$

$$\cdots \underset{\text{by } (*)_2}{\cong} H^1(\mathcal{F}_2(l))$$

$$\begin{cases} \underset{\text{by } (*)_1}{=} (0) & \text{if } l \ge 1 \quad \text{(using surjectivity of } H^0(\mathcal{O}(l-1)^{r_1}) \to H^0(\mathcal{F}_1(l))) \\ \underset{\text{by } (*)_1}{\cong} H^0(\mathcal{F}_1) \underset{\text{by } (*)_0}{=} (0) & \text{if } l = 1 \quad \text{(using injectivity of } H^0(\mathcal{O}^{r_0}) \to H^0(\mathcal{F})). \end{cases}$$

So all these groups are (0). Thus $\chi(\mathcal{F}_{n+1}(l)) = 0$, for $n+1$ distinct values $l = 0, \ldots, n$. Since $\chi(\mathcal{F}_{n+1}(l))$ is a polynomial of degree at most n, it must be identically 0. But then for $l \gg 0$, $\dim H^0(\mathcal{F}_{n+1}(l)) = \chi(\mathcal{F}_{n+1}(l)) = 0$, hence $H^0(\mathcal{F}_{n+1}(l)) = (0)$ and since these sections generate $\mathcal{F}_{n+1}(l)$, $\mathcal{F}_{n+1}(l) = (0)$ too. $\square$

Exercise. Bezout's Theorem via the Spencer resolution.

(1) If C is any abelian category, define

$$K^0(C) = \left(\begin{array}{l} \text{free abelian group on elements } [X], \text{ one for each} \\ \text{isomorphism class of objects in } C, \text{ modulo relations} \\ [X_2] = [X_1] + [X_3] \text{ for each short exact sequence:} \\ 0 \to X_1 \to X_2 \to X_3 \to 0 \\ \text{in } C. \end{array} \right)$$

If X is any noetherian scheme, define

$K_0(X) = K^0(\text{Category of coherent sheaves of } \mathcal{O}_X\text{-modules on } X)$

$K^0(X) = K^0(\text{Category of locally free finite rank sheaves of } \mathcal{O}_X\text{-modules})$.

Prove:

a) $\exists$ a natural map $K^0(X) \to K_0(X)$.

b) $K^0(X)$ is a contravariant functor in X, i.e., $\forall$ morphism $f\colon X \to Y$, we get $f^*\colon K^0(Y) \to K^0(X)$ with the usual properties.

c) $K^0(X)$ is a commutative ring via

$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] = [\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2]$$

and $K_0(X)$ is a $K^0(X)$-module via

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}].$$

d) $K_0(X)$ is a covariant functor for *proper* morphisms $f\colon X \to Y$ via

$$f_*([\mathcal{F}]) = \sum_{n=0}^{\infty} (-1)^n [R^n f_* \mathcal{F}].$$

(2) Return to the case where k is an infinite field.

a) Using the Spencer resolution, show that

$$K^0(\mathbb{P}^n_k) \longrightarrow K_0(\mathbb{P}^n_k)$$

is surjective and that they are both generated by the sheaves $[\mathcal{O}_{\mathbb{P}^n}(l)]$, $l \in \mathbb{Z}$.
Hint: On any scheme X, if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

is exact, $\mathcal{E}_i$ locally free and finitely generated, then $\mathcal{F}$ is locally free and finitely generated, and *locally* on X, the sequence *splits*, i.e., $\mathcal{E}_1 \cong \mathcal{F} \oplus \mathcal{E}_0$.

b) Consider the Koszul complex $K^{\bullet}(X_0, \dots, X_n; k[X_0, \dots, X_n])$. Take and hence show that $[\mathcal{O}_{\mathbb{P}^n}(l)] \in K^0(\mathbb{P}^n_k)$ satisfy

$$(*) \qquad \sum_{\nu=0}^{n+1} (-1)^\nu \binom{n+1}{\nu} [\mathcal{O}_{\mathbb{P}^n}(\nu + \nu_0)] = 0, \quad \forall \nu_0 \in \mathbb{Z},$$

hence $K^0(\mathbb{P}^n_k)$ is generated by $[\mathcal{O}_{\mathbb{P}^n}(\nu)]$ for any set of ν's of the form $\nu_0 \le \nu \le \nu_0 + n$. Show that $[\mathcal{O}_{L^\nu}]$, $0 \le \nu \le n$, $L^\nu =$ a fixed linear space of dimension ν, generate $K_0(\mathbb{P}^n_k)$.

c) Let

$$S_n = \begin{pmatrix} \text{group of rational polynomials } P(t) \text{ of degree } \le n \\ \text{taking integer values at integers} \end{pmatrix} = \begin{pmatrix} \text{free abelian group on the polynomials} \\ P_\nu(t) = \binom{t}{\nu}, \quad 0 \le \nu \le n \end{pmatrix}.$$

Prove that

$$[\mathcal{F}] \longmapsto \text{Hilbert polynomial of } \mathcal{F}$$

defines

$$K_0(\mathbb{P}^n_k) \longrightarrow S_n.$$

d) Combining (a), (b) and (c), show that $K^0(\mathbb{P}^n_k) \xrightarrow{\sim} K_0(\mathbb{P}^n_k)$.

e) Using the result of Part I [**76**, §6C] show that if $Z \subset \mathbb{P}^n_k$ is any subvariety, of dimension r and

$$\begin{aligned} g_\nu &= p_a(Z \cdot H_1 \cdots H_{r-\nu}) \\ &= \text{arithmetic genus of the } \nu\text{-dimensional} \\ &\qquad \text{linear section of } Z, \quad (1 \leq \nu \leq r) \end{aligned}$$

$d = \deg Z$, then in $K_0(\mathbb{P}^n_k)$:

$$\begin{aligned} [\mathcal{O}_Z] = d \cdot [\mathcal{O}_{L^n}] + (1 - d - g_1)[\mathcal{O}_{L^{r-1}}] + (g_1 + g_2)[\mathcal{O}_{L^{r-2}}] + \\ \cdots + (-1)^n (g_{r-1} + g_r)[\mathcal{O}_{L^0}]. \end{aligned}$$

(3) Because of (2), (d), $K_0(\mathbb{P}^n_k)$ inherits a ring structure. Using the sheaves $\mathcal{T}or_i$ defined in §2 as one of the applications of the "easy lemma of the double complex" (Lemma 2.4), show that this ring structure is given by

$$(*) \qquad [\mathcal{F}_1] \cdot [\mathcal{F}_2] = \sum_{i=0}^{n} (-1)^i [\mathcal{T}or_i(\mathcal{F}_1, \mathcal{F}_2)].$$

In particular, check that $\mathcal{T}or_i = (0)$ if $i > n$. (In fact, on any regular scheme X, it can be shown that $\mathcal{T}or_i = (0)$, $i > \dim X$; and that $(*)$ defines a ring structure in $K_0(X)$). Next apply this with $\mathcal{F}_1 = \mathcal{O}_{X_1}$, $\mathcal{F}_2 = \mathcal{O}_{X_2}$, X_1, X_2 subvarieties of $\mathbb{P}^n_k$ intersecting properly and transversely at generic points of the components $W_1, \ldots, W_\nu$ of $X_1 \cap X_2$ (cf. Part I [**76**, §5B]). Show by Ex. 2, §5D2???, that if $i \geq 1$,

$$\dim \operatorname{Supp}(\mathcal{T}or_i(\mathcal{O}_{X_1}, \mathcal{O}_{X_2})) < \dim X_1 \cap X_2.$$

Combining this with the results of (2), show Bezout's Theorem:

$$(\deg X_1) \cdot (\deg X_2) = \sum_{i=1}^{\nu} \deg W_i.$$

Hint: Show that $[\mathcal{O}_{L^r}] \cdot [\mathcal{O}_{L^s}] = [\mathcal{O}_{L^{r+s-n}}]$. Show next that if $i \geq 1$

$$[\mathcal{T}or_i(\mathcal{O}_{X_1}, \mathcal{O}_{X_2})] = \text{combination of } [\mathcal{O}_{L^t}] \text{ for } t < \dim(X_1 \cap X_2).$$

8. Serre's criterion for ampleness

This section gives a cohomological criterion equivalent to ampleness for an invertible sheaf introduced in §III.5. We apply it later to questions of positivity of intersections, formulated in terms of the Euler characteristic.

THEOREM 8.1. *Let X be a scheme over a noetherian ring A, embedded as a closed subscheme in a projective space over A, with canonical sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be coherent on X. Then for all $i \geq 0$, $H^i(X, \mathcal{F})$ is a finite A-module, and there exists an integer n_0 such that for $n \geq n_0$ we have*

$$H^i(X, \mathcal{F}(n)) = 0 \quad \textit{for all } i \geq 1.$$

PROOF. We have already seen in Corollary 3.8 that under a closed embedding $X \hookrightarrow \mathbb{P}^r_A$ the cohomology of $\mathcal{F}$ over X is the same as the cohomology of $\mathcal{F}$ viewed as a sheaf over projective space. Consequently we may assume without loss of generality that $X = \mathbb{P}^r_A$, which we denote by $\mathbb{P}$.

The explicit computation of cohomology $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ in Corollary 5.4 and (5.6) shows that the theorem is true when $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(n)$ for all integers n. Now let $\mathcal{F}$ be an arbitrary coherent sheaf on $\mathbb{P}$. We can represent $\mathcal{F}$ in a short exact sequence (cf. §6)

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}$ is a finite direct sum of sheaves $\mathcal{O}_{\mathbb{P}}(d)$ for appropriate positive integers d, and $\mathcal{G}$ is defined to be the kernel of $\mathcal{E} \to \mathcal{F}$. We use the cohomology sequence, and write the cohomology groups without $\mathbb{P}$ for simplicity:

$$\longrightarrow H^i(\mathcal{E}) \longrightarrow H^i(\mathcal{F}) \longrightarrow H^{i+1}(\mathcal{G}) \longrightarrow$$

We apply descending induction. For $i > r$ we have $H^i(\mathcal{F}) = 0$ because $\mathbb{P}$ can be covered by $r+1$ open affine subsets, and the Čech complex is 0 with respect to this covering in dimension $\geq r+1$ (cf. (5.5)). If, by induction, $H^{i+1}(\mathcal{G})$ is finite over A, then the finiteness of $H^i(\mathcal{E})$ implies that $H^i(\mathcal{F})$ is finite.

Furthermore, twisting by n, that is, taking tensor products with $\mathcal{O}_{\mathbb{P}}(n)$, is an exact functor, so the short exact sequence tensored with $\mathcal{O}_{\mathbb{P}}(n)$ remains exact. This gives rise to the cohomology exact sequence:

$$\to H^i(\mathcal{E}(n)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{G}(n)) \to$$

Again by induction, $H^{i+1}(\mathcal{G}(n)) = 0$ for n sufficiently large, and $H^i(\mathcal{E}(n)) = 0$ because of the special nature of $\mathcal{E}$ as a direct sum of sheaves $\mathcal{O}_{\mathbb{P}}(d)$. This implies that $H^i(\mathcal{F}(n)) = 0$ for n sufficiently large, and concludes the proof of the theorem. □

THEOREM 8.2 (Serre's criterion). *Let X be a scheme, proper over a noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if the following condition holds: For any coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that for all $n \geq n_0$ we have*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \textit{for all } i \geq 1.$$

PROOF. Suppose that $\mathcal{L}$ is ample, so $\mathcal{L}^d$ is very ample for some d. We have seen (cf. Theorem III.5.4 and §II.6) that X is projective over A. We apply Theorem 8.1 to the tensor products

$$\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \ldots, \mathcal{F} \otimes \mathcal{L}^{d-1}$$

and the very ample sheaf $\mathcal{L}^d = \mathcal{O}_X(1)$ to conclude the proof that the cohomology groups vanish for $i \geq 1$.

Conversely, assume the condition on the cohomology groups. We want to prove that $\mathcal{L}$ is ample. It suffices to prove that for any coherent sheaf $\mathcal{F}$ the tensor product $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for n sufficiently large. (cf. Definition III.5.1) By Definition III.2.1 it will suffice to prove that for every closed point P, the fibre $\mathcal{F} \otimes \Bbbk(P)$ is generated by global sections. Let $\mathcal{I}_P$ be the ideal sheaf defining the closed point P as a closed subscheme. We have an exact sequence

$$0 \longrightarrow \mathcal{I}_P\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \Bbbk(P) \longrightarrow 0.$$

Since $\mathcal{L}^n$ is locally free, tensoring with $\mathcal{L}^n$ preserves exactness, and yields the exact sequence

$$0 \longrightarrow \mathcal{I}_P\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \Bbbk(P) \otimes \mathcal{L}^n \longrightarrow 0$$

whence the cohomology exact sequence

$$H^0(\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(\mathcal{F} \otimes \Bbbk(P) \otimes \mathcal{L}^n) \longrightarrow 0$$

because $H^1(\mathcal{I}_P\mathcal{F} \otimes \mathcal{L}^n) = 0$ by hypothesis. This proves that the fibre at P of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections, and concludes the proof of the theorem. □

9. Functorial properties of ampleness

This section gives a number of conditions relating ampleness on a scheme with ampleness on certain subschemes.

PROPOSITION 9.1. *Let X be a scheme of finite type over a noetherian ring and $\mathcal{L}$ an invertible sheaf, ample on X. For every closed subscheme Y, the restriction $\mathcal{L}|_Y = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is ample on Y.*

PROOF. Taking a power of $\mathcal{L}$ we may assume without loss of generality that $\mathcal{L}$ is very ample (cf. Theorem III.5.4), so $\mathcal{O}_X(1)$ in a projective embedding of X. Then $\mathcal{O}_X|_Y = \mathcal{O}_Y(1)$ in that same embedding. Thus the proposition is immediate. □

Let X be a scheme. For each open subset U we let $\mathcal{N}il(U)$ be the ideal of nilpotnet elements in $\mathcal{O}_X(U)$. Then $\mathcal{N}il$ is a sheaf of ideals, and the quotient sheaf $\mathcal{O}_X/\mathcal{N}il$ defines a closed subscheme called the reduced scheme X_{red}. Its sheaf of rings has no nilpotent elements. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we let

$$\mathcal{F}_{\text{red}} = \mathcal{F}/\mathcal{I}\mathcal{F} \quad \text{where } \mathcal{I} = \mathcal{N}il\,.$$

Alternatively, we can say that $\mathcal{F}_{\text{red}}$ is the restriction of $\mathcal{F}$ to X_{red}.

PROPOSITION 9.2. *Let X be a scheme, proper over a noetherian ring. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{\text{red}}$ is ample on X_{red}.*

PROOF. By Proposition 9.1, it suffices to prove one side of the equivalence, namely: if $\mathcal{L}_{\text{red}}$ is ample then $\mathcal{L}$ is ample. Since X is noetherian, there exists an integer r such that if $\mathcal{N} = \mathcal{N}il$ is the sheaf of nilpotent elements, then $\mathcal{N}^r = 0$. Hence we get a finite filtration

$$\mathcal{F} \supset \mathcal{N}\mathcal{F} \supset \mathcal{N}^2\mathcal{F} \supset \cdots \supset \mathcal{N}^r\mathcal{F} = 0.$$

For each $i = 1, \ldots, r-1$ we have the exact sequence

$$0 \longrightarrow \mathcal{N}^i\mathcal{F} \longrightarrow \mathcal{N}^{i-1}\mathcal{F} \longrightarrow \mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F} \longrightarrow 0$$

whence the exact cohomology sequence

$$H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, \mathcal{N}^{i-1}\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, (\mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F}) \otimes \mathcal{L}^n).$$

For each i, $\mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F}$ is a coherent $\mathcal{O}_X/\mathcal{N}$-module, and thus is a sheaf on X_{red}. By hypothesis, and Theorem 8.2, we know that

$$H^p(X, (\mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F}) \otimes \mathcal{L}^n) = 0$$

for all n sufficiently large and all $p \geq 1$. But $\mathcal{N}^i\mathcal{F} = 0$ for $i \geq r$. We use descending induction on i. We have

$$H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \text{for all } p > 0, i \geq r,$$

and n sufficiently large. Hence inductively,

$$H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \text{for all } p > 0$$

implies that $H^p(X, \mathcal{N}^{i-1}\mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $p > 0$ and n sufficiently large. This concludes the proof. □

PROPOSITION 9.3. *Let X be a proper scheme over a noetherian ring. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}|_{X_i}$ is ample on each irreducible component X_i of X.*

PROOF. Since an irreducible component is a closed subscheme of X, Proposition 9.1 shows that it suffices here to prove one implication. So assume that $\mathcal{L}|_{X_i}$ is ample for all i. Let $\mathcal{I}_i$ be the coherent sheaf of ideals defining X_i, and say $i = 1, \dots, r$. We use induction on r. We consider the exact sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{I}_1\mathcal{F} \longrightarrow 0,$$

giving rise to the exact cohomology sequence

$$H^p(X, \mathcal{I}_1\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, (\mathcal{F}/\mathcal{I}_1\mathcal{F}) \otimes \mathcal{L}^n).$$

Since $\mathcal{L}|_{X_1}$ is ample by hypothesis, it follows that

$$H^p(X, (\mathcal{F}/\mathcal{I}_1\mathcal{F}) \otimes \mathcal{L}^n) = 0$$

for all $p > 0$ and $n \geq n_0$. Furthermore, $\mathcal{I}_1\mathcal{F}$ is a sheaf with support in $X_2 \cup \cdots \cup X_r$, so by induction we have

$$H^p(X, \mathcal{I}_1\mathcal{F} \otimes \mathcal{L}^n) = 0$$

for all $p > 0$ and $n \geq n_0$. The exact sequence then gives

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$$

for all $p > 0$ and $n \geq n_0$, thus concluding the proof. □

PROPOSITION 9.4. *Let $f\colon X \to Y$ be a finite (cf. Definition* II.6.6*) surjective morphism of proper schemes over a noetherian ring. Let $\mathcal{L}$ be an invertible sheaf on Y. Then $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample on X.*

PROOF. First note that f is affine (cf. Proposition-Definition I.7.3 and Definition II.6.6). Let $\mathcal{F}$ be a coherent sheaf on X, so $f_*\mathcal{F}$ is coherent on Y. For $p \geq 0$ we get:

$$\begin{aligned} H^p(Y, f_*(\mathcal{F}) \otimes \mathcal{L}^n) &= H^p(Y, f_*(\mathcal{F} \otimes (f^*\mathcal{L})^n)) \\ &\qquad \text{by the projection formula} \\ &= H^p(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) \end{aligned}$$

by Proposition 3.7 [9]. If $\mathcal{L}$ is ample, then the left hand side is 0 for $n \geq n_0$ and $p > 0$, so this proves that $f^*\mathcal{L}$ is ample on X.

Conversely, assume $f^*\mathcal{L}$ ample on X. We show that for any coherent $\mathcal{O}_Y$-module $\mathcal{G}$, one has

$$H^p(Y, \mathcal{G} \otimes \mathcal{L}^n) = 0, \qquad \forall p > 0 \text{ and } n \gg 0$$

by noetherian induction on $\mathrm{Supp}(\mathcal{G})$.

By Propositions 9.2 and 9.3, we may assume X and Y to be integral. We follow Hartshorne [**50**, §4, Lemma 4.5, pp. 25–27] and first prove:

[9] Let $f\colon X \to Y$ be a morphism, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{L}$ an $\mathcal{O}_Y$-module. The identity homomorphism $f^*\mathcal{L} \to f^*\mathcal{L}$ induces an $\mathcal{O}_Y$-homomorphism $\mathcal{L} \to f_*f^*\mathcal{L}$. Tensoring this with $f_*\mathcal{F}$ over $\mathcal{O}_Y$ and composing the result with a canonical homomorphism, one gets a canonical homomorphism

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L} \longrightarrow f_*\mathcal{F} \otimes_{\mathcal{O}_Y} f_*f^*\mathcal{L} \longrightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{L}).$$

This can be easily shown to be an isomophism if $\mathcal{L}$ is a locally free $\mathcal{O}_Y$-module of finite rank, giving rise to the "projection formula"

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{\sim} f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{L}).$$

LEMMA 9.5. *Let $f\colon X \to Y$ be a finite surjective morphism of degree m of noetherian integral schemes X and Y. Then for every coherent $\mathcal{O}_Y$-module $\mathcal{G}$ on Y, there exist a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-homomorphism $\xi\colon f_*\mathcal{F} \to \mathcal{G}^{\oplus m}$ that is a generic isomorphism (i.e., ξ is an isomorphism in a neighborhood of the generic point of Y).*

PROOF OF LEMMA 9.5. By assumption, the function field $\mathbb{R}(X)$ is an algebraic extension of $\mathbb{R}(Y)$ of degree m. Let $U = \operatorname{Spec} A \subset X$ be an affine open set. Since $\mathbb{R}(X)$ is the quotient field of A, we can choose $s_1, \dots, s_m \in A$ such that $\{s_1, \dots, s_m\}$ is a basis of $\mathbb{R}(X)$ as a vector space over $\mathbb{R}(Y)$. The $\mathcal{O}_X$-submodule $\mathcal{H} = \sum_{i=1}^m \mathcal{O}_X s_i$ of the constant $\mathcal{O}_X$-module $\mathbb{R}(X)$ is coherent. Since $s_1, \dots, s_m \in H^0(X, \mathcal{H}) = H^0(Y, f_*\mathcal{H})$, we have an $\mathcal{O}_Y$-homomorphism

$$\eta\colon \mathcal{O}_Y^{\oplus m} = \sum_{i=1}^m \mathcal{O}_Y e_i \longrightarrow f_*\mathcal{H}, \qquad e_i \longmapsto s_i \quad (i = 1, \dots, m),$$

which is a generic isomorphism by the choice of $s_1, \dots, s_m$. If a coherent $\mathcal{O}_Y$-module $\mathcal{G}$ is given, η induces an $\mathcal{O}_Y$-homomorphism

$$\xi\colon \mathcal{H}' = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{H}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus m}, \mathcal{G}) = \mathcal{G}^{\oplus m},$$

which is a generic isomorphism. Since $\mathcal{H}'$ is an $f_*\mathcal{O}_X$-module through the first factor of $\mathcal{H}om$ and f is finite, we have $\mathcal{H}' = f_*\mathcal{F}$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$. □

To continue the proof of Proposition 9.4, let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module $\mathcal{G}$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module as in Lemma 9.5, and let $\mathcal{K}$ and $\mathcal{C}$ be the kernel and cokernel of the $\mathcal{O}_Y$-homomorphism $\xi\colon f_*(\mathcal{F}) \to \mathcal{G}^{\oplus m}$. We have exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow f_*\mathcal{F} \longrightarrow \mathrm{Image}(\xi) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Image}(\xi) \longrightarrow \mathcal{G}^{\oplus m} \longrightarrow \mathcal{C} \longrightarrow 0.$$

$\mathcal{K}$ and $\mathcal{C}$ are coherent $\mathcal{O}_Y$-modules, and $\mathrm{Supp}(\mathcal{K}) \subsetneqq Y$ and $\mathrm{Supp}(\mathcal{C}) \subsetneqq Y$, since ξ is a generic isomorphism. Hence by the induction hypothesis, we have

$$H^p(Y, \mathcal{K} \otimes \mathcal{L}^n) = H^p(Y, \mathcal{C} \otimes \mathcal{L}^n) = 0, \qquad \forall p > 0 \text{ and } n \gg 0.$$

By the cohomology long exact sequence, we have

$$\begin{array}{c} H^p(Y, (f_*\mathcal{F}) \otimes \mathcal{L}^n) \xrightarrow{\ \sim\ } H^p(Y, \mathrm{Image}(\xi) \otimes \mathcal{L}^n) \xrightarrow{\ \sim\ } H^p(Y, \mathcal{G} \otimes \mathcal{L}^n)^{\oplus m} \\ \| \\ H^p(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) \end{array}$$

for all $p > 0$ and $n \gg 0$, the equality on the left hand side being again by the projection formula. $H^p(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) = 0$ for all $p > 0$ and $n \gg 0$, since $f^*\mathcal{L}$ is assumed to be ample. Hence

$$H^p(Y, \mathcal{G} \otimes \mathcal{L}^n) = 0, \qquad \forall p > 0 \text{ and } n \gg 0.$$

□

PROPOSITION 9.6. *Let X be a proper scheme over a noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf on X, and assume that $\mathcal{L}$ is generated by its global sections. Suppose that for every closed integral curve C in X the restriction $\mathcal{L}|_C$ is ample. Then $\mathcal{L}$ is ample on X.*

For the proof we need the following result given in Proposition VIII.1.7:

> Let C' be a geometrically irreducible curve, proper and smooth over a field k.
> An invertible sheaf $\mathcal{L}'$ on C' is ample if and only if $\deg \mathcal{L}' > 0$.

Proof. By Propositions 9.2 and 9.3 we may assume without loss of generality that X is integral. Since $\mathcal{L}$ is generated by global sections, a finite number of these define a morphism

$$\varphi\colon X \longrightarrow \mathbb{P}^n_A$$

such that $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}}(1)$. Then φ is a finite morphism. For otherwise, by Corollary V.6.5 some fiber of φ contains a closed integral curve C. Let $\varphi(C) = P$, a closed point of $\mathbb{P}^n_A$. Let $f\colon C' \to C$ be a morphism obtained as follows: C' is the normalization of C in a composite field $\overline{\Bbbk(P)}\mathbb{R}(C)$ obtained as a quotient of

$$\overline{\Bbbk(P)} \otimes_{\Bbbk(P)} \mathbb{R}(C),$$

where $\overline{\Bbbk(P)}$ is the algebraic closure of $\Bbbk(P)$. (C' is regular by Proposition V.5.11, hence is proper and smooth over $\overline{\Bbbk(P)}$.) Since $\mathcal{L}|_C$ is ample, so is $\mathcal{L}' = f^*\mathcal{L}$ by Proposition 9.4. But then the $\deg \mathcal{L}' > 0$ by Proposition VIII.1.7, while $\mathcal{L}' = f^*\mathcal{L} = f^*\varphi^*\mathcal{O}_{\mathbb{P}}(1)$. This contradicts the fact that $\varphi(C) = P$ is a point. Hence φ is finite. Propositions 9.2, 9.3 and 9.4 now conclude the proof. □

10. The Euler characteristic

Throughout this section, we let A be a local artinian ring. We let

$$X \longrightarrow \operatorname{Spec}(A)$$

be a projective morphism. We let $\mathcal{F}$ be a coherent sheaf on X.

By Theorem 8.1, the cohomology groups $H^i(X, \mathcal{F})$ are finite A-modules, and since A is artinian, they have finite length. By (5.5) and Corollary 3.8, we also have $H^i(X, \mathcal{F}) = 0$ for i sufficiently large. We define the *Euler characteristic*

$$\chi_A(X, \mathcal{F}) = \chi_A(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \operatorname{length} H^i(X, \mathcal{F}).$$

This is a generalization of what we introduced in (7.5) in the case $A = k$ a field. As a generalization of Theorem 7.7, we have:

Proposition 10.1. *Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of coherent sheaves on X. Then

$$\chi_A(\mathcal{F}) = \chi_A(\mathcal{F}') + \chi_A(\mathcal{F}'').$$

Proof. This is immediate from the exact cohomology sequence

$$\longrightarrow H^p(X, \mathcal{F}') \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}'') \longrightarrow$$

which has 0's for $p < 0$ and p sufficiently large. cf. Lang [**65**, Chapter IV]. □

We now compute this Euler characteristic in an important special case.

Proposition 10.2. *Suppose $\mathbb{P} = \mathbb{P}^r_A$. Then*

$$\chi_A(\mathcal{O}_{\mathbb{P}}(n)) = \binom{n+r}{r} = \frac{(n+r)(n+r-1)\cdots(n+1)}{r!} \quad \text{for all } n \in \mathbb{Z}.$$

PROOF. For $n > 0$, we can apply Corollary 5.4 to conclude that

$$\chi_A(\mathcal{O}_{\mathbb{P}}(n)) = \operatorname{length} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)),$$

which is the number of monomials in $T_0, \dots, T_r$ of degree n, and is therefore equal to the binomial coefficient as stated. If $n \leq -r-1$, then similarly by (5.6), we have

$$\chi_k(\mathcal{O}_{\mathbb{P}}(n)) = (-1)^r \operatorname{length} H^r(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)).$$

From the explicit determination of the cohomology in (5.6) if we put $n = -r-d$ then the length of $H^r(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ over A is equal to the number of r-tuples $(q_0, \dots, q_r)$ of integers $q_j > 0$ such that $\sum q_j = r+d$, which is equal to the number of r-tuples $(q'_0, \dots, q'_r)$ of integers ≥ 0 such that $\sum q'_j = d-1$. This is equal to

$$\binom{d-1+r}{r} = (-1)^r \binom{n+r}{r}.$$

Finally, let $-r \leq n \leq 0$. Then $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = 0$ for all $i > 0$ once more by Corollary 5.4 and (5.6). Also the binomial coefficient is 0. This proves the proposition. □

Starting with the explicit case of projective space as in Proposition 10.2, we can now derive a general result, which is a generalization of Theorem 7.7 in the case $A = k$ a field.

THEOREM 10.3. *Let A be a local artinian ring. Let X be a projective scheme over $Y = \operatorname{Spec}(A)$. Let $\mathcal{L}$ be an invertible sheaf on X, very ample over Y, and let $\mathcal{F}$ be a coherent sheaf on X. Put*

$$\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}^n \quad \text{for } n \in \mathbb{Z}.$$

i) *There exists a unique polynomial $P(T) \in \mathbb{Q}[T]$ such that*

$$\chi_A(\mathcal{F}(n)) = P(n) \quad \text{for all } n \in \mathbb{Z}.$$

ii) *For n sufficiently large, $\chi_A(\mathcal{F}(n)) = \operatorname{length} H^0(X, \mathcal{F}(n))$.*
iii) *The leading coefficient of $P(T)$ is ≥ 0.*

PROOF. By Theorem 8.1 we know that

$$H^i(\mathcal{F}(n)) = 0 \quad \text{for } i \geq 1 \text{ and } n \text{ large}.$$

Hence $\chi_A(\mathcal{F}(n))$ is the length of $H^0(\mathcal{F}(n))$ as asserted in (ii). In particular, $\chi_A(\mathcal{F}(n))$ is ≥ 0 for n large, so the leading coefficient of $P(T)$ is ≥ 0 if such polynomial exists. Its uniqueness is obvious.

To show existence, we reduce to the case of Proposition 10.2 by Jordan-Hölder techniques. Suppose we have an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

Taking the tensor product with $\mathcal{L}$ preserves exactness. It follows immediately that if (i) is true for $\mathcal{F}'$ and $\mathcal{F}''$, then (i) is true for $\mathcal{F}$. Let $\mathfrak{m}$ be the maximal ideal of A. Then there is a finite filtration

$$\mathcal{F} \supset \mathfrak{m}\mathcal{F} \supset \mathfrak{m}^2\mathcal{F} \supset \cdots \supset \mathfrak{m}^s\mathcal{F} = 0.$$

By the above remark, we are reduced to proving (i) when $\mathfrak{m}\mathcal{F} = 0$, because $\mathfrak{m}$ annihilates each factor sheaf $\mathfrak{m}^j\mathcal{F}/\mathfrak{m}^{j+1}\mathcal{F}$.

Suppose now that $\mathfrak{m}\mathcal{F} = 0$. Then $\mathcal{F}$ can be viewed as a sheaf on the fibre X_y, where y is the closed point of $Y = \operatorname{Spec}(A)$. The restriction of $\mathcal{L}$ to X_y is ample by Proposition 9.1, and the cohomology groups of a sheaf on a closed subscheme are the same as those of that same sheaf viewed on the whole scheme. The twisting operation also commutes with passing to a closed

subscheme. This reduces the proof that $\chi_A(\mathcal{F}(n))$ is a polynomial to the case when A is a field k. Thus we are done by Theorem 7.7. $\square$

When $A = k$ is a field, the length is merely the dimension over k. For any coherent sheaf $\mathcal{F}$ on X we have by definition

$$\chi(\mathcal{F}) = \sum_{i=0}^{d} (-1)^i \dim_k H^i(X, \mathcal{F}),$$

where $d = \dim X$. By Theorem 7.7, we know that

$$P(n) = \chi(\mathcal{F}(n))$$

is a polynomial of degree e where $e = \dim \operatorname{Supp} \mathcal{F}$.

11. Intersection numbers

Throughout this section we let X be a proper scheme over a field k. We let $\chi = \chi_k$.

THEOREM 11.1 (Snapper). *Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible sheaves on X and let $\mathcal{F}$ be a coherent sheaf. Let $d = \dim \operatorname{Supp}(\mathcal{F})$. Then there exists a polynomial P with rational coefficients, in r variables, such that for all integers $n_1, \ldots, n_r$ we have*

$$P(n_1, \ldots, n_r) = \chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F}).$$

This polynomial P has total degree $\leq d$.

PROOF. Suppose first that $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are very ample. Then the assertion follows by induction on r and Theorem 7.7 (generalized in Theorem 10.3). Suppose X projective. Then there exists a very ample invertible sheaf $\mathcal{L}_0$ such that $\mathcal{L}_0\mathcal{L}_1, \ldots, \mathcal{L}_0\mathcal{L}_r$ are very ample (take any very ample sheaf, raise it to a sufficiently high power and use Theorem III.5.10). Let

$$Q(n_0, n_1, \ldots, n_r) = \chi\left(\mathcal{L}_0^{n_0} \otimes (\mathcal{L}_0\mathcal{L}_1)^{n_1} \otimes \cdots \otimes (\mathcal{L}_0\mathcal{L}_r)^{n_r} \otimes \mathcal{F}\right).$$

Then

$$P(n_1, \ldots, n_r) = Q(-n_1 - \cdots - n_r, n_1, \ldots, n_r)$$

and the theorem follows.

If X is not projective, the proof is more complicated. We follow Kleiman [**62**]. The proof proceeds by induction on $d = \dim \operatorname{Supp}(\mathcal{F})$. Since the assertion is trivial if $d = -1$, i.e., $\mathcal{F} = (0)$, we assume $d \geq 0$.

Replacing X by the closed subscheme $\mathcal{S}pec_X(\mathcal{O}_X / \operatorname{Ann}(\mathcal{F}))$ defined by the annihilator ideal $\operatorname{Ann}(\mathcal{F})$, we may assume $\operatorname{Supp}(\mathcal{F}) = X$. The induction hypothesis then means that the assertion is true for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\operatorname{Supp}(\mathcal{F}) \subsetneqq X$, i.e., for torsion $\mathcal{O}_X$-modules $\mathcal{F}$.

Let $\mathbf{K}$ be the abelian category of coherent $\mathcal{O}_X$-modules, and let $\mathbf{K}' \subset \operatorname{Ob}(\mathbf{K})$ consist of those $\mathcal{F}$'s for which the assertion holds. $\mathbf{K}'$ is obviously exact in the sense of Definition II.6.11. By dévissage (Theorem II.6.12), it suffices to show that $\mathcal{O}_Y \in \mathbf{K}'$ for any closed integral subscheme Y of X. In view of the induction hypothesis, we may assume $Y = X$, that is, X itself is integral. Then by Proposition III.6.2, there exists a Cartier divisor D on X such that $\mathcal{L}_1 = \mathcal{O}_X(D)$ and that the intersections $\mathcal{I} = \mathcal{O}_X(-D) \cap \mathcal{O}_X$ as well as $\mathcal{J} = \mathcal{O}_X(D) \cap \mathcal{O}_X$ taken inside the function field $\mathbb{R}(X)$ are coherent $\mathcal{O}_X$-ideals not equal to $\mathcal{O}_X$. Obviously, we have $\mathcal{J} = \mathcal{I} \otimes \mathcal{O}_X(D) = \mathcal{I} \otimes \mathcal{L}_1$.

Tensoring the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$ (resp. $0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0$) with $\mathcal{L}_1^{n_1}$ (resp. $\mathcal{L}_1^{n_1-1}$), we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I} \otimes \mathcal{L}_1^{n_1} & \longrightarrow & \mathcal{L}_1^{n_1} & \longrightarrow & \mathcal{L}_1^{n_1} \otimes (\mathcal{O}_X/\mathcal{I}) & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & J \otimes \mathcal{L}_1^{n_1-1} & \longrightarrow & \mathcal{L}_1^{n_1-1} & \longrightarrow & \mathcal{L}_1^{n_1-1} \otimes (\mathcal{O}_X/\mathcal{J}) & \longrightarrow & 0. \end{array}$$

Thus tensoring both sequences with $\mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r}$ and taking the Euler characteristic, we have

$$\begin{aligned} &\chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r}) - \chi(\mathcal{L}_1^{n_1-1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r}) \\ &\quad = \chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes (\mathcal{O}_X/\mathcal{I})) - \chi(\mathcal{L}_1^{n_1-1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes (\mathcal{O}_X/\mathcal{J})). \end{aligned}$$

The right hand side is a polynomial with rational coefficients in $n_1, \ldots, n_r$ of total degree $< d$ since $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{J}$ are torsion $\mathcal{O}_X$-modules. Hence we are done, since $\chi(\mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r})$ is a polynomial in $n_2, \ldots, n_r$ of total degree $\leq d$ by induction on r. □

We recall here the following result on integral valued polynomials.

LEMMA 11.2. *Let $P(x_1, \ldots, x_r) \in \mathbb{Q}[x_1, \ldots, x_r] = \mathbb{Q}[x]$ be a polynomial with rational coefficients, and integral valued on $\mathbb{Z}^r$. Then P admits an expression*

$$P(x_1, \ldots, x_r) = \sum a(i_1, \ldots, i_r) \binom{x_1 + i_1}{i_1} \cdots \binom{x_r + i_r}{i_r}$$

where $a(i_1, \ldots, i_r) \in \mathbb{Z}$, the sum is taken for $i_1, \ldots, i_r \geq 0$,

$$\binom{x+i}{i} = \frac{(x+i)(x+i-1)\cdots(x+1)}{i!} \quad \text{if } i > 0,$$

and the binomial coefficients is 1 if $i = 0$, and 0 if $i < 0$.

PROOF. This is proved first for one variable by induction, and then for several variables by induction again. We leave this to the reader. □

LEMMA 11.3. *The coefficient of $n_1 \cdots n_r$ in $\chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F})$ is an integer.*

PROOF. Immediate from Lemma 11.2. □

Let us define the *intersection symbol*:

$$(\mathcal{L}_1 \ldots \mathcal{L}_r.\mathcal{F}) = \text{coefficient of } n_1 \cdots n_r \text{ in the polynomial}$$
$$\chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F}).$$

LEMMA 11.4.

(i) *The function $(\mathcal{L}_1 \ldots \mathcal{L}_r.\mathcal{F})$ is multilinear in $\mathcal{L}_1, \ldots, \mathcal{L}_r$.*
(ii) *If c is the coefficient of n^r in $\chi(\mathcal{L}^n \otimes \mathcal{F})$, then*

$$(\mathcal{L}.\mathcal{L} \ldots \mathcal{L}.\mathcal{F}) = r!c \quad (\mathcal{L} \textit{ is repeated } r \textit{ times.})$$

PROOF. Let $\mathcal{L}$, $\mathcal{M}$ be invertible sheaves. Then

$$(\mathcal{L}^n \otimes \mathcal{M}^m \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F}) = ann_2 \cdots n_r + bmn_2 \cdots n_r + \cdots$$

with rational coefficients a, b. Putting $n = 0$ and $m = 0$ shows that

$$a = (\mathcal{L}.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}) \qquad \text{and} \qquad b = (\mathcal{M}.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}).$$

Let $m = n = n_1$. It follows that

$$((\mathcal{L} \otimes \mathcal{M}).\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}) = (\mathcal{L}.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}) + (\mathcal{M}.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}).$$

Similarly, $(\mathcal{L}^{-1}.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}) = -(\mathcal{L}.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F})$. This proves the first assertion.

As to the second, let $P(n) = (\mathcal{L}^n.\mathcal{F})$ and

$$Q(n_1, \dots, n_r) = (\mathcal{L}_1^{n_1} \dots \mathcal{L}_r^{n_r}.\mathcal{F}).$$

Let ∂ be the derivative, and $\partial_1, \dots, \partial_r$ be the partial derivatives. Then the second assertion follows from the relation

$$\partial_1 \cdots \partial_r Q(0, \dots, 0) = \partial^r P(0).$$

$\square$

The next lemma gives the additivity as a function of $\mathcal{F}$, in the sense of the Grothendieck group.

LEMMA 11.5. *Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of coherent sheaves. Then

$$(\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F}) = (\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F}') + (\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F}'').$$

PROOF. Immediate since the Euler characteristic satisfies the same type of relation. $\square$

12. The criterion of Nakai-Moishezon

Let X be a proper scheme over a field k.

Let Y be a closed subscheme of X. Then Y is defined by a coherent sheaf of ideals $\mathcal{I}_Y$, and

$$\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$$

is its structure sheaf. Let $D_1, \dots, D_r$ be divisors on X, by which we always mean Cartier divisors, so they correspond to invertible sheaves $\mathcal{L}_1 = \mathcal{O}_X(D_1), \dots, \mathcal{L}_r = \mathcal{O}_X(D_r)$. *Suppose Y has dimension r.* We define the *intersection number*

$$(D_1 \dots D_r.Y) = \text{coefficient of } n_1 \cdots n_r \text{ in the polynomial}$$
$$\chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{O}_Y).$$
$$(D^r.Y) = (D \dots D.Y), \quad \text{where } D \text{ is repeated } r \text{ times.}$$

LEMMA 12.1.

(i) *The intersection number $(D_1 \dots D_r.Y)$ is an integer, and the function*

$$(D_1, \dots, D_r) \longmapsto (D_1 \dots D_r.Y)$$

is multilinear symmetric.

(ii) *If a is the coefficient of n^r in $\chi(\mathcal{L}^n \otimes \mathcal{O}_Y)$, and $\mathcal{L} = \mathcal{O}_X(D)$, then $(D^r.Y) = r!a$.*

PROOF. This is merely a repetition of Lemma 11.4 in the present context and notation. $\square$

REMARK. Suppose that Y is zero dimensional, so Y consists of a finite number of closed points. Then the higher cohomology groups are 0, and

$$(Y) = \chi(\mathcal{O}_Y) = \dim H^0(Y, \mathcal{O}_Y) > 0,$$

because $H^0(Y, \mathcal{O}_Y)$ is the vector space of global sections, and is not 0 since Y is affine. One can reduce the general intersection symbol to this case by means of the next lemma.

LEMMA 12.2. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible sheaves on X such that $\mathcal{L}_1$ is very ample. Let D_1 be a divisor corresponding to $\mathcal{L}_1$ such that D_1 does not contain any associated point of $\mathcal{O}_Y$. Let Y' be the scheme intersection of Y and D_1. Then*

$$(D_1 \ldots D_r.Y) = (D_2 \ldots D_r.Y').$$

In particular, if $D_1, \ldots, D_r$ are ample, then

$$(D_1 \ldots D_r.Y) > 0.$$

PROOF. If $\mathcal{I}_Y$ is the sheaf of ideals defining Y, and $\mathcal{I}_1$ is the sheaf of ideals defining D_1, the $(\mathcal{I}_Y, \mathcal{I}_1)$ defines $Y \cap D_1$. By §III.6 we know that $\mathcal{I}_1$ is locally principal. The assumption in the lemma implies that we have an exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{I}_1 \otimes \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y \cap D_1} \longrightarrow 0.$$

Indeed, let $\operatorname{Spec}(A)$ be an open affine subset of X containing a generic point of Y, and such that D_1 is represented in A by the local equation $f = 0$, while Y is defined by the ideal I. Then the above sequence translates to

$$0 \longrightarrow fA \otimes A/I \longrightarrow A/I \longrightarrow A/(I, f) \longrightarrow 0$$

which is exact on the left by our assumption on f.

But $\mathcal{I}_1 = \mathcal{L}_1^{-1}$ by the definitions. Tensoring the sequence $(*)$ with

$$\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r},$$

taking the Euler characteristic, and using the additivity of the Euler characteristic, we get

$$\begin{aligned}(D_1 \ldots D_r.Y)n_1 \cdots n_r - (D_1 \ldots D_r.Y)(n_1 - 1)n_2 \cdots n_r + \text{lower terms} \\ = (D_2 \ldots D_r.Y')n_2 \cdots n_r + \text{lower terms}.\end{aligned}$$

This proves the lemma. □

The intersection number $(D^r.Y)$ was taken with respect to the scheme X and it is sometimes necessary to include X in the notation, so we write

$$(D_1 \ldots D_r.Y)_X \quad \text{or} \quad (\mathcal{L}_1 \ldots \mathcal{L}_r.Y)_X.$$

On the other hand, let Z be a closed subscheme of X. Then we may induce the sheaves to Z to get $\mathcal{L}|_{1Z}, \ldots, \mathcal{L}|_{rZ}$.

LEMMA 12.3. *Let $Y \subset Z \subset X$ be inclusions of closed subschemes. Suppose Y has dimension r as before. Then*

$$(\mathcal{L}_1 \ldots \mathcal{L}_r.Y)_X = (\mathcal{L}|_{1Z} \ldots \mathcal{L}|_{rZ}.Y)_Z.$$

PROOF. In the tensor products

$$\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{O}_Y$$

we may tensor with $\mathcal{O}_Z$ each one of the factors without changing this tensor product. The cohomology of a sheaf supported by a closed subscheme is the same as the cohomology of the sheaf in the scheme itself (cf. Corollary 3.8), so the assertion of the lemma is now clear. □

THEOREM 12.4 (Criterion of Nakai-Moishezon). *Let X be a proper scheme over a field k. Then a divisor D is ample on X if and only if $(D^r.Y) > 0$ for all integral closed subschemes Y of dimension r, for all $r \leq \dim X$.*

PROOF. Suppose D is ample. Replacing D by a positive multiple, we may assume without loss of generality that D is very ample. Let $\mathcal{L} = \mathcal{O}(D)$, and let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}}(1)$ for a projective embedding $f\colon X \to \mathbb{P}$ over k. Abbreviate $\mathcal{H} = \mathcal{O}_{\mathbb{P}}(1)$. Then the Euler characteristic

$$\chi_k(\mathcal{L}^{n_1} \otimes \cdots \otimes \mathcal{L}^{n_r} \otimes \mathcal{O}_Y)$$

is the same as the Euler characteristic

$$\chi_k(\mathcal{H}^{n_1} \otimes \cdots \otimes \mathcal{H}^{n_r} \otimes \mathcal{O}_Y)$$

where $\mathcal{O}_Y$ is now viewed as a sheaf on $\mathbb{P}$. This reduces the positivity to the case of projective space, and D is a hyperplane, which is true by Lemma 12.2.

The converse is more difficult and is the essense of the Nakai-Moishezon theorem. We assume that $(D^r.Y) > 0$ for all integral closed subschemes Y of X of dimension $r \le \dim X$ and we want to prove that D is ample. By Propositions 9.2 and 9.3, we may assume that X is integral (reduced and irreducible), so X is a variety.

For the rest of the proof we let $\mathcal{L} = \mathcal{O}(D)$.

By Lemma 12.3 and induction we may assume that $\mathcal{L}|_Z$ is ample for every closed subscheme Z of X, $Z \ne X$.

LEMMA 12.5. *For n large, $H^0(X, \mathcal{L}^n) \ne 0$.*

PROOF OF LEMMA 12.5. First we remark that $\chi(\mathcal{L}^n) \to \infty$ as $n \to \infty$, for by Lemma 12.1 (ii),

$$\chi(\mathcal{L}^n) = an^d + \text{lower terms}$$

where $d = \dim X$, and $r!a = (D^d.X) > 0$ by assumption.

Next, we prove that $H^i(\mathcal{L}^n) \approx H^i(\mathcal{L}^{n-1})$ for $i \ge 2$ and $n \ge n_0$. Since X is integral, we can identify $\mathcal{L}$ as a subsheaf of the sheaf of rational functions on X. We let

$$\mathcal{I} = \mathcal{L}^{-1} \cap \mathcal{O}_X.$$

Then $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{O}_X$, defining a closed subscheme $Y \ne X$. Furthermore $\mathcal{I} \otimes \mathcal{L}$ is also a coherent sheaf of ideals, defining a closed subscheme $Z \ne X$. We have two exact sequences

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{L} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We tensor the first with $\mathcal{L}^n$ and the second with $\mathcal{L}^{n-1}$. By induction, $H^i(\mathcal{L}^n|_Y) = H^i(\mathcal{L}^{n-1}|_Z) = 0$ for $i \ge 1$ and $n \ge n_0$. Then the exact cohomology sequence gives isomorphisms for $i \ge 2$ and $n \ge n_0$:

$$H^i(\mathcal{I} \otimes \mathcal{L}^n) \approx H^i(\mathcal{L}^n) \quad \text{and} \quad H^i(\mathcal{I} \otimes \mathcal{L} \otimes \mathcal{L}^{n-1}) \approx H^i(\mathcal{L}^{n-1}).$$

This proves that $H^i(\mathcal{L}^n) \approx H^i(\mathcal{L}^{n-1})$ for $i \ge 2$. But then

$$\dim H^0(\mathcal{L}^n) \ge \chi(\mathcal{L}^n) \to \infty,$$

thus proving the lemma. □

A global section of $\mathcal{L}^n$ then implies the existence of an effective divisor $E \sim nD$, and since the intersection number depends only on the linear equivalence class (namely, on the isomorphism class of the invertible sheaves), the hypothesis of the theorem implies that $(E^r.Y) > 0$ for all closed subschemes Y of X. It will suffice to prove that E is ample. This reduces the proof of the theorem to the case when D is effective, which we now assume.

LEMMA 12.6. *Assume D effective. Then for sufficiently large n, $\mathcal{L}^n$ is generated by its global sections.*

PROOF OF LEMMA 12.6. We have $\mathcal{L} = \mathcal{O}(D)$ where D is effective, so we have an exact sequence

$$0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Tensoring with $\mathcal{L}^n$ yields the exact sequence

$$0 \longrightarrow \mathcal{L}^{n-1} \longrightarrow \mathcal{L}^n \longrightarrow \mathcal{L}^n|_D \longrightarrow 0.$$

By induction, $\mathcal{L}^n|_D$ is ample on D, so $H^1(\mathcal{L}^n|_D) = 0$ for n large. The cohomology sequence

$$H^0(\mathcal{L}^n) \longrightarrow H^0(\mathcal{L}^n|_D) \longrightarrow H^1(\mathcal{L}^{n-1}) \longrightarrow H^1(\mathcal{L}^n) \longrightarrow H^1(\mathcal{L}^n|_D)$$

shows that $H^1(\mathcal{L}^{n-1}) \to H^1(\mathcal{L}^n)$ is surjective for n large. Since the vector spaces $H^1(\mathcal{L}^n)$ are finite dimensional, there exists n_0 such that

$$H^1(\mathcal{L}^{n-1}) \longrightarrow H^1(\mathcal{L}^n) \quad \text{is an isomorphism for } n \geq n_0.$$

Now the first part of the cohomology exact sequence shows that

$$H^0(\mathcal{L}^n) \longrightarrow H^0(\mathcal{L}^n|_D) \quad \text{is surjective for } n \geq n_0.$$

Since $\mathcal{L}^n|_D$ is ample on D, it is generated by global sections. By Nakayama, it follows that $\mathcal{L}^n$ is generated by global sections. This proves Lemma 12.6 □

We return to the proof of Theorem 12.4 proper. If $\dim X = 1$, then $(D) > 0$, X is a curve, and every effective non-zero divisor on a curve is ample (cf. Proposition VIII.1.7 below).

Suppose $\dim X \geq 2$. For every integral curve (subscheme of dimension one) C on X, we know by induction that $\mathcal{L}^n|_C$ is ample on C. We can apply Proposition 9.6 to conclude the proof. □

CHAPTER VIII

Applications of cohomology

In this chapter, we hope to demonstrated the usefulness of the formidable tool that we developed in Chapter VII. We will deal with several topics that are tied together by certain common themes, although not in a linear sequence. We will start with possibly the most famous theorem in all algebraic geometry: the Riemann-Roch theorem for curves. This has always been the principal non-trivial result of an introduction to algebraic geometry and we would not dare to omit it. Besides being the key to the higher theory of curves, it also brings in differentials in an essential way — foreshadowing the central role played by the cohomology of differentials on all varieties. This theme, that of De Rham cohomology is discussed in §3. In order to be able to prove strong result there, we must first discuss in §2 Serre's cohomological approach to Chow's theorem, comparing analytic and algebraic coherent cohomologies. In §4 we discuss the application, following Kodaira, Spencer and Grothendieck, of the cohomology of Θ, the sheaf of vector fields, to deformation of varieties. Finally, in §§2, 3 and 4, we build up the tools to be able at the end to give Grothendieck's results on the partial computation of π_1 of a curve in characteristic p.

1. The Riemann-Roch theorem

As we discussed in §VII.7, cohomology, disguised in classical language, grew out of the attempt to develop formulas for the dimension of:

$$H^0(\mathcal{O}_X(D)) = \left\{ \begin{array}{l} \text{space of 0 and non-zero rational functions } f \text{ on } X \\ \text{with poles at most } D\text{, i.e., } (f) + D \geq 0 \end{array} \right\}.$$

(See also the remark in §III.6.)

Put another way, the general problem is to describe the filtration of the function field $\mathbb{R}(X)$ given by the size of the poles. This one may call the fundamental problem of the *additive* theory of functions on X (as opposed to the *multiplicative* theory dealing with the group $\mathbb{R}(X)^*$, and leading to $\mathrm{Pic}(X)$). Results on $\dim H^0(\mathcal{O}_X(D))$ lead in turn to results on the projective embeddings of X and other rational maps of X to $\mathbb{P}^n$, hence to many results on the geometry and classification of varieties X.

The first and still the most complete result of this type is the *Riemann-Roch theorem for curves.* This may be stated as follows:

Theorem 1.1 (Riemann-Roth theorem). *Let k be a field and let X be a curve, smooth and proper over k such that X is* geometrically irreducible *(also said to be absolutely irreducible, i.e., $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$ is irreducible with $\overline{k}$ = algebraic closure of k). If $\sum n_i P_i$ ($P_i \in X$, closed points) is a divisor on X, define*

$$\deg(\sum n_i P_i) = \sum n_i [\Bbbk(P_i) : k].$$

Then for any divisor D on X:

1) $\dim_k H^0(\mathcal{O}_X(D)) - \dim_k H^1(\mathcal{O}_X(D)) = \deg D - g + 1$, *where $g = \dim_k H^1(\mathcal{O}_X)$ is the genus of X, and*

2) *(weak form)* $\dim_k H^1(\mathcal{O}_X(D)) = \dim_k H^0(\Omega^1_{X/k}(-D))$.

The first part follows quickly from our general theory like this:

PROOF OF 1). Note first that $H^0(\mathcal{O}_X)$ consists only in constants in k. In fact $H^0(\mathcal{O}_X)$ is a finite-dimensional k-algebra (cf. Proposition II.6.9), without nilpotents because X is reduced and without non-trivial idempotents because X is connected. Therefore $H^0(\mathcal{O}_X)$ is a field L, finite over k. By the theorey of §IV.2, X smooth over $k \implies \mathbb{R}(X)$ separable over $k \implies L$ separable over k; and $X \times_k \overline{k}$ irreducible $\implies k$ separable algebraically closed in $\mathbb{R}(X) \implies L$ purely inseparable over k. Thus $L = k$, and (1) can be rephrased:

$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X).$$

Therefore (1) follows from:

LEMMA 1.2. *If P is a closed point on X and $\mathcal{L}$ is an invertible sheaf, then*

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-P)) + [\Bbbk(P) : k].$$

PROOF OF LEMMA 1.2. Use the exact sequence:

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \Bbbk(P) \longrightarrow 0$$

and the fact that $\mathcal{L}$ invertible $\implies \mathcal{L} \otimes_{\mathcal{O}_X} \Bbbk(P) \cong \underline{\Bbbk(P)}$ (where: $\underline{\Bbbk(P)} =$ sheaf (0) outside P, with stalk $\Bbbk(P)$ at P). Thus

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-P)) + \chi(\underline{\Bbbk(P)})$$

and since $H^0(\underline{\Bbbk(P)}) = \Bbbk(P)$, $H^1(\underline{\Bbbk(P)}) = (0)$, the result follows. □

□

To explain the rather mysterious second part, consider the first case $k = \mathbb{C}$, $D = \sum_{i=1}^d P_i$ with the P_i distinct, so that $\deg D = d$. Let $z_i \in \mathcal{O}_{P_i,X}$ vanish to first order at P_i, so that z_i is a local analytic coordinate in a small (classical) neighborhood of P_i. Then if $f \in H^0(\mathcal{O}_X(D))$, we can expand f near each P_i as:

$$f = \frac{a_i}{z_i} + \text{function regular at } P_i,$$

and we can map

$$H^0(\mathcal{O}_X(D)) \longrightarrow \mathbb{C}^d$$

$$f \longmapsto (a_1, \ldots, a_d)$$

by assigning the coefficients of their poles to each f. Since only constants have no poles, this shows right away that

$$\dim H^0(\mathcal{O}_X(D)) \leq d + 1.$$

Suppose on the other hand we start with $a_1, \ldots, a_d \in \mathbb{C}$ and seek to construct f. From elementary complex variable theory we find obstructions to the existence of this f! Namely, regarding X as a compact Riemann surface (= compact 1-dimensional complex manifold), we use the fact that if ω is a meromorphic differential on X, then the sum of the residues of ω at all its poles is zero (an immediate consequence of Cauchy's theorem). Now $\Omega^1_{X/\mathbb{C}}$ is the sheaf of *algebraic* differential forms on X and for any Zariski-open $U \subset X$ and $\omega \in \Omega^1_{X/\mathbb{C}}(U)$, ω defines a *holomorphic* differential form on U. (In fact if locally near $x \in U$,

$$\omega = \sum a_j db_j, \quad a_j, b_j \in \mathcal{O}_{x,X}$$

then a_j, b_j are holomorphic functions near x too and $\sum a_j db_j$ defines a holomorphic differential form: we will discuss this rather fine point more carefully in §3 below.) So if $\omega \in \Gamma(\Omega^1_{X/\mathbb{C}})$, then write ω near P_i as:

$$\omega = (b_i(\omega) + \text{function zero at } P_i) \cdot dz_i, \quad b_i(\omega) \in \mathbb{C}.$$

If f exists with poles a_i/z_i at P_i, then $f\omega$ is a meromorphic differential such that:

$$f\omega = a_i \cdot b_i(\omega) \cdot \frac{dz_i}{z_i} + (\text{differential regular at } P_i),$$

hence

$$\mathrm{res}_{P_i}(f\omega) = a_i \cdot b_i(\omega)$$

hence

$$0 = \sum_{i=1}^{d} \mathrm{res}_{P_i}(f\omega) = \sum_{i=1}^{d} a_i \cdot b_i(\omega).$$

This is a linear condition on $(a_1, \ldots, a_d)$ that must be satisfied if f is to exist. Now Assertion (2) of Theorem 1.1 in its most transparent form is just the converse: if $\sum a_i \cdot b_i(\omega) = 0$ for every $\omega \in \Gamma(\Omega^1_{X/\mathbb{C}})$, then f with polar parts a_i/z_i exists. How does this imply (2) as stated? Consider the pairing:

$$\mathbb{C}^d \times H^0(\Omega^1_{X/\mathbb{C}}) \longrightarrow \mathbb{C}$$

$$((a_i), \omega) \longmapsto \sum a_i \cdot b_i(\omega).$$

Clearly the null-space of this pairing on the $H^0(\Omega^1_{X/\mathbb{C}})$-side is the space of ω's zero at each P_i, i.e., $H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i))$. We have claimed that the null-space on the $\mathbb{C}^d$-side is Image $H^0(\mathcal{O}_X(\sum P_i))$. Thus we have a *non-degenerate* pairing:

$$\Big(\mathbb{C}^d / \operatorname{Image} H^0(\mathcal{O}_X(\sum P_i))\Big) \times \Big(H^0(\Omega^1_{X/\mathbb{C}})/H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i))\Big) \longrightarrow \mathbb{C}.$$

Taking dimensions,

$$(*) \qquad d - \dim H^0(\mathcal{O}_X(\sum P_i)) + 1 = \dim H^0(\Omega^1_{X/\mathbb{C}}) - \dim H^0(\Omega_{X/\mathbb{C}}(-\sum P_i)).$$

Now it turns out that if $\sum_{i=1}^d P_i$ is a large enough positive divisor, $H^1(\mathcal{O}_X(\sum P_i)) = (0)$ and $H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i)) = (0)$ and this equation reads:

$$d - \chi(\mathcal{O}_X(\sum P_i)) + 1 = \dim H^0(\Omega^1_{X/\mathbb{C}}),$$

and since by Part (1) of Theorem 1.1, $\chi(\mathcal{O}_X(\sum P_i)) = d - g + 1$, it follows that $g = \dim H^0(\Omega^1_{X/\mathbb{C}})$. Putting this back in $(*)$, and using Part (1) of Theorem 1.1 again we get

$$\begin{aligned} g - \dim H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i)) &= d + 1 - \chi(\mathcal{O}_X(\sum P_i)) - \dim H^1(\mathcal{O}_X(\sum P_i)) \\ &= g - \dim H^1(\mathcal{O}_X(\sum P_i)) \end{aligned}$$

hence Part (2) of Theorem 1.1.

A more careful study of the above residue pairing leads quite directly to a *proof* of Assertion (2) of Theorem 1.1 when $k = \mathbb{C}$. Let us first generalize the residue pairing: if D_1 and D_2 are any two divisors on X such that $D_2 - D_1$ is positive (D_1, D_2 themselves arbitrary), then we get a pairing:

$$\left(\bigoplus_x \mathcal{O}_X(D_2)_x / \mathcal{O}_X(D_1)_x\right) \times H^0(\Omega^1_{X/\mathbb{C}}(-D_1)) \longrightarrow \mathbb{C}$$

as follows: given $(\overline{f}_x)$ representing a member of the left hand side ($f_x \in \mathcal{O}_X(D_2)_x$) and $\omega \in H^0(\Omega^1_{X/\mathbb{C}}(-D_1))$, pair these to $\sum_x \operatorname{res}_x(f_x \cdot \omega)$. Here $f_x \cdot \omega$ may have a pole of order > 1 at x, but res_x still makes good sense: expand

$$f_x\omega = \left(\sum_{n=-N}^{+\infty} c_n t^n\right) dt$$

where t has a simple zero at x, and set $\operatorname{res}_x = c_{-1}$. Since

$$c_{-1} = \frac{1}{2\pi i}\oint f_x\omega$$

(taken around a small loop around x), c_{-1} is independent of the choice of t. Note that if $f'_x \in f_x + \mathcal{O}_X(D_1)_x$, then $f'_x \cdot \omega - f_x \cdot \omega \in \Omega^1_x$, hence $\operatorname{res}_x(f'_x\omega) = \operatorname{res}_x(f_x\omega)$. If $D_2 = \sum P_i$, $D_1 = 0$, we get the special case considered already. By the fact that the sum of the residues of any $\omega \in \Omega^1_{\mathbb{R}(X)/\mathbb{C}}$ is 0, the pairing factors as follows:

$$\text{(residue pairing)} \qquad \frac{\bigoplus_x \mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x}{\text{Image } H^0(\mathcal{O}_X(D_2))} \times \frac{H^0(\Omega^1_{X/\mathbb{C}}(-D_1))}{H^0(\Omega^1_{X/\mathbb{C}}(-D_2))} \longrightarrow \mathbb{C}.$$

It is trivial that this is non-degenerate on the right: i.e., if $\omega \in H^0(\Omega^1_{X/\mathbb{C}}(-D_1))\backslash H^0(\Omega^1_{X/\mathbb{C}}(-D_2))$, then for some $(\overline{f}_x)$, $\operatorname{res}_x(f_x\omega) \neq 0$. But in fact:

THEOREM 1.3 (Riemann-Roch theorem (continued)). (2)-strong form: *For every D_1, D_2 with $D_2 - D_1$ positive, the residue pairing is non-degenerate on both sides.*

PROOF OF THEOREM 1.3. First, note that the left hand side can be interpreted via H^1's: namely the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow \mathcal{O}_X(D_2) \longrightarrow \bigoplus_x \mathcal{O}_x(D_2)/\mathcal{O}_x(D_1) \longrightarrow 0,$$

where $\mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x$ is the skyscraper sheaf at x with stalk $\mathcal{O}_x(D_2)/\mathcal{O}_x(D_1)$, induces an isomorphism

$$\frac{\bigoplus_x \mathcal{O}_x(D_2)/\mathcal{O}_x(D_1)}{\text{Image } H^0(\mathcal{O}_X(D_2))} \cong \operatorname{Ker}\left[H^1(\mathcal{O}_X(D_1)) \longrightarrow H^1(\mathcal{O}_X(D_2))\right].$$

Now let D_2 increase. Whenever $D_2 < D'_2$ (i.e., $D'_2 - D_2$ a positive divisor), it follows that there are natural maps:

$$\frac{\bigoplus_x \mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x}{\text{Image } H^0(\mathcal{O}_X(D_2))} \underset{\text{injective}}{\hookrightarrow} \frac{\bigoplus_x \mathcal{O}_X(D'_2)_x/\mathcal{O}_X(D_1)_x}{\text{Image } H^0(\mathcal{O}_X(D'_2))}$$

and

$$\frac{H^0(\Omega^1_{X/\mathbb{C}}(-D_1))}{H^0(\Omega^1_{X/\mathbb{C}}(-D_2))} \underset{\text{surjective}}{\twoheadrightarrow} \frac{H^0(\Omega^1_{X/\mathbb{C}}(-D_1))}{H^0(\Omega^1_{X/\mathbb{C}}(-D'_2))}$$

compatible with the pairing. Passing to the limit, we get a pairing:

$$\frac{\bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbb{R}(X)/\mathcal{O}_X(D_1)_x}{\mathbb{R}(X) \text{ (embedded diagonally)}} \times H^0(\Omega^1_{X/\mathbb{C}}(-D_1)) \longrightarrow \mathbb{C}.$$

It follows immediately that if this is non-degenerate on the left, so is the original pairing. Note here that the left hand side can be interpreted as an H^1: namely the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow \underset{\substack{\text{constant}\\ \text{sheaf}}}{\mathbb{R}(X)} \longrightarrow \bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbb{R}(X)/\mathcal{O}_X(D_1)_x \longrightarrow 0,$$

where $\mathbb{R}(X)/\mathcal{O}_X(D_1)_x$ is the skyscraper sheaf at x with stalk $\mathbb{R}(X)/\mathcal{O}_X(D_1)_x$, induces an isomorphism:

$$\frac{\bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbb{R}(X)/\mathcal{O}_X(D_1)_x}{\mathbb{R}(X)} \cong H^1(\mathcal{O}_X(D_1)).$$

Thus we are now trying to show that we have via residue a perfect pairing:

$$H^1(\mathcal{O}_X(D_1)) \times H^0(\Omega^1_{X/\mathbb{C}}(-D_1)) \longrightarrow \mathbb{C}.$$

This pairing is known as "Serre duality". To continue, suppose

$$l\colon \bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbb{R}(X)/\mathcal{O}_X(D_1)_x \longrightarrow \mathbb{C}$$

is *any* linear function. Then $l = \sum l_x$, where

$$l_x\colon \mathbb{R}(X)/\mathcal{O}_X(D_1)_x \longrightarrow \mathbb{C}$$

is a linear function. Now if t_x has a simple zero at x, and n_x = order of x in the divisor D_1, then let

$$c_\nu = l_x(t_x^{-\nu}), \quad \text{all } \nu \in \mathbb{Z}.$$

Note that $c_\nu = 0$ if $\nu \leq -n_x$. Then we can write l_x formally:

$$l_x(f) = \operatorname{res}_x(f\cdot\omega_x)$$

where

$$\omega_x = \sum_{\nu=-n_x+1}^{+\infty} c_\nu t_x^\nu \cdot \frac{dt_x}{t_x}$$

is a *formal* differential at x; in fact

$$\omega_x \in \Omega^1_X(-D_1)_x.$$

This suggests defining, for the purposes of the proof only, *pseudo-section* of $\Omega^1_X(-D_1)$ to be a collection $(\omega_x)_{x\in X,\ \text{closed}}$, where $\omega_x \in \Omega^1_X(-D_1)_x$ are formal differentials and where

$$\sum_{\substack{x\in X\\ \text{closed}}} \operatorname{res}_x(f\cdot\omega_x) = 0, \quad \text{all } f \in \mathbb{R}(X).$$

If we let $\widetilde{H}^0(\Omega^1_X(-D_1))$ be the vector space of such pseudo-sections, then we see that

$$\frac{\bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbb{R}(X)/\mathcal{O}_X(D_1)_x}{\mathbb{R}(X)} \times \widetilde{H}^0(\Omega^1_X(-D_1)) \longrightarrow \mathbb{C}$$

is indeed a perfect pairing, and we must merely check that all pseudo-sections are true sections to establish the assertion. Now let D_1 tend to $-\infty$ as a divisor. If $D_1' < D_1$, we get a diagram:

$$\begin{array}{ccc} H^0(\Omega^1_X(-D_1')) & \subset & \widetilde{H}^0(\Omega^1_X(-D_1')) \\ \cup & & \cup \\ H^0(\Omega^1_X(-D_1)) & \subset & \widetilde{H}^0(\Omega^1_X(-D_1)) \end{array}$$

and clearly:

$$H^0(\Omega^1_X(-D_1')) \cap \widetilde{H}^0(\Omega^1_X(-D_1)) = H^0(\Omega^1_X(-D_1)).$$

Passing to the limit, we get:

$$\Omega^1_{\mathbb{R}(X)/\mathbb{C}} \subset \widetilde{\Omega}^1_{\mathbb{R}(X)/\mathbb{C}}$$

where

$$\widetilde{\Omega}^1_{\mathbb{R}(X)/\mathbb{C}} = \left\{ \begin{array}{l} \text{set of meromorphic pseudo-differentials, i.e.,} \\ \text{collections of } \omega_{x,X} \in \Omega_{x,X} \otimes_{\mathcal{O}_x} \mathbb{R}(X) \text{ such that} \\ \sum_x \operatorname{res}_x(f \cdot \omega_x) = 0, \text{ all } f \in \mathbb{R}(X) \end{array} \right\}.$$

It suffices to prove that $\Omega^1 = \widetilde{\Omega}^1$. But it turns out that if D_1' is sufficiently negative, then $-D_1'$ is very positive and

$$H^1(\Omega^1_X(-D_1')) = H^0(\mathcal{O}_X(D_1')) = (0).$$

Thus

$$\begin{aligned} \dim H^0(\Omega^1_X(-D_1')) &= \deg \Omega^1_X - \deg D_1' - g + 1 \\ \dim \widetilde{H}^0(\Omega^1_X(-D_1')) &= \dim H^1(\mathcal{O}_X(D_1')) \\ &= -\deg D_1' + g - 1 \end{aligned}$$

hence

$$\dim_{\mathbb{C}} \left(\widetilde{H}^0(\Omega^1_X(-D_1'))/H^0(\Omega^1_X(-D_1')) \right) = 2g - 2 - \deg \Omega^1_X \quad \text{(independent of } D_1'\text{)}.$$

Thus $\dim_{\mathbb{C}} \left(\widetilde{\Omega}^1_{\mathbb{R}(X)/\mathbb{C}} / \Omega^1_{\mathbb{R}(X)/\mathbb{C}} \right) < +\infty$. But $\widetilde{\Omega}^1_{\mathbb{R}(X)/\mathbb{C}}$ is an $\mathbb{R}(X)$-vector space! So if $\widetilde{\Omega}^1 \supsetneqq \Omega^1$, then $\dim_{\mathbb{C}} \widetilde{\Omega}^1/\Omega^1 = +\infty$. Therefore $\widetilde{\Omega}^1 = \Omega^1$ as required. □

All this uses the assumption $k = \mathbb{C}$ only in two ways: first in order to know that if we define the residue of a formal meromorphic differential via:

$$\operatorname{res} \left(\sum_{n=-N}^{+\infty} c_n t^n dt \right) = c_{-1},$$

then the residue remains unchanged if we take a new local coordinate $t' = a_1 t + a_2 t^2 + \cdots$, ($a_1 \neq 0$). Secondly, if $\omega \in \Omega^1_{\mathbb{R}(X)/k}$, then we need the deep fact:

$$\sum_{\substack{x \in X \\ \text{closed}}} \operatorname{res}_x \omega = 0.$$

Given these facts, our proof works over any algebraically closed ground field k (and with a little more work, over any k at all). For a long time, only rather roundabout proofs of these facts were known in characteristic p (when characteristic $= 0$, there are simple algebraic proofs or one can reduce to the case $k = \mathbb{C}$). Around the time this manuscript was being written Tate [**100**] discovered a very elementary and beautiful proof of these facts: we reproduce his proofs in an appendix to this section. Note that his "dualizing sheaf" is exactly the same as our "pseudo-differentials".

We finish the section with a few applications.

Corollary 1.4. *If X is a geometrically irreducible curve, proper and smooth over a field k, then:*

a) *For all $f \in \mathbb{R}(X)$, $\deg(f) = 0$; hence if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, then $\deg D_1 = \deg D_2$. This means we can assign a degree to an invertible sheaf $\mathcal{L}$ by requiring:*

$$\deg \mathcal{L} = \deg D \quad \textit{if } \mathcal{L} \cong \mathcal{O}_X(D).$$

b) *If $\deg D < 0$, then $H^0(\mathcal{O}_X(D)) = (0)$.*

PROOF. Multiplication by f is an isomorphism $\mathcal{O}_X \xrightarrow{\approx} \mathcal{O}_X((f))$, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X((f)))$, so by Riemann-Roch (Theorem 1.1), $\deg(f) = 0$. Secondly, if $f \in H^0(\mathcal{O}_X(D))$, $f \neq 0$, then $D + (f) \geq 0$, so

$$\deg D = \deg(D + (f)) \geq 0.$$

□

COROLLARY 1.5. *If X is a geometrically irreducible curve, proper and smooth over a field k of genus g ($g \underset{def}{=} \dim H^1(\mathcal{O}_X)$), then:*

a) $\dim_k H^0(\Omega^1_{X/k}) = g$, $\dim_k H^1(\Omega^1_{X/k}) = 1$,
b) *If K is a divisor such that $\Omega^1_{X/k} \cong \mathcal{O}_X(K)$ — a so-called* canonical divisor *— then*

$$\deg K = 2g - 2.$$

PROOF. Apply Riemann-Roch (Theorem 1.1) with $D = K$. □

COROLLARY 1.6. *If X is a geometrically irreducible curve, proper and smooth over a field k of genus g, then $\deg D > 2g - 2$ implies:*

a) $H^1(\mathcal{O}_X(D)) = (0)$
b) $\dim H^0(\mathcal{O}_X(D)) = \deg D - g + 1$.

PROOF. If $\Omega^1_{X/k} \cong \mathcal{O}_X(K)$, then $\deg(K - D) < 0$, hence $H^0(\Omega^1_{X/k}(-D)) = (0)$. Thus by Riemann-Roch (Theorem 1.1), $H^1(\mathcal{O}_X(D)) = (0)$ and $\dim H^0(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D)) = \deg D - g + 1$. □

PROPOSITION 1.7. $\boxed{\textit{Added}}$ *Let X be a geometrically irreducible curve proper and smooth over a field k. An invertible sheaf $\mathcal{L}$ on X is ample if and only if $\deg \mathcal{L} > 0$.*

PROOF. We use Serre's cohomological criterion (Theorem VII.8.2). Note that the cohomology groups H^p for $p > 1$ of coherent $\mathcal{O}_X$-modules vanish since $\dim X = 1$ (cf. Proposition VII.4.2). Thus we need to show that

$$\text{for any coherent } \mathcal{O}_X\text{-module } \mathcal{F} \text{ one has} \quad H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0, \qquad n \gg 0$$

if and only if $\deg \mathcal{L} > 0$.

Let $r = \operatorname{rk} \mathcal{F}$, i.e., the dimension of the $\mathbb{R}(X)$-vector space $\mathcal{F}_\eta$ ($\eta =$ generic point). Then we claim that $\mathcal{F}$ has a filtration

$$(0) \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = \mathcal{F}$$

by coherent $\mathcal{O}_X$-submodules such that

$$\begin{aligned} \mathcal{F}_0 \quad &= \text{torsion } \mathcal{O}_X\text{-module} \\ \mathcal{F}_j/\mathcal{F}_{j-1} \quad &= \text{invertible } \mathcal{O}_X\text{-module for } j = 1, \ldots, r. \end{aligned}$$

Indeed, $\mathcal{O}_{x,X}$ for closed points x are discrete valuation rings since X is a regular curve. Thus for the submodule $(\mathcal{F}_x)_{\rm tor}$ of torsion elements in the finitely generated $\mathcal{O}_{x,X}$-module $\mathcal{F}_x$, the quotient $\mathcal{F}_x/(\mathcal{F}_x)_{\rm tor}$ is a free $\mathcal{O}_{x,X}$-module. $\mathcal{F}_0$ is the $\mathcal{O}_X$-submodule of $\mathcal{F}$ with $(\mathcal{F}_0)_x = (\mathcal{F}_x)_{\rm tor}$ for all closed points x and $\mathcal{F}/\mathcal{F}_0$ is locally free of rank r. X is projective by Proposition V.5.11. Thus if we choose a very ample sheaf on X, then a sufficient twist $\mathcal{F}/\mathcal{F}_0$ by it has a section. Untwising the result, we get an invertible subsheaf $\mathcal{M} \subset \mathcal{F}/\mathcal{F}_0$. Let $\mathcal{F}_1 \subset \mathcal{F}$ be the $\mathcal{O}_X$-submodule containing $\mathcal{F}_0$ such that $\mathcal{F}_1/\mathcal{F}_0 \supset \mathcal{M}$ and that $(\mathcal{F}_1/\mathcal{F}_0)/\mathcal{M}$ is the $\mathcal{O}_X$-submodule of torsions of $(\mathcal{F}/\mathcal{F}_0)/\mathcal{M}$. Obviously, $\mathcal{F}_1/\mathcal{F}_0$ is an invertible submodule of $\mathcal{F}/\mathcal{F}_0$ with $\mathcal{F}/\mathcal{F}_1$ locally free of rank $r - 1$. The above claim thus follows by induction.

Since $H^1(X, \mathcal{F}_0 \otimes \mathcal{L}^n) = 0$ for any n again by Proposition VII.4.2, the proposition follows if we show that

$$\text{for } \mathcal{F} \text{ invertible one has} \quad H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0, \qquad n \gg 0$$

if and only if $\deg \mathcal{L} > 0$. But this is immediate, since the cohomology group vanishes if $\deg(\mathcal{F} \otimes \mathcal{L}^n) = \deg \mathcal{F} + n \deg \mathcal{L} > 2g - 2$ by Corollary 1.6, (a). □

Remark. $\boxed{\text{Added}}$ Using the filtration appearing in the proof above, we can generalize Theorem 1.1 (Riemann-Roch), (1) for a locally free sheaf $\mathcal{E}$ of rank r as:

$$\dim_k H^0(X, \mathcal{E}) - \dim_k H^1(X, \mathcal{E}) = \deg(\bigwedge^r \mathcal{E}) + r(1 - g).$$

Remark. Let X be a curve proper and smooth over an algebraically closed field k, and $\mathcal{L}$ an invertible sheaf on X. We can show:

- If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is generated by global sections.
- If $\deg \mathcal{L} \geq 2g + 1$, then $\mathcal{L}$ is very ample (over k).

Corollary 1.8. *If X is a geometrically irreducible curve smooth and proper over a field k of genus 0, and X has at least one k-rational point x (e.g., if k is algebraically closed; or k a finite field, cf. Proposition* IV.3.5*), then $X \cong \mathbb{P}^1_k$.*

Proof. Apply Riemann-Roch (Theorem 1.1) to $\mathcal{O}_X(x)$. It follows that

$$\dim_k H^0(\mathcal{O}_X(x)) \geq 2,$$

hence $\exists f \in H^0(\mathcal{O}_X(x))$ which is not a constant. This f defines a morphism

$$f' \colon X \longrightarrow \mathbb{P}^1_k$$

such that $(f')^{-1}(\infty) = \{x\}$, with reduced structure. Then f' must be finite; and thus $\mathcal{O}_{x,X}$ is a finite $\mathcal{O}_{\infty,\mathbb{P}^1}$-module such that

$$\begin{array}{ccc} \mathcal{O}_{\infty,\mathbb{P}^1}/\mathfrak{m}_{\infty,\mathbb{P}^1} & \longrightarrow & \mathcal{O}_{x,X}/(\mathfrak{m}_{\infty,\mathbb{P}^1} \cdot \mathcal{O}_{x,X}) \\ \| & & \| \\ k & & \Bbbk(x) \\ & & \| \\ & & k \end{array}$$

is an isomorphism. Thus $\mathcal{O}_{x,X} \cong \mathcal{O}_{\infty,\mathbb{P}^1}$, hence f' is birational, hence by Zariski's Main Theorem (§V.6), f' is an isomorphism. □

Corollary 1.9. *If X is a geometrically irreducible curve smooth and proper over a field k of genus 1, then $\Omega^1_{X/k} \cong \mathcal{O}_X$. Moreover the map*

$$X(k) = \left\{ \begin{array}{l} \textit{set of } k\textit{-rational} \\ \textit{points } x \in X \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \textit{invertible sheaves } \mathcal{L} \\ \textit{of degree } 1 \textit{ on } X \end{array} \right\}$$

$$x \longmapsto \mathcal{O}_X(x)$$

is an isomorphism, hence if $x_0 \in X(k)$ is a base point, $X(k)$ is a group via $x + y = z$ if and only if

$$\mathcal{O}_X(x) \otimes \mathcal{O}_X(y) \cong \mathcal{O}_X(z) \otimes \mathcal{O}_X(x_0).$$

PROOF. Since $H^0(\Omega^1_{X/k}) \neq (0)$, $\Omega^1_{X/k} \cong \mathcal{O}_X(D)$ for some non-negative divisor D. But then

$$\deg D = 2g - 2 = 0,$$

so $D = 0$, i.e., $\Omega^1_{X/k} \cong \mathcal{O}_X$. Next, if $\mathcal{L}$ is an invertible sheaf of degree 1, then by Corollary 1.6, $H^1(\mathcal{L}) = (0)$, hence by Riemann-Roch (Theorem 1.1), $\dim_k H^0(\mathcal{L}) = 1$. This means there is a unique non-negative divisor D such that $\mathcal{L} \cong \mathcal{O}_X(D)$. Since $\deg D = 1$, $D = x$ where $x \in X(k)$. Finally, the invertible sheaves of degree 0 form a group under $\otimes$, hence so do the sheaves of degree 1 if we multiply them by:

$$(\mathcal{L}, \mathcal{M}) \longmapsto \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{O}_X(-x_0).$$

This proves that $X(k)$ is a group. $\square$

In fact, it can be shown that X is a group scheme (in fact an abelian variety) over k (cf. §VI.1) with origin x_0: especially there is a morphism

$$\mu\colon X \times_{\operatorname{Spec} k} X \longrightarrow X$$

inducing the above addition on $X(k)$: see Mumford [**74**, Chapter I, p. 36], and compare Part I [**76**, §7D].

COROLLARY 1.10. *If $\Theta_X = \mathcal{H}om(\Omega^1_X, \mathcal{O}_X) \cong \mathcal{O}_X(-K)$ is the tangent sheaf to X, then its cohomology is:*

	$g = 0$	$g = 1$	$g > 1$
$\dim H^0(\Theta_X)$	3	1	0
$\dim H^1(\Theta_X)$	0	1	$3g - 3$

In fact, the three sections of Θ when $X = \mathbb{P}^1_k$ come from the infinitesimal section of the 3-dimensional group scheme $\mathrm{PGL}_{2,k}$ acting on $\mathbb{P}^1_k$; the one section of Θ when $g = 1$ comes from the infinitesimal action of X on itself, and the absence of sections when $g > 1$ is reflected in the fact that the group of automorphisms of such curves is finite. Thus three way division of curves, according as $g = 0$, $g = 1$, $g > 1$ is the algebraic side of the analytic division of Riemann surfaces according as whether they are a) the Gauss sphere, b) the plane modulo a discrete translation group or c) the unit disc modulo a freely acting Fuchsian group; and of the differential geometric division of compact surfaces according as they admit a metric with constant curvature K, with $K > 0$, $K = 0$, or $K < 0$.

For further study of curves, an excellent reference is Serre [**91**, Chapters 2–5]. Classical references on curves are: Hensel-Landsberg [**53**], Coolidge [**31**], Severi [**95**] and Weyl [**106**][1].

What happens in higher dimensions?[2]

The necessisity of the close analysis of all higher cohomolgy groups becomes much more apparent as the dimension increases. Part (1) of the curve Riemann-Roch theorem (Theorem 1.1) was generalized by Hirzebruch [**56**], and by Grothendieck (cf. [**24**])[3] to a formula for computing $\chi(\mathcal{O}_X(D))$ — for any smooth, projective variety X and divisor D — by a "universal polynomial" in terms of D and the Chern classes of X; this polynomial can be taken in a suitable cohomology ring of X, or else in the so-called Chow ring — a ring formed by cycles $\sum n_i Z_i$ (Z_i subvarieties of X) modulo "rational equivalence" with product given by intersection. For this theory, see Chevalley Seminar [**29**] and Samuel [**84**].

[1](Added in publication) See also Iwasawa [**57**].

[2](Added in publication) There have been considerable developments on Kodaira dimension, Minimal model program, etc.

[3](Added in publication) See also SGA6 [**9**] for further developments.

Part (2) of the curve Riemann-Roch theorem (Theorem 1.1) was generalized by Serre and Grothendieck (see Serre [**87**], Altman-Kleiman [**13**] and Hartshorne [**49**]) to show, if X is a smooth complete varietie of dimension n, that

a) a canonical isomorphism $\epsilon\colon H^n(X,\Omega^n_{X/k}) \xrightarrow{\approx} k$, and
b) that — plus cup product induces a non-degenerate pairing

$$H^i(X,\mathcal{O}_X(D)) \times H^{n-i}(X,\Omega^n_X(-D)) \to k$$

for all divisors D and all i.

Together however, these results do not give any formula in dim ≥ 2 involving H^0's alone. Thus geometric applications of Riemann-Roch requires a good deal more ingenuity (cf. for instance Shafarevitch et al. [**96**]).

Three striking examples of cases where the higher cohomology groups can be dealt with so that a geometric conclusion is deduced from a cohomological hypothesis are:

THEOREM 1.11 (Criterion of Nakai-Moishezon). *Let k be a field, X a scheme proper over k, and $\mathcal{L}$ an invertible sheaf on X. Then*

$$\left[\begin{array}{l} \mathcal{L} \textit{ is ample, i.e., } n \geq 1 \textit{ and} \\ \textit{a closed immersion} \\ \phi\colon X \to \mathbb{P}^N \textit{ such that} \\ \phi^*(\mathcal{O}_{\mathbb{P}^N}(1)) \cong \mathcal{L}^n \end{array}\right] \iff \left[\begin{array}{l} \forall \textit{ reduced and irreducible} \\ \textit{subvarieities } Y \subset X \textit{ of} \\ \textit{positive dimension,} \\ \chi(\mathcal{L}^n \otimes \mathcal{O}_Y) \to \infty \textit{ as } n \to \infty \end{array}\right].$$

(This is another form of Theorem VII.12.4. See also Kleiman [**61**]).

THEOREM 1.12 (Criterion of Kodaira). *Let X be a compact complex analytic manifold and $\mathcal{L}$ an invertible analytic sheaf on X. Then*

$$\left[\begin{array}{l} X \textit{ is a projective} \\ \textit{variety and } \mathcal{L} \textit{ is} \\ \textit{an ample sheaf on it} \end{array}\right] \iff \left[\begin{array}{l} \mathcal{L} \textit{ can be defined by transition functions } \{f_{\alpha\beta}\} \\ \textit{for a covering } \{U_\alpha\} \textit{ of } X, \textit{ where} \\ |f_{\alpha\beta}|^2 = g_\alpha/g_\beta,\ g_\alpha \textit{ positive real } C^\infty \textit{ on } U_\alpha \textit{ and} \\ (\partial^2 \log g_\alpha/\partial z_i \partial \overline{z}_j)(P) \\ \ \textit{positive definite Hermitian form at all } P \in U_\alpha \end{array}\right].$$

(For a proof, cf. Gunning-Rossi [**48**].)

THEOREM 1.13 (Vanishing theorem of Kodaira-Akizuki-Nakano). *Let X be an n-dimensional complex projective variety, $\mathcal{L}$ an ample invertible sheaf on X. Then*

$$H^p(X,\Omega^q_X \otimes \mathcal{L}) = (0), \quad \textit{if } p+q > n.$$

(For a proof, cf. Akizuki-Nakano [**11**].)

Appendix: Residues of differentials on curves by John Tate

Added

We reproduce here, in our notation, the very elementary and beautiful proof of Tate [**100**].

Here is the key to Tate's proof: Let V be a vector space over a field k. A k-linear endomorphism $\theta \in \mathrm{End}_k(V)$ is said to be *finite potent* if $\theta^n V$ is finite dimensional for a positive integer n. For such a θ, the *trace*

$$\mathrm{Tr}_V(\theta) \in k$$

is defined and has the following properties:

(T1) If $\dim V < \infty$, then $\mathrm{Tr}_V(\theta)$ is the ordinary trace.

(T2) If $W \subset V$ is a subspace with $\theta W \subset W$, then

$$\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta) + \mathrm{Tr}_{V/W}(\theta).$$

(T3) $\mathrm{Tr}_V(\theta) = 0$ if θ is nilpotent.

(T4) Suppose $F \subset \mathrm{End}_k(V)$ is a *finite potent* k-subspace, i.e., there exists a positive integer n such that $\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n V$ is finite dimensional for any $\theta_1, \ldots, \theta_n \in F$. Then

$$\mathrm{Tr}_V : F \to k \quad \text{is } k\text{-linear.}$$

(It does not seem to be known if

$$\mathrm{Tr}_V(\theta + \theta') = \mathrm{Tr}_V(\theta) + \mathrm{Tr}_V(\theta')$$

holds in general under the condition θ, θ' and $\theta + \theta'$ are finite potent.)

(T5) Let $\phi\colon V' \to V$ and $\psi\colon V \to V'$ be k-linear maps with $\phi \circ \psi\colon V \to V$ finite potent. Then $\psi \circ \phi\colon V' \to V'$ is finite potent and

$$\mathrm{Tr}_V(\phi \circ \psi) = \mathrm{Tr}_{V'}(\psi \circ \phi).$$

(T1), (T2) and (T3) characterize $\mathrm{Tr}_V(\theta)$: Indeed, by assumption, $W = \theta^n V$ is finite dimensional for some n. Then $\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta)$.

For the proof of (T4), we may assume F to be finite dimensional and compute the trace on the finite dimensional subspace $W = F^n V$.

As for (T5), ϕ and ψ induce mutually inverse isomorphisms between the subspaces $W = (\phi \circ \psi)^n V$ and $W' = (\psi \circ \phi)^n V'$ for $n \gg 0$ under which $(\psi \circ \phi)|_{W'}$ and $(\phi \circ \psi)|_W$ correspond.

Definition 1. Let A and B be k-subspaces of V.

- A is said to be "not much bigger than" B (denoted $A \prec B$) if $\dim(A + B)/B < \infty$.
- A is said to be "about the same size as" B (denoted $A \sim B$) if $A \prec B$ and $A \succ B$.

Proposition 2. *Let A be a k-subspace of V.*

(1) $E = \{\theta \in \mathrm{End}_k(V) \mid \theta A \prec A\}$ *is a k-subalgebra of* $\mathrm{End}_k(V)$.

(2) *The subspaces*

$$E_1 = \{\theta \in \mathrm{End}_k(V) \mid \theta V \prec A\}$$
$$E_2 = \{\theta \in \mathrm{End}_k(V) \mid \theta A \prec (0)\}$$
$$E_0 = E_1 \cap E_2 = \{\theta \in \mathrm{End}_k(V) \mid \theta V \prec A, \theta A \prec (0)\}$$

are two-sided ideals of E with $E = E_1 + E_2$, and E_0 is finite potent so that there is a k-linear map $\mathrm{Tr}_V\colon E_0 \to k$. Moreover, E, E_1, E_2 and E_0 depend only on the $\sim$-equivalence class of A.

(3) *Let $\phi, \psi \in \mathrm{End}_k(V)$. If either* (i) *$\phi \in E_0$ and $\psi \in E$, or* (ii) *$\phi \in E_1$ and $\psi \in E_2$, then*

$$[\phi, \psi] := \phi \circ \psi - \psi \circ \phi \in E_0$$

with $\mathrm{Tr}_V([\phi, \psi]) = 0$.

Proof. (1) is obvious. As for (2), express V as a direct sum $V = A \oplus A'$, and denote by $\varepsilon\colon V \twoheadrightarrow A$, $\varepsilon'\colon V \twoheadrightarrow A'$ the projections. Then $\mathrm{id}_V = \varepsilon + \varepsilon'$ with $\varepsilon \in E_1$ and $\varepsilon \in E_2$, so that $\theta = \theta\varepsilon + \theta\varepsilon'$ for all $\theta \in E$. Obviously, $\theta_1 \circ \theta_2 V$ is finite dimensional for any $\theta_1, \theta_2 \in E_0$. (3) follows easily from (T5). $\square$

Theorem 3 (Abstract residue). *Let K be a commutative k-algebra (with 1), V a k-vector space which is also a K-module, and $A \subset V$ a k-subspace such that $fA \prec A$ for all $f \in K$ (hence K acts on V through $K \to E \subset \mathrm{End}_k(V)$ with the image in E of each $f \in K$ denoted by the same letter f). Then there exists a unique k-linear map*

$$\mathrm{Res}_A^V \colon \Omega^1_{K/k} \longrightarrow k$$

such that for any pair $f, g \in K$, we have

$$\mathrm{Res}_A^V(f\,dg) = \mathrm{Tr}_V([f_1, g_1])$$

for $f_1, g_1 \in E$ such that

i) $f \equiv f_1 \pmod{E_2}$, $g \equiv g_1 \pmod{E_2}$

ii) *either $f_1 \in E_1$ or $g_1 \in E_1$.*

The k-linear map is called the residue *and satisfies the following properties:*

(R1) $\mathrm{Res}_A^V = \mathrm{Res}_A^{V'}$ *if $V \supset V' \supset A$ and $KV' = V'$. Moreover, $\mathrm{Res}_A^V = \mathrm{Res}_{A'}^V$ if $A \sim A'$.*

(R2) *(Continuity in f and g) We have*

$$fA + fgA + fg^2A \subset A \Longrightarrow \mathrm{Res}_A^V(f\,dg) = 0.$$

Thus $\mathrm{Res}_A^V(f\,dg) = 0$ if $fA \subset A$ and $gA \subset A$. In particular, $\mathrm{Res}_A^V = 0$ if $A \subset V$ is a K-submodule.

(R3) *For $g \in K$ and an integer n, we have*

$$\mathrm{Res}_A^V(g^n dg) = 0 \quad \textit{if} \quad \begin{cases} n \geq 0 \\ \textit{or} \\ n \leq -2 \textit{ and } g \textit{ invertible in } K. \end{cases}$$

In particular, $\mathrm{Res}_A^V(dg) = 0$.

(R4) *If $g \in K$ is invertible and $h \in K$ with $hA \subset A$, then*

$$\mathrm{Res}_A^V(hg^{-1}dg) = \mathrm{Tr}_{A/(A \cap gA)}(h) - \mathrm{Tr}_{gA/(A \cap gA)}(h).$$

In particular, if $g \in K$ is invertible and $gA \subset A$, then

$$\mathrm{Res}_A^V(g^{-1}dg) = \dim_k(A/gA).$$

(R5) *Suppose $B \subset V$ is another subspace such that $fB \prec B$ for all $f \in K$. Then $f(A+B) \prec A+B$ and $f(A \cap B) \prec A \cap B$ hold for all $f \in K$, and*

$$\mathrm{Res}_A^V + \mathrm{Res}_B^V = \mathrm{Res}_{A+B}^V + \mathrm{Res}_{A \cap B}^V.$$

(R6) *Suppose K' is a commutative K-algebra that is a free K-module of finite rank r. For a K-basis $\{v_1, \dots, v_r\}$ of K', let*

$$V' = K' \otimes_k V \supset A' = \sum_{i=1}^r v_i \otimes A.$$

Then $f'A' \prec A'$ holds for any $f' \in K'$, and the $\sim$-equivalence class of A' depends only on that of A and not on the choice of $\{v_1, \dots, v_r\}$. Moreover,

$$\mathrm{Res}_{A'}^{V'}(f'dg) = \mathrm{Res}_A^V((\mathrm{Tr}_{K'/K} f')dg), \qquad \forall f' \in K',\ \forall g \in K.$$

Proof of the existence of residue. By assumption, we have $f, g \in E = E_1 + E_2$. Thus f_1 and g_1 satisfying (i) and (ii) can be chosen. Then $[f_1, g_1] \in E_1$ by (ii), and $[f_1, g_1] \equiv [f, g] \pmod{E_2}$ by (i) with $[f, g] = 0$ by the commutativity of K. Thus $[f_1, g_1] \in E_1 \cap E_2 = E_0$ and $\mathrm{Tr}_V([f_1, g_1])$ is defined. By Proposition 2, (3), it is unaltered if f_1 or g_1 is changed by an element

of E_2 as long as the other is in E_1. Moreover by (T4), $\mathrm{Tr}_V([f_1, g_1])$ is a k-bilinear function of f and g. Thus there is a k-linear map

$$\beta\colon K \otimes_k K \longrightarrow k \qquad \text{such that } \beta(f \otimes g) = \mathrm{Tr}_V([f_1, g_1]).$$

We now show that

$$\beta(f \otimes gh) = \beta(fg \otimes h) + \beta(fh \otimes g), \qquad \forall f, g, h \in K,$$

hence r factors through the canonical surjective homomorphism

$$c\colon K \otimes_k K \longrightarrow \Omega^1_{K/k}, \qquad c(f \otimes g) = f dg.$$

Indeed, for $f, g, h \in K$, choose suitable $f_1, g_1, h_1 \in E_1$ and let $(fg)_1 = f_1 g_1$, $(fh)_1 = f_1 h_1$ and $(gh)_1 = g_1 h_1$. Then by the commutativity of K, we obviously have

$$[f_1, g_1 h_1] = [f_1 g_1, h_1] + [f_1 h_1, g_1].$$

□

We use the following lemma in proving the rest of Theorem 3:

LEMMA 4. *For $f, g \in K$, define subspaces $B, C \subset V$ by*

$$B = A + gA$$
$$C = B \cap f^{-1}(A) \cap (fg)^{-1}(A) = \{v \in B \mid fv \in A \text{ and } fgv \in A\}.$$

Then B/C is finite dimensional and

$$\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_{B/C}([\varepsilon f, g]),$$

where $\varepsilon\colon V \twoheadrightarrow A$ is a k-linear projection.

PROOF. B/C is finite dimensional, since $B/\{v \in B \mid fv \in A\}$ and $B/\{v \in B \mid fgv \in A\}$ are mapped injectively into the finite dimensional space $(A + fA + fgA + fg^2A)/A$. Moreover, $\varepsilon f \in E_1$ and $\varepsilon f \equiv f \pmod{E_2}$, hence $\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_V([\varepsilon f, g])$. On the other hand, $[\varepsilon f, g] = \varepsilon f g - g \varepsilon f$ maps V into B, and C into 0, since $fg = gf$. Thus the assertion follows by (T2), since $\mathrm{Tr}_V = \mathrm{Tr}_{V/B} + \mathrm{Tr}_{B/C} + \mathrm{Tr}_C$. □

PROOF OF THEOREM 3 CONTINUED. (R1) follows easily from Lemma 4, since $B, C \subset V'$.

As for (R2), we have $B = C$ in Lemma 4.

To prove (R3), choose $g_1 \in E_1$ such that $g_1 \equiv g \pmod{E_2}$. If $n \geq 0$, we have $\mathrm{Res}^V_A(g^n dg) = \mathrm{Tr}_V([g_1^n, g_1]) = 0$ since g_1^n and g_1 commute. If g is invertible, then $g^{-2-n} dg = -(g^{-1})^n d(g^{-1})$, whose residue is 0 if $n \geq 0$ by what we have just seen.

For the proof of (R4), let $f = hg^{-1}$ and apply Lemma 4. We have $[\varepsilon f, g] = \varepsilon h - \varepsilon_1 h$, where $\varepsilon_1 = g \varepsilon g^{-1}$ is a projection of V onto gA. Since both A and gA are stable under h, we have

$$\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_{(A+gA)/(A \cap gA)}(\varepsilon h) - \mathrm{Tr}_{(A+gA)/(A\cap gA)}(g \varepsilon g^{-1} h)$$

and we are done by computing the traces through $A + gA \supset A \supset A \cap gA$ and $A + gA \supset gA \supset A \cap gA$, respectively.

To prove (R5), choose projections $\varepsilon_A\colon V \twoheadrightarrow A$, $\varepsilon_B\colon V \twoheadrightarrow B$, $\varepsilon_{A+B}\colon V \twoheadrightarrow A + B$, $\varepsilon_{A\cap B}\colon V \twoheadrightarrow A \cap B$ such that

$$\varepsilon_A + \varepsilon_B = \varepsilon_{A+B} + \varepsilon_{A \cap B}.$$

Then $[\varepsilon_A f, g]$ and $[\varepsilon_{A+B} f, g]$ belong to

$$F = \{\theta \in E \mid \theta V \prec A + B, \theta(A + B) \prec A, \theta A \prec (0)\},$$

which is finite potent, since $\theta_1 \circ \theta_2 \circ \theta_3 V$ is finite dimensional for any $\theta_1, \theta_2, \theta_3 \in F$. Since $\varepsilon_A f \in E_1$, $\varepsilon_A f \equiv f \pmod{E_2}$, $\varepsilon_{A+B} f \in E_1$ and $\varepsilon_{A+B} f \equiv f \pmod{E_2}$, one has

$$\begin{aligned}\operatorname{Res}^V_A(fdg) - \operatorname{Res}^V_{A+B}(fdg) &= \operatorname{Tr}_V([\varepsilon_A f, g]) - \operatorname{Tr}_V([\varepsilon_{A+B} f, g]) \\ &= \operatorname{Tr}_V([(\varepsilon_A - \varepsilon_{A+B})f, g]) \\ &= \operatorname{Tr}_V([(\varepsilon_{A\cap B} - \varepsilon_B)f, g]),\end{aligned}$$

which, by a similar argument, equals $\operatorname{Res}^V_{A\cap B}(fdg) - \operatorname{Res}^V_B(fdg)$.

As for (R6), a k-endomorphism φ of V' can be expressed as an $r \times r$ matrix (φ_{ij}) of endomorphisms of V by the rule

$$\varphi(\sum_j x_j \otimes v_j) = \sum_{ij} x_i \otimes \varphi_{ij} v_j, \qquad \text{for } v_j \in V.$$

If $F \subset \operatorname{End}_k(V)$ is a finite potent subspace, then φ's such that $\varphi_{ij} \in F$ for all i, j form a finite potent subspace $F' \subset \operatorname{End}_k(V')$. We see that $\operatorname{Tr}_{V'}(\varphi) = \sum_i \operatorname{Tr}_V(\varphi_{ii})$ for all $\varphi \in F'$ by decomposing the matrix (φ_{ij}) into the sum of a diagonal matrix, a nilpotent triangular matrix having zeros on and below the diagonal, and another nilpotent triangular matrix having zeros on and above the diagonal. For $f' \in K'$, write $f' x_j = \sum_i x_i f_{ij}$ with $f_{ij} \in K$. Let $\varepsilon \colon V \twoheadrightarrow A$ be a k-linear projection and put $\varepsilon'(\sum_i x_i \otimes v_i) = \sum_i x_i \otimes \varepsilon v_i$. Then $\varepsilon' \colon V' \twoheadrightarrow A'$ is a projection, and

$$[f'\varepsilon', g]_{ij} = [f_{ij}\varepsilon, g].$$

We are done since $\operatorname{Tr}_{K'/K} f = \sum_i f_{ii}$. □

We are now ready to deal with residues of differentials on curves.

Let X be a regular irreducible curve proper over a field k, and denote by X_0 the set of closed points of X. For each $x \in X_0$ let

$$\begin{aligned}A_x &= \widehat{\mathcal{O}}_{x,X} = \mathfrak{m}_{x,X}\text{-adic completion of } \mathcal{O}_{x,X} \\ K_x &= \text{quotient field of } A_x.\end{aligned}$$

Define

$$\operatorname{Res}_x \colon \Omega^1_{\mathbb{R}(X)/k} \longrightarrow k$$

by

$$\operatorname{Res}_x(fdg) = \operatorname{Res}^{K_x}_{A_x}(fdg), \qquad f, g \in \mathbb{R}(X),$$

which makes sense since $\Bbbk(x) = A_x/\mathfrak{m}_{x,X}A_x$ is a finite dimensional k-vector space so that $A_x \sim \mathfrak{m}^n_{x,X}A_x$ for any $n \in \mathbb{Z}$ and that for any non-zero $f \in K_x$ we have $fA_x \prec A_x$ since $fA_x = \mathfrak{m}^n_{x,X}A_x$ for some n.

Theorem 5.

i) *Suppose $x \in X_0$ is k-rational so that $A_x = k[[t]]$ and $K_x = k((t))$. For*

$$f = \sum_{\nu \gg \infty} a_\nu t^\nu, \quad g = \sum_{\mu \gg \infty} b_\mu t^\mu \in K_x,$$

we have

$$\begin{aligned}\operatorname{Res}_x(fdg) &= \textit{coefficient of } t^{-1} \textit{ in } f(t)g'(t) \\ &= \sum_{\nu+\mu=0} \mu a_\nu b_\mu.\end{aligned}$$

ii) *For any subset $S \subset X_0$, let $\mathcal{O}(S) = \bigcap_{x \in S} \mathcal{O}_{x,X} \subset \mathbb{R}(X)$. Then*

$$\sum_{x \in S} \mathrm{Res}_x(\omega) = \mathrm{Res}_{\mathcal{O}(S)}^{\mathbb{R}(X)}(\omega), \qquad \forall \omega \in \Omega^1_{\mathbb{R}(X)/k}.$$

In particular

$$\sum_{x \in X_0} \mathrm{Res}_x(\omega) = 0, \qquad \forall \omega \in \Omega^1_{\mathbb{R}(X)/k}.$$

iii) *Let $\varphi \colon X' \to X$ be a finite surjective morphism of irreducible regular curves proper over k. Then*

$$\sum_{x' \in \varphi^{-1}(x)} \mathrm{Res}_{x'}(f' dg) = \mathrm{Res}_x((\mathrm{Tr}_{\mathbb{R}(X')/\mathbb{R}(X)} f')dg)$$

if $f' \in \mathbb{R}(X')$, $g \in \mathbb{R}(X)$ and $x \in X_0$, while

$$\mathrm{Res}_{x'}(f' dg) = \mathrm{Res}_x((\mathrm{Tr}_{K'_{x'}/K_x} f')dg)$$

if $x' \in X'_0$ with $\varphi(x') = x$, $f' \in K'_{x'}$ and $g \in K_x$. ($K'_{x'}$ is the quotient field of the $\mathfrak{m}_{x',X'}$-adic completion $A'_{x'}$ of $\mathcal{O}_{x',X'}$.)

Proof. (i) By the continuity (R2), we may assume that only finitely many of the a_ν and b_μ are non-zero. Indeed, express f and g as

$$f = \phi_1(t) + \phi_2(t)$$
$$g = \psi_1(t) + \psi_2(t)$$

in such a way that $\phi_1(t)$ and $\psi_1(t)$ are Laurent polynomials and that $\phi_2(t), \psi_2(t) \in t^n A_x$ for large enough n so that

$$\phi_1(t)\psi_2'(t) + \phi_2(t)\psi_1'(t) + \phi_2(t)\psi_2'(t) \in A_x.$$

Then $f dg = f(t)g'(t)dt$, and only the term in t^{-1} can give non-zero residue by (R3). By (R4) we have

$$\mathrm{Res}_{A_x}^{V_x}(t^{-1}dt) = \dim_k \Bbbk(x) = 1.$$

(Note that in positive characteristics it is not immediately obvious that the coefficient in question is independent of the choice of the uniformizing parameter t).

For (ii), let

$$A_S = \prod_{x \in S} A_x$$
$$V_S = {\prod}'_{x \in S} K_x$$
$$= \{f = (f_x) \mid f_x \in K_x,\ \forall x \in S \text{ and } f_x \in A_x \text{ for all but a finite number of } x\}.$$

Embedding $\mathbb{R}(X)$ diagonally into V_S, we see that $\mathbb{R}(X) \cap A_S = \mathcal{O}(S)$. By (R5) we have

$$\mathrm{Res}_{A_S}^{V_S} + \mathrm{Res}_{\mathbb{R}(X)}^{V_S} = \mathrm{Res}_{\mathcal{O}(S)}^{V_S} + \mathrm{Res}_{(\mathbb{R}(X)+A_S)}^{V_S}.$$

$\mathrm{Res}_{\mathbb{R}(X)}^{V_S} = 0$ by (R2), since $\mathbb{R}(X)$ is an $\mathbb{R}(X)$-module. We now show $V_S/(\mathbb{R}(X) + A_S)$ to be finite dimensional, hence $\mathrm{Res}_{(\mathbb{R}(X)+A_S)}^{V_S} = 0$ by (R1). It suffices to prove the finite dimensionality when $S = X_0$ because of the projection $V_{X_0} \twoheadrightarrow V_S$. Regarding $\mathbb{R}(X)$ as a constant sheaf on X, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \underset{\substack{\text{constant}\\\text{sheaf}}}{\mathbb{R}(X)} \longrightarrow \mathbb{R}(X)/\mathcal{O}_X = \bigoplus_{x \in X_0} K_x/A_x \longrightarrow 0,$$

where K_x/A_x is the skyscraper sheaf at x with stalk K_x/A_x. The associated cohomology long exact sequence induces an isomorphism

$$V_{X_0}/(\mathbb{R}(X)+A_{X_0}) \xrightarrow{\sim} H^1(X,\mathcal{O}_X),$$

the right hand side of which is finite dimensional since X is proper over k. To complete the proof of (ii), it remains to show

$$\operatorname{Res}_{A_S}^{V_S}(\omega)=\sum_{x\in S}\operatorname{Res}_x(\omega), \qquad \forall\omega=fdg.$$

Let $S'\subset S$ be a finite subset containing all poles of f and g. We write

$$V_S=V_{S\setminus S'}\times\prod_{x\in S'}K_x$$
$$A_S=A_{S\setminus S'}\times\prod_{x\in S'}A_x.$$

By (R5),

$$\operatorname{Res}_{A_S}^{V_S}(fdg)=\operatorname{Res}_{A_{S\setminus S'}}^{V_{S\setminus S'}}(fdg)+\sum_{x\in S'}\operatorname{Res}_x(fdg).$$

$\operatorname{Res}_{A_{S\setminus S'}}^{V_{S\setminus S'}}(fdg)=0$ and $\operatorname{Res}_x(fdg)=0$ for $x\in S\setminus S'$ by the choice of S'. The last assertion in (ii) follows, since

$$\mathcal{O}(X_0)=\bigcap_{x\in X_0}\mathcal{O}_{x,X}=H^0(X,\mathcal{O}_X)$$

is finite dimensional over k so that $\mathcal{O}(X_0)\sim(0)$ and $\operatorname{Res}_{\mathcal{O}(X_0)}^{V_{X_0}}=0$ by (R1).

To prove (iii), regard the function field $\mathbb{R}(X')$ of X' as a finite algebraic extension of $\mathbb{R}(X)$. Then (iii) follows from (R6), since the integral closure of $\mathcal{O}_x$ (resp. A_x) in $\mathbb{R}(X')$ (resp. $K'_{x'}$) is a finite module over $\mathcal{O}_x$ (resp. A_x). □

Recall that X_0 is the set of closed points of an irreducible regular curve X proper over k. Each $x\in X_0$ determines a prime divisor on X, which we denote by $[x]$. Thus a divisor D on X is of the form

$$D=\sum_{x\in X_0}n_x[x], \qquad \text{with } n_x=0 \text{ for all but a finite number of } x.$$

We denote $\operatorname{ord}_x D=n_x$.

Let

$$V=V_{X_0}={\prod}'_{x\in X_0}K_x$$
$$A=A_{X_0}=\prod_{x\in X_0}A_x.$$

For a divisor D on X, let

$$V(D)=\{f=(f_x)\in V\mid \operatorname{ord}_x f_x\geq-\operatorname{ord}_x D,\ \forall x\in X_0\}.$$

Then by an argument similar to that in the proof of Theorem 5, (ii), we get

$$H^1(X,\mathcal{O}_X(D))\cong V/(\mathbb{R}(X)+V(D)).$$

Let

$$\begin{aligned}J_{\mathbb{R}(X)/k}&=\{\lambda\in\operatorname{Hom}_k(V,k)\mid\lambda(\mathbb{R}(X)+V(D))=0,\ \exists D\text{ divisor}\}\\&=\varinjlim_D\operatorname{Hom}_k(H^1(X,\mathcal{O}_X(D)),k),\end{aligned}$$

which is nothing but the space of meromorphic "pseudo-differentials" appearing in §1.

$J_{\mathbb{R}(X)/k}$ is a vector space over $\mathbb{R}(X)$ by the action

$$(g\lambda)(f) = \lambda(gf), \qquad \forall g \in \mathbb{R}(X),\ \forall f = (f_x) \in V,$$

since obviously $(g\lambda)(\mathbb{R}(X)) = 0$, while $(g\lambda)(V((g) + D)) = 0$.

As in §1, let us assume X to be smooth and proper over k and geometrically irreducible. Then $\mathbb{R}(X)$ is a regular transcendental extension of transcendence degree one so that the module $\Omega^1_{\mathbb{R}(X)/k}$ is a one-dimensional vector space over $\mathbb{R}(X)$. Moreover, for any $x \in X_0$, the stalk $\Omega^1_{X/k,x}$ is a free $\mathcal{O}_{x,X}$-submodule of $\Omega^1_{\mathbb{R}(X)/k}$ of rank one. If t_x is a local parameter at x so that $\mathfrak{m}_{x,X} = t_x\mathcal{O}_{x,X}$, then each $\omega \in \Omega^1_{\mathbb{R}(X)/k}$ can be expressed as

$$\omega = h dt_x, \qquad \text{for some } h \in \mathbb{R}(X).$$

Let us denote $\mathrm{ord}_x(\omega) = \mathrm{ord}_x(h)$, which is independent of the choice of the local parameter t_x. We then denote

$$(\omega) = \sum_{x\in X_0} \mathrm{ord}_x(\omega)[x],$$

which is easily seen to be a divisor on X.

For any divisor D, one has

$$H^0(X, \Omega^1_{X/k}(-D)) = \{\omega \in \Omega^1_{\mathbb{R}(X)/k} \mid (\omega) \geq D\}$$

and

$$\Omega^1_{\mathbb{R}(X)/k} = \varinjlim_{D} H^0(X, \Omega^1_{X/k}(-D)).$$

The abstract residue gives rise to an $\mathbb{R}(X)$-linear map

$$\sigma\colon \Omega^1_{\mathbb{R}(X)/k} \longrightarrow J_{\mathbb{R}(X)/k}$$

defined by

$$\sigma(\omega)(f) = \sum_{x\in X_0} \mathrm{Res}_x(f_x\omega), \qquad \forall \omega \in \Omega^1_{\mathbb{R}(X)/k},\ \forall f = (f_x) \in V.$$

This makes sense, since $\sigma(\omega)(\mathbb{R}(X)) = 0$ by Theorem 5, (ii), while $\sigma(\omega)(V(D)) = 0$ for $D = (\omega)$ by (R2).

For any divisor D, we see easily that σ induces a k-linear map

$$\sigma_D\colon H^0(X, \Omega^1_{X/k}(-D)) \longrightarrow \mathrm{Hom}_k(V/(\mathbb{R}(X) + V(D)), k) = \mathrm{Hom}_k(H^1(X, \mathcal{O}_X(D)), k).$$

Theorem 6 (Serre duality).

$$\sigma\colon \Omega^1_{\mathbb{R}(X)/k} \longrightarrow J_{\mathbb{R}(X)k}$$

is an isomorphism, which induces an isomorphism

$$\sigma_D\colon H^0(X, \Omega^1_{X/k}(-D)) \xrightarrow{\sim} \mathrm{Hom}_k(V/(\mathbb{R}(X) + V(D)), k) = \mathrm{Hom}_k(H^1(X, \mathcal{O}_X(D)), k),$$

for any divisor D. Consequently, (2)*-strong form of the Riemann-Roch theorem (Theorem* 1.3*) holds, giving rinse to a non-degenerate bilinear pairing*

$$H^0(X, \Omega^1_{X/k}(-D)) \times H^1(X, \mathcal{O}_X(D)) \longrightarrow k.$$

In particular, Part (2) *of Theorem* 1.1 *holds.*

To show that σ and σ_D are isomorphisms, we follow Serre [**91**, Chapter II, §§6 and 8].

Lemma 7.

$$\dim_{\mathbb{R}(X)} J_{\mathbb{R}(X)/k} \leq 1.$$

PROOF. Suppose $\lambda, \lambda' \in J_{\mathbb{R}(X)/k}$ were $\mathbb{R}(X)$-linearly independent. Hence we have an injective homomorphism

$$\mathbb{R}(X) \oplus \mathbb{R}(X) \ni (g,h) \longmapsto g\lambda + h\lambda' \in J_{\mathbb{R}(X)/k}.$$

There certainly exists D such that $\lambda(V(D)) = 0$ and $\lambda'(V(D)) = 0$. Fix $x \in X_0$ and let $P = [x]$. For a positive integer n and $g, h \in \mathbb{R}(X)$ with $(g) + nP \geq 0$ and $(h) + nP \geq 0$, we have $(g\lambda + h\lambda')(V(D - nP)) = 0$. Thus we have an injective homomorphism

$$H^0(X, \mathcal{O}_X(nP)) \oplus H^0(X, \mathcal{O}_X(nP)) \ni (g,h) \longmapsto g\lambda + h\lambda' \in \operatorname{Hom}_k(H^1(X, \mathcal{O}_X(D - nP)), k).$$

Hence we have

$$(*) \qquad \dim H^1(X, \mathcal{O}_X(D - nP)) \geq 2 \dim_k H^0(X, \mathcal{O}_X(nP)).$$

The right hand side of $(*)$ is greater than or equal to $2(n \deg P - g + 1)$ by Theorem 1.1, (1). On the other hand, again by Theorem 1.1, (1), the left hand side of $(*)$ is equal to

$$\begin{aligned} & -\deg(D - nP) + g - 1 + \dim_k H^0(X, \mathcal{O}_X(D - nP)) \\ = \ & n \deg P + (g - 1 - \deg D) + \dim_k H^0(X, \mathcal{O}_X(D - nP)). \end{aligned}$$

However, one has $\deg(D - nP) < 0$ for $n \gg 0$, hence $H^0(X, \mathcal{O}_X(D - nP)) = 0$ by Corollary 1.4, (b). Thus $(*)$ obviously leads to a contradiction for $n \gg 0$. $\square$

LEMMA 8. *Under the $\mathbb{R}(X)$-linear map*

$$\sigma \colon \Omega^1_{\mathbb{R}(X)/k} \longrightarrow J_{\mathbb{R}(X)/k},$$

$\omega \in \Omega^1_{\mathbb{R}(X)/k}$ *belongs to* $H^0(X, \Omega^1_{X/k}(-D))$ *if* $\sigma(\omega)(V(D)) = 0$.

PROOF. Otherwise, there exists $y \in X_0$ such that $\operatorname{ord}_y(\omega) < \operatorname{ord}_y D$. Let $n = \operatorname{ord}_y(\omega) + 1$, hence $n \leq \operatorname{ord}_y D$. Define $f = (f_x) \in V$ by

$$\begin{cases} f_x = 0 & \text{if } x \neq y \\ f_y = 1/t_y^n & (t_y \text{ being a local parameter at } y). \end{cases}$$

Obviously, $\operatorname{Res}_x(f_x\omega) = 0$ for $x \neq y$, while

$$\operatorname{ord}_y(f_y\omega) = \operatorname{ord}_y((1/t_y^n)\omega) = -n + \operatorname{ord}_y(\omega) = -1,$$

hence $\sigma(\omega)(f) = \operatorname{Res}_y(f_y\omega) \neq 0$ by (R4). Since $n \leq \operatorname{ord}_y(D)$, one has $f \in V(D)$, a contradiction to the assumption $\sigma(\omega)(V(D)) = 0$. $\square$

PROOF OF THEOREM 6. σ is injective, for if $\sigma(\omega) = 0$, then $\omega \in H^0(X, \Omega^1_{X/k}(-D))$ for all D by Lemma 8, hence $\omega = 0$.

σ is surjective, since σ is a non-zero $\mathbb{R}(X)$-linear map with $\dim_{\mathbb{R}(X)} \leq 1$ by Lemma 7. Moreover, σ_D is surjective, for if $\lambda \in J_{\mathbb{R}(X)/k}$ satisfies $\lambda(V(D)) = 0$, then there exists $\omega \in \Omega^1_{\mathbb{R}(X)/k}$ with $\sigma(\omega) = \lambda$. We see that $\omega \in H^0(X, \Omega^1_{X/k}(-D))$ by Lemma 8. $\square$

2. Comparison of algebraic with analytic cohomology

In almost all of this section, we work only with complex projective space and its non-singular subvarieties. We abbreviate $\mathbb{P}^n_{\mathbb{C}}$ to $\mathbb{P}^n$ and recall that the set of closed points of $\mathbb{P}^n$ has two topologies: the Zariski topology and the much finer classical (or ordinary) topology. By ($\mathbb{P}^n$ in the classical topology) we mean the set of closed points of $\mathbb{P}^n$ in the classical topology and by ($\mathbb{P}^n$ in the Zariski topology) we mean the scheme $\mathbb{P}^n_{\mathbb{C}}$ as usual. Note that there is a continuous map

$$\epsilon \colon (\mathbb{P}^n \text{ in classical topology}) \longrightarrow (\mathbb{P}^n \text{ in the Zariski topology}).$$

We shall consider sheaves on the space on the left. The following class is very important.

DEFINITION 2.1. The holomorphic or analytic *structure sheaf* $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$ on ($\mathbb{P}^n$ in the classical topology) is the sheaf:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(U) = \text{ring of analytic functions } f\colon U \to \mathbb{C}.$$

If $U \subset \mathbb{P}^n$ is an open set, then a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules on U is called a *coherent analytic sheaf* if for all $x \in U$, there is a (classical) open neighborhood $U_x \subset U$ of x and an exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules on U_x:

$$\mathcal{O}^{n_1}_{\mathbb{P}^n,\mathrm{an}}|_{U_x} \longrightarrow \mathcal{O}^{n_0}_{\mathbb{P}^n,\mathrm{an}}|_{U_x} \longrightarrow \mathcal{F}|_{U_x} \longrightarrow 0.$$

For basic results on coherent analytic sheaves, we refer to Gunning-Rossi [**48**]. Among the standard results given there are:

(2.2) If $\phi\colon \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-module homomorphism of coherent analytic sheaves on some U, then $\operatorname{Ker}\phi$, $\operatorname{Image}\phi$ and $\operatorname{Coker}\phi$ are coherent; thus the coherent analytic sheaves on U form an abelian category.

(2.3) If $U \subset \mathbb{P}^n_{\mathbb{C}}$ is a polycylinder in some affine piece and $\mathcal{F}$ is coherent on U, then $H^i(U, \mathcal{F}|_U) = (0)$, $i > 0$, and $\mathcal{F}|_U$ is generated as $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-module by $H^0(U, \mathcal{F}|_U)$.

(2.4) If $X \subset U$ is a closed analytic subset, then the sheaf $\mathcal{I}_X$ of analytic functions vanishing on X is coherent. If $\mathcal{F}$ is coherent, then $\{x \in \mathbb{P}^n \mid \mathcal{F}_x \neq (0)\}$ is a closed analytic subset.

Now if $\mathcal{F}$ is an algebraic coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$, one can define canonically an associated analytic coherent sheaf $\mathcal{F}_{\mathrm{an}}$ as follows: for all classical open U, let:

$$\begin{aligned}\mathcal{F}_{\mathrm{an}}(U) = \ & \text{submodule of } \prod_{x\in U} (\mathcal{O}_{\mathbb{P}^n,\mathrm{an}})_x \otimes_{\mathcal{O}_{\mathbb{P}^n,x}} \mathcal{F}_x \\ & \text{consisting of families } \{s_x\} \text{ satisfying the following condition:} \\ & \text{for all } x \in U, \text{ classical neighborhood } U_1 \text{ of } x \text{ and a Zariski} \\ & \text{neighborhood } U_2 \text{ of } x,\ f_i \in \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(U_1) \text{ and } t_i \in \mathcal{F}(U_2) \text{ such that} \\ & s_x = \textstyle\sum f_i \otimes t_i, \quad x \in U_1 \cap U_2.\end{aligned}$$

This looks a bit cumbersome but, in fact, it is the natural way to define $f^*\mathcal{F}$ for any morphism f of ringed spaces, and sheaf of modules $\mathcal{F}$ on the image space. In the present situation, one has $\mathcal{F}_{\mathrm{an}} = \epsilon^*\mathcal{F}$. An elementary calculation gives the stalks of $\mathcal{F}_{\mathrm{an}}$:

$$(\mathcal{F}_{\mathrm{an}})_x = (\mathcal{O}_{\mathbb{P}^n,\mathrm{an}})_x \otimes_{\mathcal{O}_{\mathbb{P}^n,x}} \mathcal{F}_x.$$

Also, $\mathcal{F} \longmapsto \mathcal{F}_{\mathrm{an}}$ is obviously a functor, i.e., any $\mathcal{O}_{\mathbb{P}^n}$-homomorphism $\phi\colon \mathcal{F}_1 \to \mathcal{F}_2$ induces $\phi_{\mathrm{an}}\colon \mathcal{F}_{1,\mathrm{an}} \to \mathcal{F}_{2,\mathrm{an}}$. We now invoke the basic fact:

LEMMA 2.5 (Serre). $\mathbb{C}\{X_1, \ldots, X_n\}$*, the ring of convergent power series, is a flat* $\mathbb{C}[X_1, \ldots, X_n]$*-module.*

PROOF. In fact, the completion $\widehat{\mathcal{O}}$ of a noetherian local ring $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-module (Atiyah-MacDonald [**19**, (10.14) and Exercise 7, Chapter 10]), hence $\mathbb{C}[[X_1, \ldots, X_n]]$ is faithfully flat over $\mathbb{C}\{X_1, \ldots, X_n\}$ and over $\mathbb{C}[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$. Hence $\forall M \to N \to P$ over $\mathbb{C}[X_1, \ldots, X_n]$,

$$\begin{aligned} M \to N \to P \text{ exact} &\Longrightarrow M \otimes \mathbb{C}[[X]] \to N \otimes \mathbb{C}[[X]] \to P \otimes \mathbb{C}[[X]] \text{ exact} \\ &\Longrightarrow M \otimes \mathbb{C}\{X\} \to N \otimes \mathbb{C}\{X\} \to P \otimes \mathbb{C}\{X\} \text{ exact.}\end{aligned}$$

□

COROLLARY 2.6. $\mathcal{F} \longmapsto \mathcal{F}_{\mathrm{an}}$ *is an exact functor from the category of all* $\mathcal{O}_{\mathbb{P}^n}$*-modules to the category of all* $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$*-modules.*

Proof. If $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact, then by Proposition IV.4.3

$$\mathcal{F}_{\mathrm{an},x} \longrightarrow \mathcal{G}_{\mathrm{an},x} \longrightarrow \mathcal{H}_{\mathrm{an},x}$$

is exact for all x, hence $\mathcal{F}_{\mathrm{an}} \to \mathcal{G}_{\mathrm{an}} \to \mathcal{H}_{\mathrm{an}}$ is exact. □

Corollary 2.7. *If $\mathcal{F}$ is a coherent algebraic sheaf, then $\mathcal{F}_{\mathrm{an}}$ is a coherent analytic sheaf.*

Proof. If $\mathcal{F}$ is coherent, then on any affine U, there is an exact sequence

$$\mathcal{O}_{\mathbb{P}^n}^{n_1}|_U \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n_0}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

hence an exact sequence:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}^{n_1}|_U \longrightarrow \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}^{n_0}|_U \longrightarrow \mathcal{F}_{\mathrm{an}}|_U \longrightarrow 0.$$

□

Note that covering the identity map

$$\epsilon\colon (\mathbb{P}^n \text{ in the classical topology}) \longrightarrow (\mathbb{P}^n \text{ in the Zariski topology})$$

there is a map $\epsilon^*\colon \mathcal{F} \to \mathcal{F}_{\mathrm{an}}$ of sheaves. This induces a canonical map on cohomology $H^i(\mathcal{F}) \to H^i(\mathcal{F}_{\mathrm{an}})$.

Our goal is now the following fundamental theorem:

Theorem 2.8 (Serre). (Fundamental "GAGA"[4] comparison theorem)

i) *For every coherent algebraic $\mathcal{F}$, and every i,*

$$H^i(\mathbb{P}^n \text{ in the Zariski topology}, \mathcal{F}) \cong H^i(\mathbb{P}^n \text{ in the classical topology}, \mathcal{F}_{\mathrm{an}}).$$

ii) *The categories of coherent algebraic and coherent analytic sheaves are equivalent, i.e., every coherent analytic $\mathcal{F}'$ is isomorphic to $\mathcal{F}_{\mathrm{an}}$, some $\mathcal{F}$, and*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}}(\mathcal{F}_{\mathrm{an}}, \mathcal{G}_{\mathrm{an}}).$$

We will omit the details of the first and most fundamental step in the proof (for these we refer the reader to Gunning-Rossi [**48**, Chapter VIII A]). This is the finiteness assertion: given a coherent analytic $\mathcal{F}$, then $\dim_{\mathbb{C}} H^i(\mathbb{P}^n, \mathcal{F}) < +\infty$, for all i. The proof goes as follows:

a) For all $C > 1$, $0 \le i \le n$, let

$$U_{i,C} = \left\{ x \in \mathbb{P}^n \;\middle|\; x \notin V(X_i) \text{ and } \left|\frac{X_j}{X_i}(x)\right| < C,\ 0 \le j \le n \right\}.$$

b) Then $\bigcup_{i=0}^n U_{i,C} = \mathbb{P}^n$ so we have an open covering $\mathcal{U}_C = \{U_{0,C}, \cdots, U_{n,C}\}$ of $\mathbb{P}^n$.

c) Note that each intersection $U_{i_1,C} \cap \cdots \cap U_{i_k,C}$ can be mapped bihilomorphically onto a closed analytic subset Z of a high-dimensional polycylinder D by means of the set of functions X_j/X_{i_l}, $0 \le j \le n$, $1 \le l \le k$. Therefore, every coherent analytic $\mathcal{F}$ on $U_{i_1,C} \cap \cdots U_{i_k,C}$ corresponds to a sheaf $\mathcal{F}'$ on Z and, extending it to $D \setminus Z$ by (0) outside Z, a coherent analytic $\mathcal{F}'$ on D. Then

$$H^i(U_{i_1,C} \cap \cdots U_{i_k,C}, \mathcal{F}) \cong H^i(Z, \mathcal{F}') \cong H^i(D, \mathcal{F}') \cong (0),\ i > 0.$$

[4] Short for "géométrie analytique et géométrie algébrique"

d) Therefore by Proposition VII.2.2 it follows that

$$H^i(\mathbb{P}^n, \mathcal{F}) \cong H^i(\mathcal{U}_C, \mathcal{F})$$

and that the refinement maps (for $C > C' > 1$):

$$\mathrm{ref}^i_{C,C'}\colon C^i(\mathcal{U}_C, \mathcal{F}) \longrightarrow C^i(\mathcal{U}_{C'}, \mathcal{F})$$

induce an isomorphism on cohomology.

e) The key step is to show that the space of sections:

$$\mathcal{F}(U_{i_1,C} \cap \cdots U_{i_k,C})$$

is a *topological* vector space, in fact, a Fréchet space in a natural way; and that all restriction maps such as

$$\mathcal{F}(U_{i_2,C} \cap \cdots U_{i_k,C}) \longrightarrow \mathcal{F}(U_{i_1,C} \cap \cdots U_{i_k,C})$$

are continuous, and that restriction to a relatively compact open subset, as in

$$\mathcal{F}(U_{i_1,C} \cap \cdots U_{i_k,C}) \longrightarrow \mathcal{F}(U_{i_1,C'} \cap \cdots U_{i_k,C'}) \qquad (C > C')$$

is *compact*. This last is a generalization of Montel's theorem that

$$\mathrm{res}\colon \left\{ \begin{array}{l} \text{holomorphic functions on} \\ \text{disc } |z| < C \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{holomorphic functions on} \\ \text{disc } |z| < C' \end{array} \right\}$$

is compact. It follows that $C^i(\mathcal{U}_C, \mathcal{F})$ is a complex of Fréchet spaces and continuous maps, and that $\mathrm{ref}^i_{C,C'}$ is compact.

f) By step (d),

$$\begin{array}{ccc} Z^i(\mathcal{U}_C, \mathcal{F}) \oplus C^{i-1}(\mathcal{U}_{C'}) & \xrightarrow{\mathrm{ref}+\delta} & Z^i(\mathcal{U}_{C'}, \mathcal{F}) \\ (a,b) & \longmapsto & \mathrm{ref}^i_{C,C'} a + \delta b \end{array}$$

is surjective. A standard fact in the theory of Fréchet spaces is that if

$$\alpha, \beta\colon V_1 \longrightarrow V_2$$

are two continuous maps of Fréchet spaces, with α surjective and β compact, then $\alpha+\beta$ has closed image of finite codimension. Apply this with $\alpha = \mathrm{ref}+\delta$, $\beta = -\mathrm{ref}$ and we find that $\mathrm{Coker}(\delta) = H^i(\mathcal{U}_{C'}, \mathcal{F})$ is finite-dimensional.

The second step in the proof is the vanishing theorem — if $\mathcal{F}$ is coherent analytic, then for $i > 0$, $m \gg 0$, $H^i(\mathbb{P}^n, \mathcal{F}(m)) = (0)$. (Here $\mathcal{F}(m) \underset{\text{def}}{=} \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)$ as usual.) We prove this by induction on n, the complex dimension of the ambient projective space, since it is obvious for $n = 0$. As in §VII.7, we use the $\otimes L\colon \mathcal{F}(m) \to \mathcal{F}(m+1)$, where $L = \sum c_i X_i$ is a linear form. This induces exact sequences:

(2.9)
$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathcal{G}_L(m) \arrow[r] & \mathcal{F}(m) \arrow[rr] \arrow[dr] & & \mathcal{F}(m+1) \arrow[r] & \mathcal{H}_L(m) \arrow[r] & 0 \\ & & & \mathcal{F}'_L(m) \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

where both $\mathcal{G}_L$ and $\mathcal{H}_L$ are annihilated by L/X_i on $\mathbb{P}^n \setminus V(X_i)$. Therefore they are coherent analytic sheaves on $V(L) \cong \mathbb{P}^{n-1}$, and the induction assumption applies to them, i.e., $\exists m_0(L)$ such that

$$\left.\begin{array}{rcl} H^i(\mathbb{P}^n, \mathcal{G}_L(m)) & \cong & (0) \\ H^i(\mathbb{P}^n, \mathcal{H}_L(m)) & \cong & (0) \end{array}\right. \qquad \text{if } m \geq m_0(L),\ 1 \leq i \leq n.$$

The cohomology sequence of (2.9) then gives us:

$$\otimes L\colon H^i(\mathbb{P}^n, \mathcal{F}(m)) \xrightarrow{\sim} H^i(\mathbb{P}^n, \mathcal{F}(m+1)), \quad m \geq m_0(L).$$

In particular, $\dim H^i(\mathbb{P}^n, \mathcal{F}(m)) = N_i$, independent of m for $m \geq m_0(L)$. Now fix one linear form L and consider the maps:

$$\otimes F\colon H^i(\mathcal{F}(m_0(L))) \longrightarrow H^i(\mathcal{F}(m_0(L)+d))$$

for all homogeneous F of degree d. If R_d is the vector space of such F's, then choosing fixed bases of the above cohomology groups, we have a linear map:

$$\begin{aligned} R_d &\longrightarrow \text{vector space of } (N_i \times N_i)\text{-matrices} \\ F &\longmapsto \text{matrix for } \otimes F. \end{aligned}$$

Let I_d be the kernel. It is clear that $I = \sum_{d=1}^{\infty} I_d$ is an ideal in $R = \bigoplus_{d=0}^{\infty} R_d$ and that $\dim R_d/I_d \leq N_i^2$. Thus the degree of the Hilbert polynomial of R/I is 0, hence the subscheme $V(I) \subset \mathbb{P}^n$ with structure sheaf $\mathcal{O}_{\mathbb{P}^n}/\widetilde{I}$ is 0-dimensional. If $V(I) = \{x_1, \ldots, x_t\}$, it follows that the only associated prime ideals of I can be either

$$\mathfrak{m}_{x_i} = \text{ideal of forms } F \text{ with } F(x_i) = 0,\ 1 \leq i \leq t$$

or

$$(X_0, \ldots, X_n).$$

By the primary decomposition theorem, it follows that

$$I \supset (X_0, \ldots, X_n)^{d_0} \cap \bigcap_{i=1}^{t} \mathfrak{m}_{x_i}^{d_i}$$

for some $d_0, \ldots, d_t$. Now fix linear forms L_i such that $L_i(x_i) = 0$. Let

$$F = L^{\max(d_0, m_0(L_i) - m_0(L))} \cdot \prod_{i=1}^{t} L_i^{d_i}.$$

On the one hand, we see that $F \in I$, hence $\otimes F$ on $H^i(\mathcal{F}(m_0(L)))$ is 0. But on the other hand, if $m_1 = \max(d_0, m_0(L_i) - m_0(L))$, then $\otimes F$ factors:

$$\begin{aligned} H^i(\mathcal{F}(m_0(L))) \xrightarrow[\approx]{\otimes L} \cdots \xrightarrow[\approx]{\otimes L} H^i(\mathcal{F}(m_0(L)+m_1)) \\ \xrightarrow[\approx]{\otimes L_1} \cdots \xrightarrow[\approx]{\otimes L_t} H^i(\mathcal{F}(m_0(L)+m_1+\textstyle\sum d_i)) \end{aligned}$$

which is an isomorphism. It follows that $H^i(\mathcal{F}(m_0(L))) = (0)$ as required.

The third step is to show that if $\mathcal{F}$ is coherent analytic, then $\mathcal{F}(\nu)$ is generated by its sections if $\nu \gg 0$. For each $x \in \mathbb{P}^n$, let $\mathfrak{m}_x$ = sheaf of functions zero at x, and consider the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}_x \cdot \mathcal{F}(\nu) \arrow[r] \arrow[d, equal] & \mathcal{F}(\nu) \arrow[r] & \mathcal{F}(\nu)/\mathfrak{m}_x \cdot \mathcal{F}(\nu) \arrow[r] & 0 \\ & (\mathfrak{m}_x \cdot \mathcal{F})(\nu). & & & \end{tikzcd}$$

There exists ν_x such that if $\nu \geq \nu_x$, then $H^1(\mathfrak{m}_x \cdot \mathcal{F}(\nu)) = (0)$, hence $H^0(\mathcal{F}(\nu)) \to H^0(\mathcal{F}(\nu)/\mathfrak{m}_x \cdot \mathcal{F}(\nu))$ is onto. Let $\mathcal{G}$ be the cokernel:

$$H^0(\mathcal{F}(\nu_x)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}} \longrightarrow \mathcal{F}(\nu_x) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then $\mathcal{G}$ is coherent analytic and

$$\mathcal{G}_x/\mathfrak{m}_x \cdot \mathcal{G}_x \cong \mathcal{F}(\nu)_x / \big(\mathfrak{m}_x \cdot \mathcal{F}(\nu)_x + \mathrm{Image}\, H^0(\mathcal{F}(\nu)\big) = (0).$$

Therefore by Nakayama's lemma, $\mathcal{G}_x = (0)$ and by coherency, $\exists$ a neighborhood U_x of x in which $\mathcal{G} \equiv (0)$. It follows that $\mathcal{F}(\nu_x)$ is generated by $H^0(\mathcal{F}(\nu_x))$ in U_x and hence $\mathcal{F}(\nu)$ is generated by $H^0(\mathcal{F}(\nu))$ in U_x for $\nu \geq \nu_x$ too! By compactness $\mathbb{P}^n$ is covered by finitely many of these U_x's, say $U_{x_1}, \ldots, U_{x_t}$. Then if $\nu \geq \max(\nu_{x_i})$, $\mathcal{F}(\nu)$ is generated everywhere by $H^0(\mathcal{F}(\nu))$.

The fourth step is to show that

$$H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)) = \begin{array}{l}\text{vector space of homogeneous forms of}\\ \text{degree } m \text{ in } X_0, \ldots, X_n\end{array}$$

just as in the algebraic case. We do this by induction first on n, since it is clear for $n = 0$; and then by a second induction on m, since it is also clear for $m = 0$, i.e., by the maximum principle the only global analytic functions on the compact space $\mathbb{P}^n$ are constants. The induction step uses the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-1) \xrightarrow{\otimes X_n} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1},\mathrm{an}}(m) \longrightarrow 0$$

which gives:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \left\{\begin{array}{l}\text{Polynomials in}\\ X_0, \ldots, X_n\\ \text{of degree } m-1\end{array}\right\} & \xrightarrow{X_n} & \left\{\begin{array}{l}\text{Polynomials in}\\ X_0, \ldots, X_n\\ \text{of degree } m\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{Polynomials in}\\ X_0, \ldots, X_{n-1}\\ \text{of degree } m\end{array}\right\} & \longrightarrow 0\\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-1)) & \xrightarrow{\otimes X_n} & H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{n-1},\mathrm{an}}(m)). &
\end{array}$$

"Chasing" this diagram shows the required assertion for $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)$.

The fifth step is that every coherent analytic $\mathcal{F}'$ is isomorphic to $\mathcal{F}_{\mathrm{an}}$, some coherent algebraic $\mathcal{F}$. By the third step there is a surjection:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}^{n_0} \longrightarrow \mathcal{F}'(m_0) \longrightarrow 0$$

for suitable n_0 and m_0, hence a surjection:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_0)^{n_0} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Applying the same reasoning to the kernel, we get a presentation:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_1)^{n_1} \xrightarrow{\phi'} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_0)^{n_0} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Now ϕ' is given by an $(n_0 \times n_1)$-matrix of sections ϕ'_{ij} of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m_1 - m_0)$, hence by an $(n_0 \times n_1)$-matrix F_{ij} of polynomials of degree $m_1 - m_0$. Thus the F_{ij} defines ϕ, with cokernel $\mathcal{F}$:

$$\mathcal{O}_{\mathbb{P}^n}(-m_1)^{n_1} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(-m_0)^{n_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

By exactness of the functor $\mathcal{G} \mapsto \mathcal{G}_{\mathrm{an}}$, it follows that $\mathcal{F}' \cong \mathcal{F}_{\mathrm{an}}$. Using the same set-up, we can also conclude that $H^0(\mathcal{F}(m)) \cong H^0(\mathcal{F}_{\mathrm{an}}(m))$ for $m \gg 0$. In fact, twist enough so that the H^1 of the kernel and image of both ϕ and ϕ' are all (0): then the usual sequences show that the two rows below are exact:

$$\begin{array}{ccccccc}
H^0(\mathcal{O}_{\mathbb{P}^n}(m-m_1)^{n_1}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(m-m_0)^{n_0}) & \longrightarrow & H^0(\mathcal{F}(m)) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-m_1)^{n_1}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-m_0)^{n_0}) & \longrightarrow & H^0(\mathcal{F}_{\mathrm{an}}(m)) & \longrightarrow & 0.
\end{array}$$

Thus $H^0(\mathcal{F}(m)) \to H^0(\mathcal{F}_{\mathrm{an}}(m))$ is an isomorphism.

The sixth step is to compare the cohomologies of $\mathcal{F}(m)$ and $\mathcal{F}_{\mathrm{an}}(m)$ for all m. We know that for $m \gg 0$, all their cohomology groups are isomorphic and we may assume by induction on n that we know the result for sheaves on $\mathbb{P}^{n-1}$. We use a second induction on m, i.e., assuming

the result for $H^i(\mathcal{F}(m+1))$, all i, deduce it for $H^i(\mathcal{F}(m))$, all i. Use the diagram (2.9) above for any linear form L. We get

$$\begin{array}{ccccccccc}
H^{i-1}(\mathcal{F}(m+1)) & \longrightarrow & H^{i-1}(\mathcal{H}_L(m)) & \longrightarrow & H^i(\mathcal{F}'_L(m)) & \longrightarrow & H^i(\mathcal{F}(m+1)) & \longrightarrow & H^i(\mathcal{H}_L(m)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{i-1}(\mathcal{F}_{\mathrm{an}}(m+1)) & \to & H^{i-1}(\mathcal{H}_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{F}'_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{F}_{\mathrm{an}}(m+1)) & \to & H^i(\mathcal{H}_{L,\mathrm{an}}(m))
\end{array}$$

and

$$\begin{array}{ccccccccc}
H^{i-1}(\mathcal{F}'_L(m)) & \longrightarrow & H^i(\mathcal{G}_L(m)) & \longrightarrow & H^i(\mathcal{F}(m)) & \longrightarrow & H^i(\mathcal{F}'_L(m)) & \longrightarrow & H^{i+1}(\mathcal{G}_L(m)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{i-1}(\mathcal{F}'_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{G}_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{F}_{\mathrm{an}}(m)) & \to & H^i(\mathcal{F}'_{L,\mathrm{an}}(m)) & \to & H^{i+1}(\mathcal{G}_{L,\mathrm{an}}(m)).
\end{array}$$

By the 5-lemma, the result for $H^i(\mathcal{F}(m+1))$ and $H^{i-1}(\mathcal{F}(m+1))$ implies it for $H^i(\mathcal{F}'_L(m))$. And the result for $H^i(\mathcal{F}'_L(m))$ and $H^{i-1}(\mathcal{F}'_L(m))$ implies it for $H^i(\mathcal{F}(m))$.

The seventh step is to compare $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and $\mathrm{Hom}(\mathcal{F}_{\mathrm{an}},\mathcal{G}_{\mathrm{an}})$. Presenting $\mathcal{F}$ as before, we get:

$$\begin{aligned}
\mathrm{Hom}(\mathcal{F},\mathcal{G}) &\cong \mathrm{Ker}\left[\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-m_0)^{n_0},\mathcal{G}) \xrightarrow{\circ\phi} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-m_1)^{n_1},\mathcal{G})\right] \\
&\cong \mathrm{Ker}\left[H^0(\mathcal{G}(m_0)^{n_0}) \xrightarrow{\otimes F_{ij}} H^0(\mathcal{G}(m_1)^{n_1})\right] \\
&\cong \mathrm{Ker}\left[H^0(\mathcal{G}_{\mathrm{an}}(m_0)^{n_0}) \xrightarrow{\otimes F_{ij}} H^0(\mathcal{G}_{\mathrm{an}}(m_1)^{n_1})\right] \\
&\cong \mathrm{Ker}\left[\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_0)^{n_0},\mathcal{G}_{\mathrm{an}}) \xrightarrow{\circ\phi} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_1)^{n_1},\mathcal{G}_{\mathrm{an}})\right] \\
&\cong \mathrm{Hom}(\mathcal{F}_{\mathrm{an}},\mathcal{G}_{\mathrm{an}}).
\end{aligned}$$

□

COROLLARY 2.10. *A new proof of Chow's theorem (*Part I [**76**, (4.6)]*): If $X \subset \mathbb{P}^n$ is a closed analytic subset, then X is a closed algebraic subset.*

PROOF. If $X \subset \mathbb{P}^n$ is a closed analytic subset, then $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$ is a coherent analytic sheaf, so $\mathcal{I}_X = \mathcal{J}_{\mathrm{an}}$ for some coherent algebraic $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^n}$. So $X = \mathrm{Supp}\,\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}/\mathcal{I}_X = \mathrm{Supp}\,\mathcal{O}_{\mathbb{P}^n}/\mathcal{J}$ is a closed algebraic subset. □

COROLLARY 2.11. *If X_1 and X_2 are two complete verieties over $\mathbb{C}$, then every holomorphic map $f\colon X_1 \to X_2$ is algebraic, i.e., a morphism.*

PROOF. Apply Chow's lemma (Theorem II.6.3) to find proper birational $\pi_1\colon X'_1 \to X_1$ and $\pi_2\colon X'_2 \to X_2$ with X'_i projective. Let $\Gamma \subset X_1 \times X_2$ be the graph of f. Then $(\pi_1 \times \pi_2)^{-1}\Gamma \subset X'_1 \times X'_2$ is a closed analytic subset of projective space, hence is algebraic by Chow's theorem. Since $\pi_1 \times \pi_2$ is proper, $\Gamma = (\pi_1 \times \pi_2)[(\pi_1 \times \pi_2)^{-1}\Gamma]$ is also a closed algebraic set. In order to see that it is the graph of a morphism, we must check that $p_1\colon \Gamma \to X_1$ is an isomorphism. This follows from:

LEMMA 2.12. *Let $f\colon X \to Y$ be a bijective morphism of varieties. If f is an analytic isomorphism, then f is an algebraic isomorphism.*

PROOF OF LEMMA 2.12. Note that f is certainly birational since $\#f^{-1}(y) = 1$ for all $y \in Y$. Let $x \in X$, $y = f(x)$. We must show that $f^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective. The local rings of

analytic functions on X and Y at x and y and the formal completions of these rings are related by:

$$\begin{array}{ccc}
\mathcal{O}_{Y,y} & \xrightarrow{f^*} & \mathcal{O}_{X,x} \\
\downarrow & & \downarrow \\
(\mathcal{O}_{Y,\mathrm{an}})_y & \xrightarrow{f^*_{\mathrm{an}}} & (\mathcal{O}_{X,\mathrm{an}})_x \\
\downarrow & & \downarrow \\
\widehat{\mathcal{O}}_{Y,y} & \xrightarrow{f^*} & \widehat{\mathcal{O}}_{X,x}.
\end{array}$$

Now f^*_{an} is an isomorphism by assumption. If $a \in \mathcal{O}_{X,x}$, write $a = b/c$, $b, c \in \mathcal{O}_{Y,y}$ using the fact that f is birational. Then

$$\begin{aligned}
f^*_{\mathrm{an}} \text{ isomorphism} &\Longrightarrow \exists a' \in (\mathcal{O}_{Y,\mathrm{an}})_y \text{ with } b = c \cdot a' \\
&\Longrightarrow b \in \mathcal{O}_{Y,y} \cap c \cdot \widehat{\mathcal{O}}_{Y,y}.
\end{aligned}$$

But for any ideal $\mathfrak{a} \subset \mathcal{O}_{Y,y}$, $\mathfrak{a} = \mathcal{O}_{Y,y} \cap \mathfrak{a} \cdot \widehat{\mathcal{O}}_{Y,y}$, so $b \in c \cdot \mathcal{O}_{Y,y}$, i.e., $a \in \mathcal{O}_{Y,y}$. □

□

COROLLARY 2.13 (Projective case of Riemann's Existence Theorem). *Let X be a complex projective variety*[5]*. Let $\widetilde{Y}$ be a compact topological space and*

$$\widetilde{\pi}\colon \widetilde{Y} \longrightarrow (X \textit{ in the classical topology})$$

a covering map (since $\widetilde{Y}$ is compact, this amounts merely to requiring that $\widetilde{\pi}$ is a local homeomorphism). Then there is a unique scheme Y and étale proper morphism $\pi\colon Y \to X$ such that there exists a homeomorphism ρ:

$$\begin{tikzcd}
\widetilde{Y} \arrow[rr, "\approx", "\rho"'] \arrow[dr] & & (Y \textit{ in the classical topology}) \arrow[dl] \\
& (X \textit{ in the classical topology}). &
\end{tikzcd}$$

PROOF. Given $\widetilde{Y}$, note first that since $\widetilde{\pi}$ is a local homeomorphism we can put a unique analytic structure on it making $\widetilde{\pi}$ into a local analytic isomorphism. Let $\mathcal{B} = \widetilde{\pi}_*(\mathcal{O}_{\widetilde{Y}})$: this is a sheaf of $\mathcal{O}_{X,\mathrm{an}}$-algebras. Now every $x \in X$ has a neighborhood U such that $\widetilde{\pi}^{-1}(U) \cong$ disjoint union of l copies of U; hence $\mathcal{B}|_U \cong \bigoplus_{i=1}^{l} \mathcal{O}_{X,\mathrm{an}}$ as a sheaf of algebras. In particular, $\mathcal{B}$ is a coherent analytic sheaf of $\mathcal{O}_{X,\mathrm{an}}$-modules. Recall that we can identify sheaves of $\mathcal{O}_{X,\mathrm{an}}$-modules with sheaves of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules, (0) outside X and killed by multiplication by $\mathcal{I}_X$. Therefore by the fundamental GAGA Theorem 2.8, $\mathcal{B} \cong \mathfrak{B}_{\mathrm{an}}$ for some algebraic coherent sheaf of $\mathcal{O}_X$-modules $\mathfrak{B}$. Multiplication in $\mathcal{B}$ defines an $\mathcal{O}_{X,\mathrm{an}}$-module homomorphism

$$\mu\colon \mathcal{B} \otimes_{\mathcal{O}_{X,\mathrm{an}}} \mathcal{B} \longrightarrow \mathcal{B},$$

hence by the GAGA Theorem 2.8 again this is induced by some $\mathcal{O}_X$-module homomorphism:

$$\nu\colon \mathfrak{B} \otimes_{\mathcal{O}_X} \mathfrak{B} \longrightarrow \mathfrak{B}.$$

The associative law for μ implies it for ν and so this makes $\mathfrak{B}$ into a sheaf of $\mathcal{O}_X$-algebras. The unit in $\mathcal{B}$ similarly gives a unit in $\mathfrak{B}$. We now define $Y = \mathcal{S}pec_X(\mathfrak{B})$, with $\pi\colon Y \to X$ the canonical map (proper since $\mathfrak{B}$ is coherent by Proposition II.6.5). How are Y and $\widetilde{Y}$ related? We have

[5]The theorem is true in fact for any variety X and finite-sheeted covering $\pi^*\colon Y^* \to X$, but this is harder, cf. SGA4 [**7**, Theorem 4.3, Exposé 11], where Artin deduces the general case from Grauert-Remmert [**40**]; or SGA1 [**4**, Exposé XII, Théorème 5.1, p. 332], where Grothendieck deduces it from Hironaka's resolution theorems [**55**].

i) a continuous map $\zeta\colon \widetilde{Y} \to$ (closed points of X)
ii) a map backwards covering ζ:

$$\zeta^*\colon \mathfrak{B} \longrightarrow (\text{sheaf of continuous } \mathbb{C}\text{-valued functions on } \widetilde{Y})$$

such that $\forall x \in \widetilde{Y}$ and $\forall f \in \mathfrak{B}_{\zeta(x)}$,

$$\zeta^* f(x) = f(\zeta(x)),$$

defined as the composite

$$\mathfrak{B}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{O}_{\widetilde{Y},\mathrm{an}}(\widetilde{\pi}^{-1}U) \longrightarrow [\text{continuous functions on } \widetilde{\pi}^{-1}U].$$

These induce a continuous map:

$$\eta\colon \widetilde{Y} \longrightarrow (\text{closed points of } Y)$$

by

$$\eta(x) = \text{ point corresponding to maximal ideal } \frac{\{f \in \mathfrak{B}_{\zeta(x)} \mid \zeta^* f(x) = 0\}}{\mathfrak{m}_{X,\zeta(x)} \cdot \mathfrak{B}_{\zeta(x)}}$$

via the correspondence
$\pi^{-1}(\zeta(x)) \leftrightarrows$ maximal ideals in $\mathfrak{B}_{\zeta(x)}/\mathfrak{m}_{\zeta(x)} \cdot \mathfrak{B}_{\zeta(x)}$.

Now η has the property

2.14. $\forall f \in \mathcal{O}_Y(V)$, *the composite map*

$$\eta^{-1}(V) \xrightarrow{\eta} (\textit{closed points of } V) \xrightarrow{f} \mathbb{C}$$

is a continuous function on $\eta^{-1}(V)$ *(in the classical topology).*

But a basis for open sets in the classical topology on Y is given by finite intersections of the sets:

$$\begin{aligned} &V \text{ Zariski open, } f \in \mathcal{O}_Y(V), \text{ let} \\ &W_{f,\epsilon} = \{x \in V \mid x \text{ closed and } |f(x)| < \epsilon\}. \end{aligned}$$

Because of (2.14), $\eta^{-1}(W_{f,\epsilon})$ is open in $\widetilde{Y}$, i.e., η is a continuous map from $\widetilde{Y}$ to (Y in the classical topology). Now in fact η is bijective too. In fact, if $U \subset X$ is a classical open so that $\pi^{-1}(U) =$ (disjoint union of n copies of U) and $\mathcal{B}|_U = \bigoplus_{i=1}^l \mathcal{O}_{X,\mathrm{an}}|_U$, then for all $x \in U$,

$$\mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x \cong \bigoplus_{i=1}^{l} \mathbb{C}$$

and the correspondence between points of $\pi^{-1}(x)$ and maximal ideals of $\mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x$ given by $y \mapsto \{f \mid f(y) = 0\}$ is bijective. On the other hand, since $\mathcal{B}_x \cong \mathfrak{B}_x \otimes_{\mathcal{O}_{x,x}} (\mathcal{O}_{X,\mathrm{an}})_x$, it follows that $\mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x \cong \mathfrak{B}_x/\mathfrak{m}_x \cdot \mathfrak{B}_x$. Thus η is a continuous bijective map from a compact space $\widetilde{Y}$ to (Y in the classical topology). Thus η is a homeomorphism. Finally $\mathcal{B}_x$ is a free $(\mathcal{O}_{X,\mathrm{an}})_x$-module, hence it follows that $\mathfrak{B}_x$ is a free $\mathcal{O}_{X,x}$-module: Hence $\pi\colon Y \to X$ is a flat morphism. And the scheme-theoretic fibre is:

$$\begin{aligned} \pi^{-1}(x) &= \operatorname{Spec} \mathfrak{B}_x/\mathfrak{m}_x \cdot \mathfrak{B}_x \\ &\cong \operatorname{Spec} \mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x \\ &\cong \operatorname{Spec} \bigoplus_{i=1}^{l} \mathbb{C} = l \text{ reduced points.} \end{aligned}$$

Thus π is étale.

As for the uniqueness of Y, it is a consequence of the stronger result: say

$$\begin{tikzcd} Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

are two étale proper morphisms. Then any map continuous in the classical topology:

$$(Y_1 \text{ in the classical topology}) \xrightarrow{f} (Y_2 \text{ in the classical topology})$$

with $\pi_2 \circ f = \pi_1$ is a morphism. To see this, note that π_i are local analytic isomorphisms, hence f is analytic, hence by Corollary 2.11, f is a morphism. $\square$

This Corollary 2.13 implies profound connections between topology and field theory. To explain these, we must first define the *algebraic fundamental group* $\pi_1^{\mathrm{alg}}(X)$ for any normal noetherian scheme X[6]. We have seen in §V.6 that morphisms:

$$\text{(a)} \qquad \begin{array}{l} \pi\colon Y \longrightarrow X \\ Y \text{ normal irreducible} \\ \pi \text{ proper, surjective, } \pi^{-1}(x) \text{ finite for all } x, \\ \pi \text{ generically smooth} \end{array}$$

are uniquely determined by the function field extension $\mathbb{R}(Y) \supset \mathbb{R}(X)$, which is necessarily separable; and that conversely, given any finite separable $K \supset \mathbb{R}(X)$, we obtain such a π by setting $Y =$ the normalization of X in K. In particular, suppose we start with a morphism:

$$\text{(b)} \qquad \begin{array}{l} \pi\colon Y \longrightarrow X \\ Y \text{ connected} \\ \pi \text{ proper and étale.} \end{array}$$

Then Y is smooth over a normal X, hence is normal by Proposition V.5.5. Being connected, Y is also irreducible. Thus $Y =$ normalization of X in $\mathbb{R}(Y)$. Now choose a specific separable algebraic closure $\overline{\mathbb{R}(X)}$ of $\mathbb{R}(X)$ and let

$$\begin{aligned} G &= \mathrm{Gal}(\overline{\mathbb{R}(X)}/\mathbb{R}(X)), \text{ the Galois group} \\ &\cong \varprojlim_K \mathrm{Aut}(K/\mathbb{R}(X)) \\ &\qquad \text{where } \mathbb{R}(X) \subset K \subset \overline{\mathbb{R}(X)},\ K \text{ normal over } \mathbb{R}(X) \text{ with } [K : \mathbb{R}(X)] < +\infty. \end{aligned}$$

[6]Normality is not necessary and noetherian can be weakened. For a discussion of the results below in more general case, see SGA1 [**4**, Exposés V and XII].

As usual, G, being an inverse limit of finite groups, has a natural structure of compact, totally disconnected topological group. One checks easily[7] that there is an intermediate field:

$$\mathbb{R}(X) \subset \widetilde{\mathbb{R}(X)} \subset \overline{\mathbb{R}(X)}$$

such that for all $K \subset \overline{\mathbb{R}(X)}$, finite over $\mathbb{R}(X)$:

$$\left.\begin{array}{l}\text{the normalization } Y_K \text{ of } X\\ \text{in } K \text{ is étale over } X\end{array}\right\} \iff K \subset \widetilde{\mathbb{R}(X)}.$$

Note that because of its defining property, $\widetilde{\mathbb{R}(X)}$ is invariant under all automorphisms of $\overline{\mathbb{R}(X)}$, i.e., it is normal over $\mathbb{R}(X)$ and its Galois group over $\mathbb{R}(X)$ is a quotient G/N of G. By Galois theory, the closed subgroups of finite index in G are in one-to-one correncepondence with the subfields $K \subset \overline{\mathbb{R}(X)}$ finite over $\mathbb{R}(X)$. So the closed subgroups of finite index in G/N are in one-to-one correspondence with the subfields $K \subset \widetilde{\mathbb{R}(X)}$ finite over $\mathbb{R}(X)$, hence with the set of schemes Y_K étale over X. It is therefore reasonable to call G/N the *algebraic fundamental group* of X, or $\pi_1^{\text{alg}}(X)$:

$$\pi_1^{\text{alg}}(X) = \text{Gal}(\widetilde{\mathbb{R}(X)}/\mathbb{R}(X)). \tag{2.15}$$

Next in the complex projective case again choose a universal covering space Ω of X in the classical topology. Then the topological fundamental group is:

$$\pi_1^{\text{top}}(X) = \text{group of homeomorphisms of } \Omega \text{ over } X,$$

and its subgroups of finite index are in one-to-one correspondence with the compact covering spaces $\widetilde{Y}$ dominated by Ω:

$$\Omega \longrightarrow \widetilde{Y} \xrightarrow{\widetilde{\pi}} X,$$

which give, by algebraization (Corollary 2.13), connected normal complete varieties Y, étale over X. This must simply force a connection between the two groups and, in fact, it implies this:

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[7]This follows from two simple facts:

a) $K_1 \subset K_2$, Y_{K_2} étale over $X \implies Y_{K_1}$ étale over X,
b) Y_{K_1} and Y_{K_2} étale over $X \implies Y_{K_1 \cdot K_2}$ étale over X.

To prove (a), note that we have a diagram

$$Y_{K_2} \longrightarrow Y_{K_1} \longrightarrow X.$$

Now Y_{K_2} étale over $X \implies \Omega_{Y_{K_2}/X} = (0) \implies \Omega_{Y_{K_2}/Y_{K_1}} = (0) \implies Y_{K_2}$ étale over Y_{K_1} by Criterion 4.1 for smoothness in §V.4. In particular, Y_{K_2} is flat over Y_{K_1}, hence if $y_2 \in Y_{K_2}$ has images y_1 and x in Y_{K_1} and X, then $\mathcal{O}_{y_2}$ is flat over $\mathcal{O}_{y_1}$, hence $\mathfrak{m}_x \cdot \mathcal{O}_{y_2} \cap \mathcal{O}_{y_1} = \mathfrak{m}_x \cdot \mathcal{O}_{y_1}$. Thus

$$\mathcal{O}_{y_1}/\mathfrak{m}_x \cdot \mathcal{O}_{y_1} \subset \mathcal{O}_{y_2}/\mathfrak{m}_x \cdot \mathcal{O}_{y_2} \cong \text{product of separable field extensions of } \Bbbk(x)$$

hence $\mathcal{O}_{y_1}/\mathfrak{m}_x \cdot \mathcal{O}_{y_1}$ is also a product of separable field extensions of $\Bbbk(x)$. This shows $\Omega_{Y_{K_1}/X} \otimes \Bbbk(y_1) = (0)$, hence by Nakayama's lemma, $\Omega_{Y_{K_1}/X} = (0)$ near y_1, hence by Criterion 4.1 for smoothness in §V.4, Y_{K_1} is étale over X at y_1.

To prove (b), note that $Y_{K_1} \times_X Y_{K_2}$ will be étale over X, hence normal. We get a morphism

$$Y_{K_1 \cdot K_2} \xrightarrow{\phi} Y_{K_1} \times_X Y_{K_2}$$

and if $Z =$ component of $Y_{K_1} \times_X Y_{K_2}$ containing Image ϕ, then $\phi\colon Y_{K_1 \cdot K_2} \to Z$ is birational. Since Z is normal and the fibres of ϕ are finite, ϕ is an isomorphism by Zariski's Main Theorem in §V.6.

THEOREM 2.16. *Let X be a normal subvariety of $\mathbb{P}^n_{\mathbb{C}}$ and let*

$$\begin{aligned}\widehat{\pi}_1^{\mathrm{top}}(X) &= \varprojlim_H \pi_1^{\mathrm{top}}(X)/H, \text{ over all } H \subset \pi_1^{\mathrm{top}}(X) \text{ of finite index}\\ &= \text{“pro-finite completion” of } \pi_1^{\mathrm{top}}(X).\end{aligned}$$

Then $\widehat{\pi}_1^{\mathrm{top}}(X)$ and $\pi_1^{\mathrm{alg}}(X)$ are isomorphic as topological groups, the isomorphism being canonical up to an inner automorphism.

PROOF. Choose a sequence $\{H_\nu\}$ of normal subgroups of π_1^{top} of finite index, with $H_{\nu+1} \subset H_\nu$, such that for any H of finite index, $H_\nu \subset H$ for some ν. Let $\pi_1^{\mathrm{top}}/H_\nu = G_\nu$ and let H_ν define $\widetilde{Y}_\nu \to X$. Then

$$\widehat{\pi}_1^{\mathrm{top}} \cong \varprojlim_\nu G_\nu$$

$$\text{and} \qquad G_\nu \cong \text{group of homeomorphisms of } \widetilde{Y}_\nu \text{ over } X.$$

Algebraize $\widetilde{Y}_\nu$ to a scheme Y_ν étale over X by Corollary 2.13. Then the map $\widetilde{Y}_{\nu+1} \to \widetilde{Y}_\nu$ comes from a morphism $Y_{\nu+1} \to Y_\nu$ and we get a tower of function field extensions:

$$\cdots \longleftarrow \mathbb{R}(Y_{\nu+1}) \longleftarrow \mathbb{R}(Y_\nu) \longleftarrow \cdots \longleftarrow \mathbb{R}(X).$$

Note that

$$\mathrm{Aut}_{\mathbb{R}(X)}(\mathbb{R}(Y_\nu)) \cong \mathrm{Aut}_X(Y_\nu) \cong \mathrm{Aut}_X(\widetilde{Y}_\nu) \cong G_\nu$$

and since $\#G_\nu =$ degree of the covering $(\widetilde{Y}_\nu \to X) = [\mathbb{R}(Y_\nu) : \mathbb{R}(X)]$, this shows that $\mathbb{R}(Y_\nu)$ is a normal extension of $\mathbb{R}(X)$. The fact that $Y_\nu \cong Y_{\mathbb{R}(Y_\nu)}$ is étale over X shows that $\mathbb{R}(Y_\nu)$ is isomorphic to a subfield of $\widetilde{\mathbb{R}(X)}$. Now choose an $\mathbb{R}(X)$-isomorphism:

$$\phi\colon \bigcup_{\nu=1}^{\infty} \mathbb{R}(Y_\nu) \longrightarrow \widetilde{\mathbb{R}(X)}.$$

It is easy to see that ϕ is surjective by going backwards from an étale $Y_K \to X$ to a topological covering $\widetilde{Y}_K \to X$ and dominating this by Ω. So we get the sought for isomorphism:

$$\begin{aligned}\widehat{\pi}_1^{\mathrm{top}} &\cong \varprojlim_\nu G_\nu\\ &\cong \varprojlim_\nu \mathrm{Gal}(\mathbb{R}(Y_\nu)/\mathbb{R}(X))\\ &\cong \mathrm{Gal}(\widetilde{\mathbb{R}(X)}/\mathbb{R}(X))\\ &\cong \pi_1^{\mathrm{alg}}.\end{aligned}$$

The only choice here is of ϕ and varying ϕ changes the above isomorphism by an inner automorphism. □

As a final topic I would like to discuss Grothendieck's *formal analog* of Serre's fundamental theorem. His result is this:

Let $R =$ noetherian ring, complete in the topology defined by the powers of an ideal I.

Let $X \longrightarrow \operatorname{Spec} R$ be a proper morphism.

Consider the schemes:

$$X_n = X \times_{\operatorname{Spec} R} \operatorname{Spec} R/I^{n+1}$$

i.e.,

$$\begin{array}{ccccccccc} X_0 & \subset & \cdots & \subset & X_n & \subset & \cdots\cdots & \subset & X \\ \downarrow & & & & \downarrow & & & & \downarrow \\ \operatorname{Spec} R/I & \subset & \cdots & \subset & \operatorname{Spec} R/I^{n+1} & \subset & \cdots\cdots & \subset & \operatorname{Spec} R \end{array}$$

Define: a *formal coherent sheaf* $\mathcal{F}$ on X is a set of coherent sheaves $\mathcal{F}_n$ on X_n plus isomorphisms:

$$\mathcal{F}_{n-1} \cong \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}.$$

Note that every coherent $\mathcal{F}$ on X induces a formal $\mathcal{F}_{\text{for}}$ by letting

$$\mathcal{F}_{\text{for},n} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}.$$

Then:

THEOREM 2.17 (Grothendieck). (Fundamental "GFGA"[8] comparison theorem)

i) *For every coherent algebraic $\mathcal{F}$ on X and every i,*

$$H^i(X, \mathcal{F}) \cong \varprojlim_n H^i(X_n, \mathcal{F}_n)$$

where $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.

ii) *The categories of formal and algebraic coherent sheaves are equivalent, i.e., every formal $\mathcal{F}'$ is isomorphic to $\mathcal{F}_{\text{for}}$, some $\mathcal{F}$, and*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \text{Formal } \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_{\text{for}}, \mathcal{G}_{\text{for}}).$$

The result for H^0 is essentially due to Zariski, whose famous [**108**] proving this and applying it to prove the connectedness theorem (see (V.6.3) Fundamental theorem of "holomorphic functions") started this whole development. A complete proof of Theorem 2.17 can be found in EGA [**1**, Chapter 3, §§4 and 5].[9] Here we will prove only the special case:

R complete local, $I =$ maximal ideal, $k = R/I$
X projective over $\operatorname{Spec} R$

(which suffices for most applications). If X is projective over $\operatorname{Spec} R$, we can embed X in $\mathbb{P}^m_R$ for some m, and extend all sheaves from X to $\mathbb{P}^m_R$ by (0): thus it suffices to prove Theorem 2.17 for $X = \mathbb{P}^m_R$.

Before beginning the proof, we need elementary results on the category of coherent formal sheaves. For details, we refer the reader to EGA [**1**, Chapter 0, §7 and Chapter 1, §10]; however none of these facts are very difficult and the reader should be able to supply proofs.

2.18. *If A is a noetherian ring, complete in its I-adic topology and $U_n = \operatorname{Spec} A/I^{n+1}$, then there is an equivalence of categories between:*

a) *sets of coherent sheaves $\mathcal{F}_n$ on U_n plus isomorphism*

$$\mathcal{F}_{n-1} \cong \mathcal{F}_n \otimes_{\mathcal{O}_{U_n}} \mathcal{O}_{U_{n-1}}$$

b) *finitely generated A-modules M*

given by:

$$M = \varprojlim_n \Gamma(U_n, \mathcal{F}_n)$$

$$\mathcal{F}_n = \widetilde{M/I^{n+1}M}.$$

[8] Short for "géométrie formelle et géométrie algébrique".

[9] (Added in publication) Illusie's account in FAG [**3**, Chapter 8] "provides an introduction, explaining the proofs of the key theorems, discussing typical applications, and updating when necessary."

In particular, Category (a) is abelian: [but kernel is *not* the usual sheaf-theoretic kernel because $M_1 \subset M_2$ does *not* imply $M_1/I^{n+1}M_1 \subset M_2/I^{n+1}M_2$!].

2.19. *Given A as above, and $f \in A$, then*

$$A_f = \varprojlim_n A_f/I^n \cdot A_f = \varprojlim_n (A/I^n A)_f$$

is flat over A.

Corollary 2.20. *The category of coherent formal sheaves $\{\mathcal{F}_n\}$ on a scheme X, proper over* $\operatorname{Spec} R$ *(R as above) is abelian with*

$$\operatorname{Coker}[\{\mathcal{F}_n\} \longrightarrow \{\mathcal{G}_n\}] = \{\operatorname{Coker}(\mathcal{F}_n \longrightarrow \mathcal{G}_n)\}_{n=0,1,\dots}$$

but

$$\operatorname{Ker}[\{\mathcal{F}_n\} \longrightarrow \{\mathcal{G}_n\}] = \{\mathcal{H}_n\}$$

where for each affine $U \subset X$:

$$\mathcal{H}_n(U) = \mathcal{H}(U)/I^{n+1}\mathcal{H}(U)$$

$$\mathcal{H}(U) = \operatorname{Ker}\left[\varprojlim_n \mathcal{F}_n(U) \longrightarrow \varprojlim_n \mathcal{G}_n(U)\right].$$

Proof of Corollary 2.20. Applying (2.18) with $A = \varprojlim_n \mathcal{O}_{X_n}(U)$ we construct kernels of $\{\mathcal{F}_n|_U\} \to \{\mathcal{G}_n|_U\}$ for each affine U as described above. Use (2.19) to check that on each distinguished open $U_f \subset U$, the restriction of the kernel on U is the kernel on U_f. □

2.21. *If A is any ring and*

$$0 \longrightarrow K_n \longrightarrow L_n \longrightarrow M_n \longrightarrow 0$$

are exact sequences of A-modules for each $n \geq 0$ fitting into an inverse system

$$\begin{array}{ccccccccc} 0 & \to & K_{n+1} & \to & L_{n+1} & \to & M_{n+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_n & \longrightarrow & L_n & \longrightarrow & M_n & \longrightarrow & 0 \end{array}$$

and if for each n, the decreasing set of submodules $\operatorname{Image}(K_{n+k} \to K_n)$ *of K_n is stationary for k large enough, then*

$$0 \longrightarrow \varprojlim_n K_n \longrightarrow \varprojlim_n L_n \longrightarrow \varprojlim_n M_n \longrightarrow 0$$

is exact.

Proof of Theorem 2.17. We now begin the proof of GFGA. To start off, say $\mathcal{F} = \{\mathcal{F}_n\}$ is a coherent formal sheaf on $\mathbb{P}^m_R$. Introduce

$$\begin{aligned} \operatorname{gr} R = \bigoplus_{n=0}^{\infty} I^n/I^{n+1} : &\quad \text{a finitely generated graded } k\text{-algebra} \\ S = \operatorname{Spec}(\operatorname{gr} R) : &\quad \text{an affine scheme of finite type over } k \\ \operatorname{gr} \mathcal{F} = \bigoplus_{n=0}^{\infty} I^n \cdot \mathcal{F}_n : &\quad \text{a quasi-coherent sheaf on } \mathbb{P}^m_k. \end{aligned}$$

Note that $\operatorname{gr} \mathcal{F}$ is in fact a sheaf of $(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}) \otimes \mathcal{O}_{\mathbb{P}^m_k}$-modules and since

$$I^n/I^{n+1} \otimes_k \mathcal{F}_0 \longrightarrow I^n \cdot \mathcal{F}_n$$

is surjective, $\operatorname{gr}\mathcal{F}$ is finitely generated as a sheaf of $(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}) \otimes \mathcal{O}_{\mathbb{P}^m_k}$-modules. In other words, we can form a *coherent* sheaf $\widetilde{\operatorname{gr}\mathcal{F}}$ on

$$\mathcal{S}pec_S\left(\left(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}\right) \otimes \mathcal{O}_{\mathbb{P}^m_k}\right) = \mathbb{P}^m_S.$$

Moreover,

$$H^q(\mathbb{P}^m_S, \widetilde{\operatorname{gr}\mathcal{F}}) = \bigoplus_{n=0}^{\infty} H^q(\mathbb{P}^m_k, I^n \cdot \mathcal{F}_n).$$

This same holds after twisting $\mathcal{F}$ by the standard invertible sheaf $\mathcal{O}(l)$, hence:

$$H^q(\mathbb{P}^m_S, \widetilde{\operatorname{gr}\mathcal{F}}(l)) = \bigoplus_{n=0}^{\infty} H^q(\mathbb{P}^m_k, I^n \cdot \mathcal{F}_n(l)).$$

But since $\operatorname{gr} R$ is a noetherian ring, the left hand side is (0) if $l \geq l_0$ (for some l_0) and $q \geq 1$. Thus:

$$H^q(\mathbb{P}^m_k, I^n \cdot \mathcal{F}_n(l)) = (0), \text{ if } q \geq 1,\ n \geq 0,\ l \geq l_0.$$

Now look at the exact sequences:

$$0 \longrightarrow I^n \cdot \mathcal{F}_n(l) \longrightarrow \mathcal{F}_n(l) \longrightarrow \mathcal{F}_{n-1}(l) \longrightarrow 0.$$

It follows from the cohomology sequences by induction on n that:

$$H^q(\mathbb{P}^m_R, \mathcal{F}_n(l)) = (0), \text{ if } q \geq 1,$$
$$\text{and } H^0(\mathbb{P}^m_R, \mathcal{F}_n(l)) \longrightarrow H^0(\mathbb{P}^m_R, \mathcal{F}_{n-1}(l)) \text{ surjective for all } n \geq 0,\ l \geq l_0.$$

The next step (like the third step of the GAGA Theorem 2.8) is that for some l_1 $\{\mathcal{F}_n(l)\}$ is generated by its sections for all $l \geq l_1$: i.e., there is a set of surjections:

$$\mathcal{O}^N_{\mathbb{P}^m_R}/I^{n+1} \cdot \mathcal{O}^N_{\mathbb{P}^m_R} \longrightarrow \mathcal{F}_n(l) \longrightarrow 0 \tag{2.22}$$

commuting with restriction from $n+1$ to n. To see this, take $l_1 \geq l_0$ so that $\mathcal{F}_0(l)$ is generated by its sections for $l \geq l_1$. This means there is a surjection:

$$\mathcal{O}^N_{\mathbb{P}^m_k} \longrightarrow \mathcal{F}_0(l) \longrightarrow 0.$$

By (2.21), this lifts successively to compatible surjections as in the third step of the GAGA Theorem 2.8. In other words, we have a surjection of formal coherent sheaves:

$$\mathcal{O}^N_{\mathbb{P}^m_R}(-l)_{\text{for}} \longrightarrow \{\mathcal{F}_n\}. \tag{2.23}$$

Next, as in the fourth step of the GAGA Theorem 2.8, we prove

$$\begin{aligned}\varprojlim_n H^0(\mathcal{O}_{\mathbb{P}^m_R}(l)/I^{n+1} \cdot \mathcal{O}_{\mathbb{P}^m_R}(l)) &\cong (R\text{-module of homogeneous forms of degree } l)\\ &\cong H^0(\mathcal{O}_{\mathbb{P}^m_R}(l)).\end{aligned}$$

This is obvious since $\mathcal{O}_{\mathbb{P}^m_R}(l)/I^{n+1} \cdot \mathcal{O}_{\mathbb{P}^m_R}(l)$ is just the structure sheaf of $\mathbb{P}^m_{R_n}$, where $R_n = R/I^{n+1} \cdot R$. Then the fifth step follows GAGA in Theorem 2.8 precisely: given $\{\mathcal{F}_n\}$, we take the kernel of Corollary 2.20 and repeat the construction, obtaining a presentation:

$$\mathcal{O}^{N_1}_{\mathbb{P}^m_R}(-l_1)_{\text{for}} \xrightarrow{\phi} \mathcal{O}^{N_0}_{\mathbb{P}^m_R}(-l_0)_{\text{for}} \longrightarrow \{\mathcal{F}_n\} \longrightarrow 0.$$

By the fourth step, ϕ is given by a matrix of homogeneous forms, hence we can form the *algebraic* coherent sheaf:

$$\mathcal{F} = \operatorname{Coker}\left[\phi\colon \mathcal{O}^{N_1}_{\mathbb{P}^m_R}(-l_1) \longrightarrow \mathcal{O}^{N_0}_{\mathbb{P}^m_R}(-l_0)\right]$$

and it follows immediately that $\mathcal{F}_n \cong \mathcal{F}/I^{n+1} \cdot \mathcal{F}$, i.e., $\{\mathcal{F}_n\} \cong \mathcal{F}_{\mathrm{for}}$.

The rest of the proof follows that of GAGA in Theorem 2.8 precisely with $H^q(\mathcal{F}_{\mathrm{an}})$ replaced by $\varprojlim_n H^q(\mathcal{F}/I^{n+1} \cdot \mathcal{F})$, once one checks that

$$\mathcal{F} \longmapsto \varprojlim_n H^q(\mathcal{F}/I^n \cdot \mathcal{F})$$

is a "cohomological δ-functor" of coherent algebraic sheaves $\mathcal{F}$, i.e., if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact, then one has a long exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim_n H^0(\mathcal{F}/I^n \cdot \mathcal{F}) \longrightarrow \varprojlim_n H^0(\mathcal{G}/I^n \cdot \mathcal{G}) \longrightarrow \varprojlim_n H^0(\mathcal{H}/I^n \cdot \mathcal{H}) \\ \xrightarrow{\delta} \varprojlim_n H^1(\mathcal{F}/I^n \cdot \mathcal{F}) \longrightarrow \cdots\cdots . \end{aligned}$$

But this follows by looking at the exact sequences:

$$0 \longrightarrow \mathcal{F}/(\mathcal{F} \cap I^n \cdot \mathcal{G}) \longrightarrow \mathcal{G}/I^n \cdot \mathcal{G} \longrightarrow \mathcal{H}/I^n \cdot \mathcal{H} \longrightarrow 0.$$

By (2.21), the cohomology groups

$$\begin{aligned} &\varprojlim_n H^q(\mathcal{F}/(\mathcal{F} \cap I^n \cdot \mathcal{G})) \\ &\varprojlim_n H^q(\mathcal{G}/I^n \cdot \mathcal{G}) \\ &\varprojlim_n H^q(\mathcal{H}/I^n \cdot \mathcal{H}) \end{aligned}$$

fit into a long exact sequence (since for each n, the n-th terms of these limits are finitely generated $(R/I^n \cdot R)$-modules, hence are of finite length). But by the Artin-Rees lemma (Zariski-Samuel [**109**, vol. II, Chapter VIII, §2, Theorem 4′, p. 255]), the sequence of subsheaves $\mathcal{F} \cap I^n \cdot \mathcal{G}$ of $\mathcal{F}$ is cofinal with the sequence of subsheaves $I^n \cdot \mathcal{F}$: in fact $\exists l$ such that for all $n \geq l$:

$$I^n \cdot \mathcal{F} \subset \mathcal{F} \cap I^n \cdot \mathcal{G} = I^{n-l} \cdot (\mathcal{F} \cap I^l \cdot \mathcal{G}) \subset I^{n-l} \cdot \mathcal{F}.$$

Therefore

$$\varprojlim_n H^q(\mathcal{F}/(\mathcal{F} \cap I^n \cdot \mathcal{G})) = \varprojlim_n H^q(\mathcal{F}/I^n \cdot \mathcal{F}).$$

□

COROLLARY 2.24. *Every formal closed subscheme Y_{for} of X (i.e., the set of closed subschemes $Y_n \subset X_n$ such that $Y_{n-1} = Y_n \times_{X_n} X_{n-1}$) is induced by a unique closed subscheme Y of X (i.e., $Y_n = Y \times_X X_n$).*

COROLLARY 2.25. *Every formal étale covering $\pi\colon Y_{\mathrm{for}} \to X$ (i.e., a set of coverings $\pi_n\colon Y_n \to X_n$ plus isomorphisms $Y_{n-1} \cong Y_n \times_{X_n} X_{n-1}$) is induced by a unique étale covering $\pi\colon Y \to X$ (i.e., $Y_n \cong Y \times_X X_n$).*

In fact, it turns out that an étale covering $\pi_0\colon Y_0 \to X_0$ already defines uniquely the whole formal covering, so that it follows that $\pi_1^{\mathrm{alg}}(X_0) \cong \pi_1^{\mathrm{alg}}(X)$: See Corollary 5.9 below.[10] Another remarkable fact is that the GAGA and GFGA comparison theorems are closer than it would

[10](Added in publication) See §5 for other applications of GFGA in connection with deformations (e.g., Theorem 5.5 on algebraization). See also Illusie's account in FAG [**3**, Chapter 8].

seem at first. In fact, if R is a complete discrete valuation ring with absolute value $|\ \ |$, note that for $\mathbb{A}^m_R$:

$$\begin{aligned}\varprojlim_n H^0(\mathbb{A}^m_{(R/I^{n+1})}, \mathcal{O}_{\mathbb{A}^m}) &= \varprojlim_n (R/I^{n+1})[X_1, \dots, X_m] \\ &\cong \text{ring of "convergent power series" } \sum c_\alpha X^\alpha \\ &\quad \text{where } c_\alpha \in R \text{ and } |c_\alpha| \to 0 \text{ as } |\alpha| \to \infty.\end{aligned}$$

This is the basis of a connection between the above formal geometry and a so-called "rigid" or "global" analytic geometry over the quotient field K of R. For an introduction to this, see Tate [**101**] and ?????.

Exercise.

(1) Let X be a normal irreducible noetherian scheme and let $L \supset \mathbb{R}(X)$ be a separable *Galois* extension such that the normalization Y_L of X in L is étale over X. Let $\pi\colon Y_L \to X$ be the canonical morphism. Let $G = \mathrm{Gal}(L/\mathbb{R}(X))$. Then G acts on Y_L over X: show that for all $y \in Y_L$, if $x = \pi(y)$, then:
   a) G acts transitively on $\pi^{-1}(x)$.
   b) If $G_y \subset G$ is the subgroup leaving y fixed, then G_y acts naturally on $\Bbbk(y)$ leaving $\Bbbk(x)$ fixed.
   c) $\Bbbk(y)$ is Galois over $\Bbbk(x)$ and, via the action in (b),
$$G_y \xrightarrow{\approx} \mathrm{Gal}(\Bbbk(y)/\Bbbk(x)).$$
[*Hint*: Let $n = [L\colon \mathbb{R}(X)]$. Using the fact that $L \otimes_{\mathbb{R}(X)} L \cong \overbrace{L \times \cdots \times L}^{n \text{ times}}$ and that $Y_L \times_X Y_L$ is normal, prove that $Y_L \times_X Y_L =$ disjoint union of n copies of Y_L. Prove that if G acts on $Y_L \times_X Y_L$ non-trivially on the first factor but trivially on the second, then it permutes these components simply transitively.]

(2) Note that the first part of the GFGA theorem (Theorem 2.17) would be trivial *if* the following were true:
$$\begin{aligned}&X \text{ a scheme over } \operatorname{Spec} A\\ &\mathcal{F} \text{ a quasi-coherent sheaf of } \mathcal{O}_X\text{-modules}\\ &B \text{ an } A\text{-algebra.}\end{aligned}$$
Then for all i, the canonical map
$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, \mathcal{F} \otimes_A B)$$
is an isomorphism. Show that if B is flat over A, this is correct.

(3) Using (2), deduce the more elementary form of GFGA:
$$\begin{gathered} f\colon Z \longrightarrow X \text{ proper, } X \text{ noetherian}\\ \mathcal{F} \text{ a coherent sheaf of } \mathcal{O}_X\text{-modules.}\end{gathered}$$
Then for all i, and for all $x \in X$,
$$\varprojlim_n R^i f_*(\mathcal{F})_x / \left(\mathfrak{m}_x^n \cdot R^i f_*(\mathcal{F})_x\right) \cong \varprojlim_n H^i(f^{-1}(x), \mathcal{F}/\mathfrak{m}_x^n \cdot \mathcal{F}).$$

3. De Rham cohomology

As in §2 we wish to work in this section only with varieties X over $\mathbb{C}$. For any such X, we have the topological space (X in the classical topology) and for any group G, we can consider the "constant sheaf G_X" on this:

$$G_X(U) = \left\{ \begin{array}{l} \text{functions } f\colon U \to G, \text{ constant on each} \\ \text{connected component of } U. \end{array} \right\}$$

It is a standard fact from algebraic topology (cf. for instance, Spanier [**98**, Chapter 6, §9]; or Warner [**104**]) that if a topological space Y is nice enough — e.g., if it is a finite simplicial complex — then the sheaf cohomology $H^i(Y, G_Y)$ and the singular cohomology computed by G-valued cochains on all singular simplices of Y as in Part I [**76**, §5C] are canonically isomorphic. One may call these the *classical cohomology groups of* Y. I would like in this part to indicate the basic connection between these groups for $G = \mathbb{C}$, and the coherent sheaf cohomology studied above. This connection is given by the ideas of De Rham already mentioned in Part I [**76**, §5C].

We begin with a completely general definition: let $f\colon X \to Y$ be a morphism of schemes. We have defined the Kähler differentials $\Omega_{X/Y}$ in Chapter V. We now go further and set:

$$\Omega^k_{X/Y} \underset{\text{def}}{=} \bigwedge^k(\Omega_{X/Y}), \text{ i.e., the sheafification of the pre-sheaf}$$

$$U \mapsto \bigwedge^k \text{ of the } \mathcal{O}_X(U)\text{-module } \Omega_{X/Y}(U).$$

One checks by the methods used above (e.g., $\boxed{\text{Ex. ????}}$) that this is quasi-coherent and that

$$\Omega^k_{X/Y}(U) = \bigwedge^k \text{ over } \mathcal{O}_X(U) \text{ of } \Omega_{X/Y}(U) \text{ for } U \text{ affine.}$$

In effect, this means that for U affine in X lying over V affine in Y:

$$\begin{aligned}
\Omega^k_{X/Y}(U) = &\text{free } \mathcal{O}_X(U)\text{-module on generators } dg_1 \wedge \cdots \wedge dg_k, \\
&(g_i \in \mathcal{O}_X(U)), \text{ modulo} \\
&\text{a) } d(g_1 + g_1') \wedge \cdots \wedge dg_k = dg_1 \wedge \cdots \wedge dg_k + dg_1' \wedge \cdots \wedge dg_k \\
&\text{b) } d(g_1 g_1') \wedge \cdots \wedge dg_k = g_1 dg_1' \wedge \cdots \wedge dg_k + g_1' dg_1 \wedge \cdots \wedge dg_k \\
&\text{c) } dg_{\epsilon 1} \wedge \cdots \wedge dg_{\epsilon k} = \operatorname{sgn}(\epsilon) \cdot dg_1 \wedge \cdots \wedge dg_k \ (\epsilon = \text{permutation}) \\
&\text{d) } dg_1 \wedge dg_2 \wedge \cdots \wedge dg_k = 0 \text{ if } g_1 = g_2 \\
&\text{d) } dg_1 \wedge \cdots \wedge dg_k = 0 \text{ if } g_1 \in \mathcal{O}_Y(V).
\end{aligned}$$

The derivation $d\colon \mathcal{O}_X \to \Omega_{X/Y}$ extends to maps:

$$d\colon \Omega^k_{X/Y} \longrightarrow \Omega^{k+1}_{X/Y} \qquad (\textit{not } \mathcal{O}_X\text{-linear})$$

given on affine U by:

$$d(f dg_1 \wedge \cdots \wedge dg_k) = df \wedge dg_1 \wedge \cdots \wedge dg_k, \quad f, g_i \in \mathcal{O}_X(U).$$

(Check that this is compatible with relations (a)–(e) on Ω^k and Ω^{k+1}, hence d is well-defined.) It follows immediately from the definition that $d^2 = 0$, i.e.,

$$\Omega^\bullet_{X/Y}\colon 0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \xrightarrow{d} \cdots$$

is a *complex*. Therefore as in §VII.3 we may define the hypercohomology $\mathbb{H}^i(X, \Omega^\bullet_{X/Y})$ of this complex, which is known as the De Rham cohomology $H^i_{\mathrm{DR}}(X/Y)$ of X over Y. Grothendieck

[**41**], putting together more subtly earlier ideas of Serre, Atiyah and Hodge with Hironaka's resolution theorems [**55**] has proven the very beautiful:

THEOREM 3.1 (De Rham comparison theorem). *If X is a variety smooth (but not necessarily proper) over $\mathbb{C}$, then there is a canonical isomorphism:*

$$H^i_{\mathrm{DR}}(X/\mathbb{C}) \cong H^i((X \textit{ in the classical topology}), \mathbb{C}_X).$$

We will only prove this for projective X referring the reader to Grothendieck's elegant paper [**41**] for the general case. Combining Theorem 3.1 with the spectral sequence of hypercohomology gives:

COROLLARY 3.2. *There is a spectral sequence with*

$$E_1^{pq} = H^q(X, \Omega^p_{X/\mathbb{C}})$$

and d_1^{pq} being induced by $d\colon \Omega^p \to \Omega^{p+1}$ abutting to $H^\nu((X \textit{ in the classical topology}), \mathbb{C})$. In particular, if X is affine, then

$$\frac{\{\textit{closed } \nu\textit{-forms}\}}{\{\textit{exact } \nu\textit{-forms}\}} \cong H^\nu((X \textit{ in the classical topology}), \mathbb{C}).$$

To prove the theorem in the projective case, we must simply combine the GAGA comparison theorem (Theorem 2.8) with the so-called Poincaré lemma on analytic differentials. First we recall the basic facts about analytic differentials. If X is an n-dimensional complex manifold, then the tangent bundle T_X has a structure of a rank n complex analytic vector bundle over X, i.e.,

$$T_X \cong \{(P, D) \mid P \in X,\ D\colon (\mathcal{O}_{X,\mathrm{an}})_P \to \mathbb{C} \text{ a derivation over } \mathbb{C} \text{ centered at } P\}$$

(cf. Part I [**76**, SS1A, 5C, 6B]). Thus if $U \subset X$ is an open set with analytic coordinates $z_1, \ldots, z_n$, then the inverse image of U in T_X has coordinates

$$(P, D) \longmapsto (z_1(P), \ldots, z_n(P), D(z_1), \ldots, D(z_n))$$

under which it is analytically isomorphic to $U \times \mathbb{C}^n$. We then define the sheaves $\Omega^p_{X,\mathrm{an}}$ of holomorphic p-forms by:

$$\Omega^p_{X,\mathrm{an}}(U) = \{\text{holomorphic sections over } U \text{ of the complex vector bundle } \textstyle\bigwedge^p(T_X^*)\}.$$

(Here $E^* = \mathrm{Hom}(E, \mathbb{C})$ is the dual bundle.) Locally such a section ω is written as usual by an expression

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} c_{i_1,\ldots,i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad c_{i_1,\ldots,i_p} \in \mathcal{O}_{X,\mathrm{an}}(U),$$

and we get the first order differential operators:

$$d\colon \Omega^p_{X,\mathrm{an}} \longrightarrow \Omega^{p+1}_{X,\mathrm{an}}$$

given by

$$d\omega = \sum_{1 \le i_1 < \cdots < i_{p+1} \le n} \sum_{k=1}^{p+1} (-1)^{k+1} \frac{\partial c_{i_1,\ldots,\widehat{i_k},\ldots,i_{p+1}}}{\partial z_{i_k}} dz_{i_1} \wedge \cdots \wedge dz_{i_{p+1}}.$$

The map $(\omega, \eta) \mapsto \omega \wedge \eta$ makes $\bigoplus_p \Omega^p_{X,\mathrm{an}}$ into a skew-commutative algebra in which d is a derivation.

LEMMA 3.3 (Poincaré's lemma). *The sequence of sheaves:*

$$0 \longrightarrow \mathcal{O}_{X,\mathrm{an}} \xrightarrow{d} \Omega^1_{X,\mathrm{an}} \xrightarrow{d} \Omega^2_{X,\mathrm{an}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X,\mathrm{an}} \longrightarrow 0$$

is exact, except at the 0*-th place where* $\mathrm{Ker}(d\colon \mathcal{O}_{X,\mathrm{an}} \to \Omega^1_{X,\mathrm{an}})$ *is the sheaf of constant functions* $\mathbb{C}_X$*.*

For an elementary proof of this see Hartshorne [**51**, Remark after Proposition (7.1), p. 54]. (See also Wells [**105**, Chapter II, §2, Example 2.13, p. 49] as well as the proof of the Delbeault Lemma in Gunning-Rossi [**48**, Chapter I, §D, 3.Theorem, p. 27].) Now if X is a variety smooth over $\mathbb{C}$, an essential point to check is that the general functor $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{an}}$ of §2 takes the Kähler p-forms $\Omega^p_{X/\mathbb{C}}$ to the above-defined sheaf of holomorphic p-forms $\Omega^p_{X,\mathrm{an}}$. This is virtually a tautology but to tie things together properly, we can proceed like this. For the sake of this argument, we write $\Omega^p_{X,\mathrm{alg}}$ for Kähler differentials on the scheme X, parallel to $\Omega^p_{X,\mathrm{an}}$ defined above:

a) For all $U \subset X$ affine, $\bigoplus_p \Omega^p_{X,\mathrm{alg}}(U)$ is the universal skew-commutative $\mathcal{O}_X(U)$-algebra with derivation (i.e., the free algebra on elements df, $f \in \mathcal{O}_X(U)$, modulo the standard identities); since $\bigoplus_p \Omega^p_{X,\mathrm{an}}(U)$ is another skew-commutative algebra with derivation over $\mathcal{O}_X(U)$ (via the inclusion $\mathcal{O}_X(U) \subset \mathcal{O}_{X,\mathrm{an}}(U)$), there is a unique collection of maps:
$$\Omega^p_{X,\mathrm{alg}}(U) \longrightarrow \Omega^p_{X,\mathrm{an}}(U)$$
commuting with $\wedge$ and d.

b) From a general sheaf theory argument, such a collection of maps factors through a map of sheaves of $\mathcal{O}_{X,\mathrm{an}}$-modules (on X in the classical topology):
$$\left(\Omega^p_{X,\mathrm{alg}}\right)_{\mathrm{an}} \longrightarrow \Omega^p_{X,\mathrm{an}}.$$

c) If $z_1, \ldots, z_n \in \mathfrak{m}_{X,x}$ induce a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$, then we have:
$$\left(\Omega_{X,\mathrm{alg}}\right)_x \cong \bigoplus_{i=1}^{n} \mathcal{O}_{X,x} \cdot dz_i$$
hence
$$\left(\Omega^p_{X,\mathrm{alg}}\right)_x \cong \bigoplus_{1 \le i_i < \cdots < i_p \le n} \mathcal{O}_{X,x} \cdot dz_{i_1} \wedge \cdots \wedge dz_{i_p},$$
hence
$$\left(\Omega^p_{X,\mathrm{alg}}\right)_{\mathrm{an},x} \cong \bigoplus_{1 \le i_1 < \cdots < i_p \le n} \left(\mathcal{O}_{X,\mathrm{an}}\right)_x dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

While $z_1, \ldots, z_n$ are local analytic coordinates near x, so
$$\left(\Omega^p_{X,\mathrm{an}}\right)_x \cong \bigoplus_{1 \le i_1 < \cdots < i_p \le n} \left(\mathcal{O}_{X,\mathrm{an}}\right)_x dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$
too! So we have the following situation: with respect to the identity map
$$\epsilon\colon (X \text{ in the classical topology}) \longrightarrow (X \text{ in the Zariski topology})$$
we have a map backwards from the De Rham complex $(\Omega^{\bullet}_X, d)$ of the scheme X to the analytic De Rham complex $(\Omega^{\bullet}_{X,\mathrm{an}}, d)$ of the analytic manifold X. This induces:

a) a map of hypercohomology
$$\mathbb{H}^i(X \text{ in the Zariski topology}, \Omega^{\bullet}_{X/\mathbb{C}}) \longrightarrow \mathbb{H}^i(X \text{ in the classical topology}, \Omega^{\bullet}_{X,\mathrm{an}})$$

and

b) a map of the spectral sequences abutting to these too:

$$\begin{array}{ccc} \text{algebraic } E_1^{pq} & \longrightarrow & \text{analytic } E_1^{pq} \\ \| & & \| \\ H^q(X \text{ in the Zariski topology}, \Omega^p_{X/\mathbb{C}}) & & H^q(X \text{ in the classical topology}, \Omega^p_{X,\mathrm{an}}). \end{array}$$

But by the GAGA comparison theorem (Theorem 2.8), the map on E_1^{pq}'s is an isomorphism. Now quite generally, if

$$E_2^{pq} \Longrightarrow E^\nu$$
$$\widetilde{E}_2^{pq} \Longrightarrow \widetilde{E}^\nu$$

are two spectral sequences, and

$$\phi^{p,q} \colon E_2^{pq} \longrightarrow \widetilde{E}_2^{pq}$$
$$\phi^\nu \colon E^\nu \longrightarrow \widetilde{E}^\nu$$

are homomorphisms "compatible with the spectral sequences", i.e., commuting with the d's, taking $F^l(E^\nu)$ into $F^l(\widetilde{E}^\nu)$ and commuting with the isomorphisms of E_∞^{pq} with $F^p(E^{p+q})/F^{p+1}(E^{p+q})$, then it follows immediately that

$$\phi^{p,q} \text{ isomorphisms, all } p,q \Longrightarrow \phi^\nu \text{ isomorphisms, all } \nu.$$

In our case, this means that the map in (a) is an isomorphism.

Now compute $\mathbb{H}^\nu(X \text{ in the classical topology}, \Omega^{\boldsymbol{\cdot}}_{X,\mathrm{an}})$ by the second spectral sequence of hypercohomology (cf. (VII.3.11)). Since X in its classical topology is paracompact Hausdorff, we get (cf. §VII.1)

$$H^p\left(X \text{ in the classical topology, sheaf } \frac{\mathrm{Ker}\left(d\colon \Omega^q_{\mathrm{an}} \to \Omega^{q+1}_{\mathrm{an}}\right)}{\mathrm{Image}\left(d\colon \Omega^{q-1}_{\mathrm{an}} \to \Omega^q_{\mathrm{an}}\right)}\right) \Longrightarrow \mathbb{H}^\nu(X, \Omega^{\boldsymbol{\cdot}}_{X,\mathrm{an}}).$$

By Poincaré's lemma (Lemma 3.3), all but one of these sheaves are (0) and the spectral sequence degenerates to an isomorphism:

$$H^\nu(X \text{ in the classical topology}, \mathbb{C}_X) \cong \mathbb{H}^\nu(X \text{ in the classical topology}, \Omega^{\boldsymbol{\cdot}}_{X,\mathrm{an}}).$$

This proves Theorem 3.1 in the projective case.

In the projective case and more generally for any complete variety X, the spectral sequence of Corollary 3.2:

$$E_1^{pq} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Longrightarrow H^\nu(X \text{ in the classical topology}, \mathbb{C})$$

simplifies quite remarkably. In fact the Theory of Hodge implies:

Fact. I: *All d_r^{pq}'s are* 0.

This implies that

$$H^q(X, \Omega^p_{X/\mathbb{C}}) \cong E_\infty^{pq} \cong p\text{-th graded piece: } F^p(H^{p+q})/F^{p+1}(H^{p+q}) \text{ of } H^{p+q}(X, \mathbb{C}).$$

Note that $H^{p+q}(X,\mathbb{C}) \cong H^{p+q}(X,\mathbb{Z}) \otimes \mathbb{C}$, hence there is a natural complex conjugation $x \mapsto \overline{x}$ on $H^{p+q}(X,\mathbb{C})$.

Fact. II: *In $H^{p+q}(X,\mathbb{C})$, $\overline{F^{q+1}(H^{p+q})}$ is a complement to the subspace $F^p(H^{p+q})$.*

This implies that H^{p+q} splits canonically into a direct sum:

$$H^\nu(X,\mathbb{C}) = \bigoplus_{\substack{p+q=\nu \\ p,q\geq 0}} H^{p,q}$$

such that

a) $H^{q,p} = \overline{H^{p,q}}$.
b) $F^p(H^{p+q}) = \bigoplus_{p'\geq p} H^{p',q'}$.

Combining both facts,

$$H^{p,q} \cong H^q(X, \Omega^p_{X/\mathbb{C}})$$

hence

$$H^\nu(X,\mathbb{C}) \cong \bigoplus_{p+q=\nu} H^q(X,\Omega^p_{X/\mathbb{C}}). \tag{3.4}$$

Fact. III: *If we calculate $H^\nu(X,\mathbb{C})$ by C^∞ differential forms, then*

$$H^{p,q} \cong \left\{ \begin{array}{l} \text{set of cohomology classes representable by forms } \omega \\ \text{of type } (p,q), \text{ i.e., in local coordinates } z_1,\dots,z_n, \\ \omega = \displaystyle\sum_{\substack{1\leq i_1<\cdots<i_p\leq n \\ 1\leq j_1<\ldots<j_q\leq n}} c_{i_1,\dots,i_p,j_1,\dots,j_q} dz_{i_1}\wedge\cdots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q} \end{array} \right\}.$$

(See Kodaira-Morrow [**64**] or ?????).

4. Characteristic p phenomena

The theory of De Rham cohomology in characteristic p is still in its infancy[11] and rather than trying to discuss the situation at all generality, I would like instead to fix on one of the really new features of characteristic p and discuss this: namely the *Hasse-Witt matrix*. To set the stage, if X is a complete non-singular variety over a field k of characteristic $p > 0$, then the De Rham groups

$$\mathbb{H}^\nu(X, \Omega^{\cdot}_{X/k})$$

are finite-dimensional k-vector spaces which usually behave quite like their counterparts in characteristic 0 and are "reasonable" candidates for the cohomology of X with coefficients in k[12].

For instance, if X is a complete non-singular curve of genus g, then $\dim H^1(\mathcal{O}_X) = \dim H^0(\Omega^1) = g$, $\dim \mathbb{H}^1(\Omega^{\cdot}_X) = 2g$ in all characteristics. However the De Rham groups have a much richer structure in characteristic p even in the case of curves. The simplest examples of this are the cohomology operations:

$$F\colon H^n(X,\mathcal{O}_X) \longrightarrow H^n(X,\mathcal{O}_X),$$
$$X = \text{any scheme in which } p\cdot\mathcal{O}_X \equiv 0$$

given by

$$F(\{a_{i_0,\dots,i_n}\}) = \{a^p_{i_0,\dots,i_n}\}$$

[11] (Added in publication) There have been considerable advances, since the original manuscript was written. See the footnote at the end of this section.

[12] There are some cases where their dimension is larger than the expected n-th Betti number B_n and there are also cases where the spectral sequence

$$H^q(\Omega^p) \Longrightarrow \mathbb{H}^n(\Omega^{\cdot})$$

does not degenerate: cf. ???, ???, ???. This is apparently connected with the presence of p-torsion on X. And if X is affine instead of complete, these groups are not even finite-dimensional.

on the cocycle level. Note that if X is a scheme over k, $\operatorname{char} k = p$, so that $H^\nu(X, \mathcal{O}_X)$ is a k-vector space, then F is not k-linear; in fact $F(\alpha \cdot x) = \alpha^p \cdot F(x)$, $\forall \alpha \in k,\ x \in H^\nu(\mathcal{O}_X)$. Such a map we call p-linear. Expanded in terms of a basis of $H^\nu(\mathcal{O}_X)$, F is given by a matrix which is called the ν-th Hasse-Witt matirix of X. p-linear maps do *not* have eigenvalues; instead they have the following canonical form:

LEMMA 4.1. *Let k be an algebraically closed field of characteristic p, let V be a finite-dimensional vector space over k and let $T\colon V \to V$ be a p-linear transformation. Then V has a unique decomposition:*

$$V = V_s \oplus V_n$$

where

a) *$T(V_n) \subset V_n$ and T is nilpotent on V_n.*
b) *$T(V_s) \subset V_s$ and V_s has a basis $e_1, \ldots, e_n$ such that $T(e_i) = e_i$. Furthermore,*

$$\{e \in V_s \mid Te = e\} = \{\sum m_i e_i \mid m_i \in \mathbb{Z}/p\mathbb{Z}\}.$$

PROOF. Let $V_s = \bigcap_{\nu=1}^\infty \operatorname{Image} T^\nu$ and $V_n = \bigcup_{\nu=1}^\infty \operatorname{Ker} T^\nu$. Since $\dim V < +\infty$, $V_s = \operatorname{Image} T^\nu$, $V_n = \operatorname{Ker} T^\nu$ for $\nu \gg 0$. Now if $\nu \gg 0$:

$$\begin{aligned} x \in V_s \cap V_n &\Longrightarrow T^\nu x = 0 \text{ and } x = T^\nu y \\ &\Longrightarrow T^{2\nu} y = 0 \\ &\Longrightarrow T^\nu y = 0 \\ &\Longrightarrow x = 0 \end{aligned}$$

and since $\dim V = \dim \operatorname{Ker} T^\nu + \dim \operatorname{Image} T^\nu$, it follows that $V \cong V_s \oplus V_n$. Then $T|_{V_n}$ is nilpotent and $T|_{V_s}$ is bijective. Now choose $x \in V_s$ and take ν minimal such that there is a relation

$$T^\nu x = a_0 x + a_1 T(x) + \cdots + a_{\nu-1} T^{\nu-1}(x).$$

If $a_0 = 0$, then

$$T^{\nu-1} x = a'_1 x + \cdots + a'_{\nu-1} T^{\nu-2}(x)$$

and ν would not be minimal. Now try to solve the equation:

$$T(\lambda_0 x + \cdots + \lambda_{\nu-1} T^{\nu-1}(x)) = \lambda_0 x + \cdots + \lambda_{\nu-1} T^{\nu-1}(x).$$

This leads to

$$\begin{aligned} \lambda_{\nu-1}^p \cdot a_0 &= \lambda_0 \\ \lambda_0^p + \lambda_{\nu-1}^p \cdot a_1 &= \lambda_1 \\ &\cdots\cdots \\ \lambda_{\nu-2}^p + \lambda_{\nu-1}^p \cdot a_{\nu-1} &= \lambda_{\nu-1}. \end{aligned}$$

By substitution, we get:

$$\lambda_{\nu-1}^{p^\nu} \cdot a_0^{p^{\nu-1}} + \lambda_{\nu-1}^{p^{\nu-1}} \cdot a_1^{p^{\nu-2}} + \cdots + \lambda_{\nu-1}^p \cdot a_{\nu-1} - \lambda_{\nu-1} = 0$$

which has a non-zero solution. Solving backwards, we find $\lambda_{\nu-2}, \ldots, \lambda_0$ as required, hence an $x \in V_s$ with $Tx = x$. Now take a maximal independent set of solutions $e_1, \ldots, e_l$ to the equation $Tx = x$. If $W = \sum k \cdot e_i$, then $T\colon W \to W$ is bijective, hence $T\colon V_s/W \to V_s/W$ is also bijective. If $W \subsetneqq V_s$, the argument above then shows $\exists\, \overline{x} \in V_s/W$ such that $T\overline{x} = \overline{x}$. Lifting $\overline{x}$ to $x \in V_s$, we find

$$Tx = x + \sum \lambda_i e_i.$$

Let $\mu_i \in k$ satisfy $\mu_i^p - \mu_i = \lambda_i$. Then $e_{l+1} = x - \sum \mu_i e_i$ also lifts $\overline{x}$ but it satisfies $Te_{l+1} = e_{l+1}$. This proves that e_i span V_s. $\square$

We can apply this decomposition in particular to $H^1(X, \mathcal{O}_X)$ and we find the following interpretations of the eigenvectors:

THEOREM 4.2. *Let X be a complete variety over an algebraically closed field k of characteristic p. Consider F acting on $H^1(X, \mathcal{O}_X)$. Then:*

a) *There is a one-to-one correspondence between $\{\alpha \in H^1(\mathcal{O}_X) \mid F\alpha = \alpha\}$ and pairs (π, ϕ):*

$$\begin{tikzcd} Y \arrow[loop right, "\phi"] \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

*π étale, proper, $\pi \circ \phi = \pi$, $\phi^p = 1_Y$, such that $\forall x \in X$ closed, $\#\pi^{-1}(x) = p$ and ϕ permutes these points cyclically: we call this, for short, a p-*cyclic étale convering[13].

b) *If X is non-singular, there is an isomorphism:*

$$\{a \in H^1(\mathcal{O}_X) \mid F\alpha = 0\} \cong \{\omega \in H^0(\Omega^1_X) \mid \omega = df, \ \textit{some } f \in \mathbb{R}(X)\}.$$

PROOF. (a) Given α with $F\alpha = \alpha$, represent α by a cocycle $\{f_{ij}\}$. Then $F\alpha$ is represented by $\{f_{ij}^p\}$ and since this is cohomologous to α:

$$\begin{aligned} f_{ij} &= f_{ij}^p + g_i - g_j \\ g_i &\in \mathcal{O}_X(U_i). \end{aligned}$$

But then define a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras by:

$$\begin{aligned} \mathcal{A}|_{U_i} &= \mathcal{O}_X[t_i]/(f_i) \\ f_i(t_i) &= t_i^p - t_i + g_i \end{aligned}$$

and by the glueing:

$$t_i = t_j + f_{ij}$$

over $U_i \cap U_j$. Let $Y_\alpha = \mathcal{S}pec_X(\mathcal{A})$. Since $(df_i/dt_i)(t_i) = -1$, Y_α is étale over X. Since $\mathcal{A}$ is integral and finitely generated over $\mathcal{O}_X$, Y_α is proper over X (cf. Corollary II.6.7). Define $\phi_\alpha \colon Y_\alpha \to Y_\alpha$ by

$$\phi_\alpha^*(t_i) = t_i + 1.$$

For all closed points $x \in U_i$, let a be one solution of $t_i^p - t_i + g_i(x) = 0$. Then $\pi^{-1}(x)$ consists of the p points $t_i = a, a+1, \ldots, a+p-1$ which are permuted cyclically by ϕ_α. Finally, and this is where we use the completeness of X, note that Y_α depends only on α:

$$\begin{aligned} \text{if} \quad f'_{ij} &= f_{ij} + h_i - h_j \\ \text{and} \quad f'_{ij} &= (f'_{ij})^p + g'_i - g'_j \end{aligned}$$

is another solution to the above requirements, then

$$\begin{aligned} g'_i - g'_j &= f'_{ij} - (f'_{ij})^p \\ &= f_{ij} - f_{ij}^p + h_i - h_j - (h_i - h_j)^p \\ &= (g_i + h_i - h_i^p) - (g_j + h_j - h_j^p), \end{aligned}$$

hence

$$g'_i = g_i + h_i - h_i^p + \xi, \qquad \xi \in \Gamma(\mathcal{O}_X).$$

[13]Compare the statement of (a) and the proof with provisionally, Exercise (5) after §III.6, which treats p-cyclic coverings for $p \neq \operatorname{char} k$.

Thus $\xi \in k$, hence $\xi = \eta - \eta^p$ for some $\eta \in k$ and we get an isomorphism

$$\begin{aligned} \mathcal{O}_X[t_i]/(f_i^p - t_i + g_i) &\xrightarrow{\approx} \mathcal{O}_X[t_i']/((t_i')^p - t_i' + g_i') \\ t_i &\longmapsto t_i' - (h_i + \eta). \end{aligned}$$

We leave it to the reader to check that $(Y_\alpha, \phi_\alpha) \cong (Y_\beta, \phi_\beta)$ only if $\alpha = \beta$.

Conversely, suppose $\pi\colon Y \to X$ and $\phi\colon Y \to Y$ are a p-cyclic étale covering. By Proposition II.6.5, $Y = \mathcal{S}pec_X\, \mathcal{A}$, $\mathcal{A}$ a coherent sheaf of $\mathcal{O}_X$-algebras.

Now π étale $\implies$ π flat $\implies$ $\mathcal{A}_x$ is a flat $\mathcal{O}_{x,X}$-module. Now a finitely presented flat module over a local ring is free (cf. Bourbaki [**26**, Chapter II, §3.2]), hence $\mathcal{A}_x$ is a free $\mathcal{O}_{x,X}$-module. In fact

$$\mathcal{A}_x/\mathfrak{m}_{x,X} \cdot \mathcal{A}_x \cong \Gamma(\mathcal{O}_{\pi^{-1}(x)}) \cong \bigoplus_{y \in \pi^{-1}(x)} \Bbbk(y)$$

so $\mathcal{A}_x$ is free of rank p, and the function $1 \in \mathcal{A}_x$, since it is not in $\mathfrak{m}_{x,X} \cdot \mathcal{A}_x$, may be taken as a part of a basis. Moreover, ϕ induces an automorphism $\phi^*\colon \mathcal{A} \to \mathcal{A}$ in terms of which we can characterize the subsheaf $\mathcal{O}_X \subset \mathcal{A}$:

$$\mathcal{O}_X(U) = \{f \in \mathcal{A}(U) \mid \phi^* f = f\}.$$

In fact for all closed points $x \in X$, we get an inclusion:

$$\begin{array}{ccc} \mathcal{O}_{x,X}/\mathfrak{m}_{x,X} & \hookrightarrow & \mathcal{A}_x/\mathfrak{m}_{x,X} \cdot \mathcal{A}_x \\ \| & & \| \\ \Bbbk(x) & & \Gamma(\mathcal{O}_{\pi^{-1}(x)}) \\ & & \| \\ & & \bigoplus_{y \in \pi^{-1}(x)} \Bbbk(y) \end{array}$$

and clearly $\Bbbk(x)$ is characterized as the set of ϕ^*-invariant functions in $\bigoplus_{y \in \pi^{-1}(x)} \Bbbk(y)$. So if U is affine and $f \in \mathcal{A}(U)$ is ϕ^*-invariant, then

$$(*) \qquad f \in \bigcap_{\substack{x \in U \\ \text{closed}}} [\mathcal{O}_X(U) + \mathfrak{m}_{x,X} \cdot \mathcal{A}(U)].$$

But if U is small enough, $\mathcal{A}|_U$ has a free basis:

$$\mathcal{A}|_U = \mathcal{O}_X|_U \oplus \sum_{i=2}^{p} \mathcal{O}_X|_U \cdot e_i$$

and if we expand $f = f_1 + \sum_{i=2}^p f_i \cdot e_i$, then $(*)$ means that $f_i(x) = 0$, $2 \le i \le p$, $\forall x \in U$ closed. By the Nullstellensatz, $f_i = 0$, hence $f \in \mathcal{O}_X(U)$. Let $x \in X$ be a closed point and let $\pi^{-1}(x) = \{y, \phi y, \ldots, \phi^{p-1} y\}$. We can find a function $e_x \in \mathcal{A}_x$ such that $e_x(y) = 1$, $e_x(\phi^i y) = 0$, $1 \le i \le p-1$. Then

- $\operatorname{Tr} e_x = \sum_{i=0}^{p-1} (\phi^i)^* e_x$ satisfies $\phi^*(\operatorname{Tr} e_x) = \operatorname{Tr} e_x$ and has value 1 at all points of $\pi^{-1}(x)$, hence is invertible in $\mathcal{A}_x$.
- Set

$$f_x = \frac{-1}{\operatorname{Tr} e_x} \cdot \sum_{i=0}^{p-1} i \cdot (\phi^i)^* e_x.$$

A small calculation shows that $\phi^*(f_x) = f_x + 1$ and $f_x(\phi^i y) = i$. Let $g_x = f_x - f_x^p$. Then $\phi^* g_x = g_x$, hence $g_x \in \mathcal{O}_{x,X}$. Define a homomorphism

$$\begin{aligned} \lambda_x \colon \mathcal{O}_{x,X}[t_x]/(t_x^p - t_x + g_x) &\longrightarrow \mathcal{A}_x \\ t_x &\longmapsto f_x. \end{aligned}$$

Note that since f_x has *distinct* values at all points $\phi^i y$, f_x generates

$$\mathcal{A}_x/\mathfrak{m}_{x,X} \cdot \mathcal{A}_x \cong \bigoplus_{i=0}^{p-1} \Bbbk(\phi^i y),$$

hence by Nakayama's lemma, λ_x is surjective. But as λ_x is a homomorphism of free $\mathcal{O}_{x,X}$-modules of rank p, it must be injective too. Now λ_x extends to an isomorphism in some neighborhood of x and covering X by such neighborhoods, we conclude that X has a covering $\{U_i\}$ such that

$$\mathcal{A}|_{U_i} \cong \mathcal{O}_X|_{U_i}[t_i]/(t_i^p - t_i + g_i), \quad g_i \in \mathcal{O}_X(U_i).$$

Over $U_i \cap U_j$, $\phi^*(t_i - t_j) = t_i - t_j$, hence $t_i = t_j + f_{ij}$, $f_{ij} \in \mathcal{O}_X(U_i \cap U_j)$. Then

$$\begin{aligned} f_{ij} - f_{ij}^p &= (t_i - t_j) - (t_i - t_j)^p \\ &= g_i - g_j, \end{aligned}$$

so $\alpha = \{f_{ij}\}$ is a cohomology class in $\mathcal{O}_X$ such that $F\alpha = \alpha$. This completes the proof of (a).

(b) Given α with $F\alpha = 0$, represent α by a cocycle $\{f_{ij}\}$. Then

$$\begin{aligned} f_{ij}^p &= g_i - g_j \\ g_i &\in \mathcal{O}_X(U_i) \end{aligned}$$

hence $dg_i = dg_j$ on $U_i \cap U_j$. Therefore the dg_i's define a global section ω_α of Ω_X^1 of the form df, $f \in \mathbb{R}(X)$. If

$$\begin{aligned} f'_{ij} &= f_{ij} + h_i - h_j \\ (f'_{ij})^p &= g'_i - g'_j \end{aligned}$$

is another solution to the above requirements, then

$$\begin{aligned} g'_i - g'_j &= (f'_{ij})^p \\ &= f_{ij}^p + h_i^p - h_j^p \\ &= (g_i + h_i^p) - (g_j + j_j^p) \end{aligned}$$

hence

$$g'_i = g_i + h_i^p + \xi, \qquad \xi \in \Gamma(\mathcal{O}_X) = k.$$

Thus $dg'_i = dg_i$ and ω_α depends only on α. Conversely, if we are given $\omega \in \Gamma(\Omega_X^1)$, $\omega = df$, $f \in \mathbb{R}(X)$, the first step is to show that for all $x \in X$, $\omega = df_x$ for some $f_x \in \mathcal{O}_{x,X}$ too. We use the following important lemma:

LEMMA 4.3. *Let X be a non-singular n-dimensional variety over an algebraically closed field k, and assume $\exists z_1, \ldots, z_n \in \Gamma(\mathcal{O}_X)$ such that*

$$\Omega_{X/k}^1 \cong \bigoplus_{i=1}^n \mathcal{O}_X \cdot dz_i.$$

Consider $\mathcal{O}_X$ as a sheaf of $\mathcal{O}_X^p$-modules: $\mathcal{O}_X$ is a free $\mathcal{O}_X^p$-module with basis consisting of monomials $\prod_{i=1}^n z_i^{\alpha_i}$, $0 \le \alpha \le p-1$.

Another way to view this is to consider the pair $Y = (X, \mathcal{O}_X^p)$ consisting of the topological space X and the sheaf of rings $\mathcal{O}_X^p$: this is a scheme too, in fact it is isomorphic to X as scheme — but not as scheme over k — via:

$$\text{identity}\colon X \xrightarrow{\approx} Y$$

$$p\text{-th power}\colon \mathcal{O}_X \xrightarrow{\approx} \mathcal{O}_X^p = \mathcal{O}_Y.$$

Thus Y is in fact an irreducible regular scheme, ant it is of finite type over k, i.e., a non-singular k-variety. Now

$$\text{identity}\colon X \xrightarrow{\approx} X$$

$$\text{inclusion}\colon \mathcal{O}_Y = \mathcal{O}_X^p \longrightarrow \mathcal{O}_X$$

induces a k-morphism

$$\pi\colon X \longrightarrow Y$$

which is easily seen to be bijective and proper. Thus $\pi_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module, and we are asserting that it is free with basis $\prod_{i=1}^n z_i^{\alpha_i}$, $0 \le \alpha_i \le p-1$.

Proof of Lemma 4.3. To check that $\prod z_i^{\alpha_i}$ generate $\pi_*\mathcal{O}_X$, it suffices to prove that for all closed points $x \in Y$, $\prod z_i^{\alpha_i}$ generate $(\pi_*\mathcal{O}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\mathcal{O}_X)_x$ over k. But identifying $\mathcal{O}_{x,Y}$ with $\mathcal{O}_{x,X}^p$, $\mathfrak{m}_{x,Y} = \{f^p \mid f \in \mathfrak{m}_{x,X}\}$: write this $\mathfrak{m}_{x,X}^{[p]}$. Then

$$(\pi_*\mathcal{O}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\mathcal{O}_X)_x \cong \mathcal{O}_{x,X}/\mathfrak{m}_{x,X}^{[p]}\cdot\mathcal{O}_{x,X}.$$

Let $a_i = z_i(x)$ and $y_i = z_i - a_i$. Then $y_1, \dots, y_n$ generate $\mathfrak{m}_{x,X}$ and $\widehat{\mathcal{O}}_{x,X} \cong k[[y_1, \dots, y_n]]$ by Proposition V.3.8. Thus

$$(\pi_*\widehat{\mathcal{O}}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\widehat{\mathcal{O}}_X)_x \cong k[[y_1, \dots, y_n]]/(y_1^p, \dots, y_n^p)$$

and the latter has a basis given by the monomials $\prod y_i^{\alpha_i}$, $0 \le \alpha_i \le p-1$, hence by $\prod z_i^{\alpha_i}$, $0 \le \alpha_i \le p-1$.

But now suppose there was a relation over $U \subset X$:

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} c_\alpha^p \cdot z^\alpha = 0, \qquad c_\alpha \in \mathcal{O}_X(T) \text{ not all zero.}$$

Then for some closed point $x \in U$, $c_\alpha(x) \ne 0$ for some α, hence there would be relation over k:

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} c_\alpha(x)^p \cdot z^\alpha = 0$$

in $(\pi_*\widehat{\mathcal{O}}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\widehat{\mathcal{O}}_X)_x$. But the above proof showed that the z^α were k-independent in $(\pi_*\widehat{\mathcal{O}}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\widehat{\mathcal{O}}_X)_x$. $\square$

To return to the proof of Theorem 4.2, let $x \in X$, $f \in \mathbb{R}(X)$ and suppose $df \in (\Omega^1_{X/k})_x$. Write $f = g/h^p$, $g, h \in \mathcal{O}_{x,X}$, and by Lemma 4.3 expand:

$$g = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} c_\alpha^p z^\alpha, \qquad \{z_1, \dots, z_n\} \text{ a generator of } \mathfrak{m}_{x,X}.$$

Then

$$df = \sum_{l=1}^{n} \left(\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} \left(\frac{c_\alpha}{h}\right)^p \alpha_l z_1^{\alpha_1} \cdots z_l^{\alpha_l - 1} \cdots z_n^{\alpha_n} \right) dz_l$$

hence

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1\\ 0<\alpha_l}} c_\alpha^p \alpha_l z_1^{\alpha_1} \cdots z_l^{\alpha_l-1} \cdots z_n^{\alpha_n} = h^p \cdot b_l, \quad b_l \in \mathcal{O}_{x,X}.$$

Expanding b_l by Lemma 4.3, and equating coefficients of z^α, it follows that $c_\alpha^p \in h^p \cdot \mathcal{O}_{x,X}^p$ if $\alpha_l > 0$. Since this is true for all $l = 1, \dots, n$, it follows:

$$g = c_{(0,\dots,0)}^p + h^p \cdot f_x, \qquad f_x \in \mathcal{O}_{x,X}.$$

Therefore

$$df = d(g/h^p) = df_x.$$

Now we can find a covering $\{U_i\}$ of X and $f_i \in \mathcal{O}_X(U_i)$ such that $\omega = df_i$. Then in $U_i \cap U_j$, $d(f_i - f_j) = 0$, hence $f_i - f_j = g_{ij}^p$, $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ (prove this either by Lemma 4.3 again, or by field theory since $d\colon \mathbb{R}(X)/\mathbb{R}(X)^p \to \Omega^1_{X/k}$ is injective and $\mathcal{O}_x \cap \mathbb{R}(X)^p = \mathcal{O}_x^p$ by the normality of X). Then $\{g_{ij}\}$ defines $\alpha \in H^1(\mathcal{O}_X)$ such that $F\alpha = 0$. This completes the proof of Theorem 4.2. □

The astonishing thing about (b) is that any $f \in [\mathbb{R}(X) \setminus k]$ must have poles and *in characteristic* 0,

$$f \notin \mathcal{O}_{x,X} \Longrightarrow df \notin (\Omega^1_{X/k})_x.$$

In fact, if f has an l-fold pole along an irreducible divisor D, then df has an $(l+1)$-fold pole along D. But *in characteristic* p, if $p \mid l$ then the expected pole of df may sometimes disappear! Nonetheless, this is relatively rare phenomenon even in characteristic p.

For instance, in char $\neq 2$, consider a hyperelliptic curve C. This is defined to be the normalization of $\mathbb{P}^1$ in a quadratic field extension $k(X, \sqrt{f(X)})$. Explicitly, if we take $f(X)$ to be a polynomial with no multiple roots and assume its degree is odd: say $2n+1$, then C is covered by two affine pieces:

$$\begin{aligned} C_1 &= \operatorname{Spec} k[X,Y]/(Y^2 - f(X)) \\ C_2 &= \operatorname{Spec} k[\widetilde{X},\widetilde{Y}]/(\widetilde{Y}^2 - g(\widetilde{X})) \end{aligned}$$

where

$$\begin{aligned} \widetilde{X} &= 1/X \\ \widetilde{Y} &= Y/X^{n+1} \\ g(\widetilde{X}) &= (\widetilde{X})^{2n+2} \cdot f(1/\widetilde{X}). \end{aligned}$$

Then consider $\omega = dX/Y$:

On C_1: $2Y\,dY = f'(X)\cdot dX$, so

$\omega = dX/Y = 2dY/f'(X)$ and since $Y, f'(X)$ have no common zeroes, ω has no poles.

On C_2: $2\widetilde{Y}\,d\widetilde{Y} = g'(\widetilde{X})\cdot d\widetilde{X}$, and one checks

$\omega = -(\widetilde{X})^{n-1}d\widetilde{X}/\widetilde{Y} = -2(\widetilde{X})^{n-1}d\widetilde{Y}/g'(\widetilde{X})$ and since $\widetilde{Y}, g'(\widetilde{X})$ have no common zeroes, ω has no poles.

But now say

$$f(X) = h(X)^p + X.$$

Then $f'(X) = 1$, so $\omega = d(2Y)$ is exact!

The area of characteristic p De Rham theory is far from being completely understood. For further developments, see Serre [**90**, p.24] (from which our theorem has been taken), Grothendieck [**42**] and Monsky [**71**, p.451][14].

5. Deformation theory

We want to study here some questions of a completely new type: given an artin local ring R, with maximal ideal M, residue field $k = R/M$ and some other ideal I such that $I\cdot M = (0)$, we get

$$\operatorname{Spec} R \supset \operatorname{Spec} R/I \supset \operatorname{Spec} k.$$

Then

a) Suppose X_1 is a scheme smooth and of finite type over R/I. How many schemes X_2 are there, smooth and of finite type over R, such that $X_1 \cong X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$?

$$\begin{array}{ccc} X_2 & \supset & X_1 \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \supset & \operatorname{Spec} R/I \end{array}$$

Such an X_2 we call a *deformation of* X_1 *over* R.

b) Suppose X_2, Y_2 are two schemes smooth and of finite type over R, and let $X_1 = X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$, $Y_1 = Y_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$. Suppose $f_1\colon X_1 \to Y_1$ is an R/I-morphism. How many R-morphisms $f_2\colon X_2 \to Y_2$ are there lifting f_1?

In fact the methods that we use to study these questions can be extended to the case where the X's and Y's are merely *flat* over R or R/I (this is another reason why flat is such an important concept). (For this, see ?????.) We can state the results in the smooth case as follows:

In case (a), let $X_0 = X_1 \times_{\operatorname{Spec} R/I} \operatorname{Spec} k$. As in §V.3, let

$$\begin{aligned} \Theta_{X_0} &= \mathcal{H}om(\Omega^1_{X_0/k}, \mathcal{O}_{X_0}) \\ &\cong \text{sheaf of } k\text{-derivations from } \mathcal{O}_{X_0} \text{ to } \mathcal{O}_{X_0} \end{aligned}$$

be the tangent sheaf to X_0. Then

a$_{\text{i}}$) In order that at lease one X_2 exist, it is necessary and sufficient that a canonically defined obstruction $\alpha \in H^2(X_0, \Theta_{X_0}) \otimes_k I$ vanishes. (α will be denoted by $\operatorname{obstr}(X_1)$ below.)

[14] (Add in publication) There have been considerable developments since the manuscript was written. See, for instance, Chambert-Lior [**47**], Astérisque volumes [**45**], [**46**] on "p-adic cohomology" related to "crystalline cohomology" initiated by Grothendieck [**42**].

a$_{\mathrm{ii}}$) If one X_2 exists, consider the set of pairs (X_2, ϕ), with X_2 as above and $\phi\colon X_1 \xrightarrow{\approx} X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$ an isomorphism, modulo the equivalence relation

$$(X_2, \phi) \sim (X_2', \phi') \quad \text{if } \exists \text{an } R\text{-isomorphism } X_2 \underset{\psi}{\xrightarrow{\approx}} X_2'$$

such that

$$\begin{tikzcd} X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I \arrow[rr, "\psi \times 1_{R/I}"] & & X_2' \times_{\operatorname{Spec} R} \operatorname{Spec} R/I \arrow[dl] \\ & X_1 \arrow[from=1-1] & \end{tikzcd}$$

commutes.

Denote this set $\operatorname{Def}(X_1/R)$: then $\operatorname{Def}(X_1/R)$ is a principal homogeneous space over the group $H^1(X_0, \Theta_{X_0}) \otimes_k I$: i.e., the group acts freely and transitively on the set.

a$_{\mathrm{iii}}$) Given two smooth schemes X_1 and Y_1 over R/I and a morphism over R/I:

$$X_1 \xrightarrow{f} Y_1$$

the obstructions to deforming X_1 and Y_1 are connected by having the same image in $H^2(X_0, f^*\Theta_{Y_0}) \otimes_k I$:

$$\begin{tikzcd} \operatorname{obstr}(X_1) \in H^2(X_0, \Theta_{X_0}) \otimes_k I \arrow[dr, "df_0"] & \\ & H^2(X_0, f_0^*\Theta_{Y_0}) \otimes_k I \\ \operatorname{obstr}(Y_1) \in H^2(Y_0, \Theta_{Y_0}) \otimes_k I \arrow[ur, "f_0^*"'] & \end{tikzcd}$$

where $f_0 = f \otimes_{R/I} k\colon X_0 \to Y_0$ and $df_0\colon \Theta_{X_0} \to f_0^*\Theta_{Y_0}$ is the differential of f_0.

In case (b), let $X_0 = X_1 \times_{\operatorname{Spec} R/I} \operatorname{Spec} k$, $Y_0 = Y_1 \times_{\operatorname{Spec} R/I} \operatorname{Spec} k$ and let f_1 induce $f_0\colon X_0 \to Y_0$. We have:

b$_{\mathrm{i}}$) In order that at least one lifting f_2 exist, it is necessary and sufficient that a canonically defined obstruction $\alpha \in H^1(X_0, f_0^*\Theta_{Y_0}) \otimes_k I$ vanishes.

b$_{\mathrm{ii}}$) If one lifting f_2 exists, denote the set of all lifts by $\operatorname{Lift}(f_1/R)$. Then $\operatorname{Lift}(f_1/R)$ is a principal homogeneous space over the group $H^0(X_0, f_0^*\Theta_{Y_0}) \otimes_k I$.

b$_{\mathrm{iii}}$) The action of $H^1(X_0, \Theta_{X_0}) \otimes_k I$ on $\operatorname{Def}(X_1/R)$ is a special case of the obstructions in (i): namely, if X_2, X_2' are two deformations of X_1 over R, then the element of $H^1(X_0, \Theta_{X_0}) \otimes_k I$ by which they differ is the obstruction to lifting $1_{X_1}\colon X_1 \to X_1$ to a morphism from X_2 to X_2'.

b$_{\mathrm{iv}}$) Given three schemes and two morphisms:

$$X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1,$$

the obstructions to lifting compose as follows: if

$$\begin{aligned} \alpha &= (\text{obstruction for } f_1) \in H^1(X_0, f_0^*\Theta_{Y_0}) \\ \beta &= (\text{obstruction for } g_1) \in H^1(Y_0, g_0^*\Theta_{Z_0}) \\ \gamma &= (\text{obstruction of } g_1 \circ f_1) \in H^1(X_0, (g_0 \circ f_0)^*\Theta_{Z_0}) \end{aligned}$$

then

$$\gamma = dg_0(\alpha) + f_0^*(\beta)$$

where $dg_0\colon \Theta_{Y_0} \to g_0^*\Theta_{Z_0}$ is the differential of g_0.

Note, in particular, what these say in the affine case[15]

Affine a) If X_0 is affine, $\exists!$ deformation X_2 of X_1 smooth over R.

Affine b) If X_0 and Y_0 are affine, then every f_1 lifts to some $f_2\colon X_2 \to Y_2$ and if $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$ then these liftings are a principal homogeneous space under:

$$\Gamma(X_0, f_0^*\Theta_{Y_0}) \otimes_k I \cong \operatorname{Der}_k(B_0, A_0) \otimes_k I.$$

If one is interested only in the existence of a lifting in (b), then the smoothness of X_2 is irrelevant and one can prove:

Lifting Property for smooth morphisms: : If X_2, Y_2 are of finite type over R, Y_2 smooth and X_2 affine, then any $f_1\colon X_1 \to Y_1$ lifts to an $f_2\colon X_2 \to Y_2$.

Variants of this lifing property have been used by Grothendiek to *characterize* smooth morphisms (cf. "formal smoothness" in Criterion V.4.10, EGA [**1**, Chapter IV, §17] and SGA1 [**4**, Exposé III]). Our method of proof will be to analyze the deformation problem in an even more local case and then to analyze the patching problem via Čech cocycles. In fact if Z is smooth over $\operatorname{Spec} R'$, then we know that locally Z is isomorphic to U where

$$\begin{array}{c} U = (\operatorname{Spec} R'[X_1, \dots, X_{n+l}]/(f_1, \dots, f_l))_g \\ \downarrow \\ \operatorname{Spec} R' \end{array}$$

where in $R'[X_1, \dots, X_{n+l}]$

$$\det_{1\le i,j\le l}\left(\frac{\partial f_i}{\partial X_{n+j}}\right) \cdot h = g, \qquad \text{some } h \in R'[X].$$

Let's call such U *special smooth affine* schemes over R'.

Step. I: If X_1 is a special smooth affine over R/I, then $\exists$ a deformation X_2 of X_1 over R which is again a special smooth affine.

Proof. Write $X_1 = (\operatorname{Spec}(R/I)[X]/(f))_g$ as above, with $\det \cdot h = g$. Simply choose any polynomials f_i', h' with coefficients in R which reduce mod I to f_i, h. Let $X_2 = (\operatorname{Spec} R[X]/(f'))_{g'}$, where $g' = \det' \cdot h'$. □

Step. II: If X_2 is any affine over R (not even necessarily smooth) and Y_2 is a special smooth affine over R, then any $f_1\colon X_1 \to Y_1$ lifts to an $f_2\colon X_2 \to Y_2$.

Proof. If $X_2 = \operatorname{Spec} A_2$ and $Y_2 = (\operatorname{Spec} R[X]/(f))_g$ as above, then the problem is to define a homomorphism ϕ_2 indicated by the dotted arrow.

$$\begin{tikzcd} & & R[X]_g \arrow[dl] \\ A_2 \arrow[d] & R[X]_g/(f) \arrow[l, dashed, "\phi_2"'] \arrow[d] & \\ A_2/I \cdot A_2 & (R/I)[X]_g/(f). \arrow[l, "\phi_1"'] & \end{tikzcd}$$

If we choose any element $a_j \in A_2$ which reduce mod I to $\phi_1(X_j)$, then we get a homomorphism

$$\phi_2'\colon R[X]_g \longrightarrow A_2$$

[15]It is a theorem that for any noetherian scheme X, X affine $\iff X_{\text{red}}$ affine (EGA [**1**, Chapter I, (5.1.10)]). Hence in our case, X_2 affine $\iff X_1$ affine $\iff X_0$ affine. We will not The last line is illegible.

by setting $\phi_2'(X_j) = a_j$ (since $\phi_2'(g)$ mod $I \cdot A_2$ equals $\phi_1(g)$ which is a unit; hence $\phi_2'(g)$ is a unit in A_2). However $\phi_2'(f_i) = f_i(a)$ may not be zero. But we may alter a_j to $a_j + \delta a_j$ provided $\delta a_j \in I \cdot A_2$. Then since $I^2 = (0)$, $\phi_2'(f_i)$ changes to

$$f_i(a + \delta a) = f_i(a) + \sum_{j=1}^{n+l} \frac{\partial f_i}{\partial X_j}(a) \cdot \delta a_j.$$

Note that since $\delta a_j \in I \cdot A_2$ and $I \cdot M = (0)$, $\dfrac{\partial f_i}{\partial X_j}(a) \cdot \delta a_j$ depends only on the image of $\dfrac{\partial f_i}{\partial X_j}(a)$ in $k[X]$. Multiplying the adjoint matrix to $\left(\dfrac{\partial f_i}{\partial X_{n+j}}\right)_{1 \le i,j \le l}$ by h, we obtain an $(l \times l)$-matrix $(h_{ij}) \in k[X]$ such that

$$\sum_{j=1}^{l} \frac{\partial f_i}{\partial X_{n+j}} \cdot h_{jq} = g \cdot \delta_{iq}.$$

Now set

$$\begin{aligned} \delta a_j &= 0, \quad 1 \le j \le n \\ \delta a_{n+j} &= -g(a)^{-1} \sum_{q=1}^{l} h_{jq} f_q(a), \quad 1 \le j \le l. \end{aligned}$$

Then:

$$\begin{aligned} f_i(a + \delta a) &= f_i(a) - \sum_{n+j}^{l} \frac{\partial f_i}{\partial X_{n+j}}(a) \cdot g(a)^{-1} \cdot \sum_{q=1}^{l} h_{jq} f_q(a) \\ &= f_i(a) - g(a)^{-1} \sum_{q=1}^{l} f_q(a) \cdot g(a) \delta_{iq} \\ &= 0. \end{aligned}$$

Therefore if we define ϕ_2 by $\phi_2(X_j) = a_j + \delta a_j$, we are through. □

STEP. III: Suppose X_2 and Y_2 are affines over R, $X_2 = \operatorname{Spec} A_2$, $Y_2 = \operatorname{Spec} B_2$. $A_0 = A_2/M \cdot A_2$, $B_0 = B_2/M \cdot B_2$. Let $f_2 \colon X_2 \to Y_2$ be a morphism and let $f_1 = \operatorname{res}_{X_1} f_2$. Then $\operatorname{Lift}(f_1/R)$ is a principal homogeneous space over

$$\operatorname{Der}_k(B_0, I \cdot A_2).$$

PROOF. We are given a homomorphism $\phi_1 \colon B_1 \to A_1$ and we wish to study

$$L = \{\phi_2 \colon B_2 \longrightarrow A_2 \mid \phi_2 \bmod I = \phi_1\}$$

which we assume is non-empty. If $\phi_2, \phi_2' \in L$, then $\phi_2' - \phi_2$ factors via D:

$$\begin{tikzcd} B_2 \arrow[rr, "\phi_2' - \phi_2"] \arrow[dr] & & A_2 \\ & B_2/M \cdot B_2 \arrow[r, dashed, "D"] \arrow[d, equal] & I \cdot A_2 \arrow[u, phantom, "\cup"] \\ & B_0 & \end{tikzcd}$$

One checks immediately that D is a derivation. And conversely for any such derivation D, $\phi_2 \in L \Longrightarrow \phi_2 + D \in L$. □

Step. IV: Globalize Step III: Let X_2, Y_2 be two schemes of finite type over R. Let $f_2\colon X_2 \to Y_2$ be a morphism, and let $f_1 = \operatorname{res}_{X_1} f_2$. Then $\operatorname{Lift}(f_1/R)$ is a principal homogeneous space over

$$\Gamma(X_2, \mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I\cdot\mathcal{O}_X)).$$

(Note that $I\cdot\mathcal{O}_X$ is really an $\mathcal{O}_{X_0}$-module).

Proof. Take affine coverings $\{U_\alpha\}$, $\{V_\alpha\}$ of X_2 and Y_2 such that $f_2(U_\alpha)\subset V_\alpha$. If $U_\alpha = \operatorname{Spec} A_2^{(\alpha)}$, $V_\alpha = \operatorname{Spec} B_2^{(\alpha)}$, $f_1^{(\alpha)} = \operatorname{res}_{U_\alpha} f_1$, then as in Step III,

$$\begin{aligned}\operatorname{Lift}(f_1^{(\alpha)}/R) = \text{principal homogeneous space under }\ & \operatorname{Der}_k(B_0^{(\alpha)}, I\cdot A_2^{(\alpha)})\\ & \quad\Vert\\ & \operatorname{Hom}_{B_0^{(\alpha)}}(\Omega^1_{B_0^{(\alpha)}/k}, I\cdot A_2^{(\alpha)})\\ & \quad\Vert\\ & \operatorname{Hom}_{A_0^{(\alpha)}}(\Omega^1_{B_0^{(\alpha)}/k}\otimes_{B_0^{(\alpha)}} A_0^{(\alpha)}, I\cdot A_2^{(\alpha)})\\ & \quad\Vert\\ & \Gamma(U_\alpha, \mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I\cdot\mathcal{O}_{X_2})).\end{aligned}$$

Therefore on the one hand, one can "add" a morphism $f_2\colon X_2\to Y_2$ and a global section D of $\mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I\cdot\mathcal{O}_{X_2})$ by adding them locally on the U_α's and noting that the "sums" agree on overlaps $U_\alpha\cap U_\beta$. Again given two lifts f_2, f_2', their "difference" $f_2 - f_2'$ defines locally on the U_α's a section D_α of $\mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I\cdot\mathcal{O}_{X_2})$, hence a global section D.

Note that if Y_0 is smooth over k, $\Omega^1_{Y_0/k}$ is locally free with dual Θ_{Y_0}, hence

$$\mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, \mathcal{F}) \cong f_0^*\Theta_{Y_0}\otimes_{\mathcal{O}_{X_0}}\mathcal{F}\quad\text{for any sheaf }\mathcal{F};$$

and if X_2 is flat over R, then $I\cdot\mathcal{O}_{X_2}\cong I\otimes_k\mathcal{O}_{X_0}$. Thus case ($\mathrm{b_{ii}}$) of our main result is proven! □

Step. V: Proof of case ($\mathrm{b_i}$): viz. construction of the obstruction to lifting $f_1\colon X_1\to Y_1$.[16]

Proof. Choose affine open coverings $\{U_\alpha\}$, $\{V_\alpha\}$ of X_2, Y_2 such that

- $f_1(U_\alpha)\subset V_\alpha$
- V_α is a special smooth affine.

Then by Step II, there exists a lift $f_2^{(\alpha)}\colon U_\alpha\to V_\alpha$ of $\operatorname{res}_{U_\alpha} f_1$. By Step III, $\operatorname{res} f_2^{(\alpha)}\colon U_\alpha\cap U_\beta\to V_\alpha\cap V_\beta$ and $\operatorname{res} f_2^{(\beta)}\colon U_\alpha\cap U_\beta\to V_\alpha\cap V_\beta$ differ by an element

$$D_{\alpha\beta}\in\Gamma(U_\alpha\cap U_\beta, f_0^*\Theta_{Y_0}\otimes_k I).$$

But on $U_\alpha\cap U_\beta\cap U_\gamma$ we may write somewhat loosely:

$$\begin{aligned}D_{\alpha\beta}+D_{\beta\gamma} &= [\operatorname{res} f_2^{(\alpha)} - \operatorname{res} f_2^{(\beta)}] + [\operatorname{res} f_2^{(\beta)} - \operatorname{res} f_2^{(\gamma)}]\\ &= \operatorname{res} f_2^{(\alpha)} - \operatorname{res} f_2^{(\gamma)}\\ &= D_{\alpha\gamma}.\end{aligned}$$

(Check the proof in Step III to see that this does make sense.) Thus

$$\{D_{\alpha\beta}\}\in Z^1(\{U_\alpha\}, f_0^*\Theta_{Y_0}\otimes_k I).$$

Now if the lifts $f_2^{(\alpha)}$ are changed, this can only be done by adding to them elements $E_\alpha\in\Gamma(U_\alpha, f_0^*\Theta_{Y_0}\otimes_k I)$ and then $D_{\alpha\beta}$ is changed to $D_{\alpha\beta}+E_\alpha-E_\beta$. Moreover, if the covering $\{U_\alpha\}$

[16]Note that we use, in fact, only that Y_2 is smooth over R and that the same proof gives the Lifting Property for smooth morphisms.

is refined and one restricts the lifts $f_2^{(\alpha)}$, then the cocycle we get is just the refinement of $D_{\alpha\beta}$. Thus we have a well-defined element of $H^1(X_0, f_0^*\Theta_{Y_0} \otimes_k I)$. Moreover it is zero if and only if for some coverings $\{U_\alpha\}$, $\{V_\alpha\}$, $D_{\alpha\beta}$ is homologous to zero, i.e.,

$$D_{\alpha\beta} = E_\alpha - E_\beta, \qquad E_\alpha \in \Gamma(U_\alpha, f_0^*\Theta_{Y_0} \otimes_k I).$$

Then changing $f_2^{(\alpha)}$ by E_α as in Step III, we get $\widetilde{f}_2^{(\alpha)}$'s, lifting f_1 such that on $U_\alpha \cap U_\beta$, $\widetilde{f}_2^{(\alpha)} - \widetilde{f}_2^{(\beta)}$ is represented by the zero derivation, i.e., the $\widetilde{f}_2^{(\alpha)}$'s agree on overlaps and give an f_2 lifting f_1. □

The assertion ($\mathrm{b_{iv}}$) is a simple calculation that we leave to the reader.

Step. VI: Proof of ($\mathrm{a_{ii}}$) and ($\mathrm{b_{ii}}$) simultaneously.

Proof. Suppose we are given X_1 smooth over $\operatorname{Spec} R/I$ and at least one deformation X_2 of X_1 over R exists. If X_2, X_2' are any two deformations, we can apply the construction of Step V to the lifting of $1_{X_1}\colon X_1 \to X_1$ to an R-morphism $X_2 \to X_2'$, getting an obstruction in $H^1(X_0, \Theta_{X_0}) \otimes_k I$. This gives us a map:

$$\operatorname{Def}(X_1/R) \times \operatorname{Def}(X_1/R) \longrightarrow H^1(X_0, \Theta_{X_0}) \otimes_k I$$

which we write:

$$(X, X') \longmapsto X - X'.$$

The functorial property ($\mathrm{b_{iv}}$) proves that:

$$(*) \qquad (X - X') + (X' - X'') = (X - X'').$$

Moreover, $X - X' = 0 \Longrightarrow X = X'$: because if $1_{X_1}\colon X_1 \to X_1$ lifts to an R-morphism $f\colon X_2 \to X_2'$, f is automatically an isomorphism in view of the easy:

Lemma 5.1. *Let A and B be R-algebras, B flat over R. If $\phi\colon A \to B$ is an R-homomorphism such that*

$$\overline{\phi}\colon A/I \cdot A \xrightarrow{\approx} B/I \cdot B$$

is an isomorphism, then ϕ is an isomorphism.

(Proof left to the reader.)

If we now show that $\forall$ deformation X_2 and $\forall \alpha \in H^1(X_0, \Theta_{X_0}) \otimes_k I$, $\exists$ a deformation X_2' with $X_2' - X_2 = \alpha$, we will have proven that $\operatorname{Def}(X_1/R)$ is a principal homogeneous space over $H^1(X_0, \Theta_{X_0}) \otimes_k I$ as required. To construct X_2', represent α by a Čech cocycle $\{D_{ij}\}$, for any open covering $\{U_i\}$ of X_2, where

$$D_{ij} \in \Gamma(U_i \cap U_j, \Theta_{X_0} \otimes_k I).$$

As in Step IV, we then have an automrophism of $U_i \cap U_j$ (as a *subscheme* of X_2):

$$1_{U_i \cap U_j} + D_{ij}\colon U_i \cap U_j \longrightarrow U_i \cap U_j.$$

X_2' is obtained by glueing together the subschemes U_i of X_2 by these new automorphisms between $U_i \cap U_j$ regarded as part of U_i and $U_i \cap U_j$ regarded as part of U_j. The cocycle condition $D_{ij} + D_{ji} = D_{ik}$ guarantees that these glueings are consistent and one checks easily that for this X_2', $X_2' - X_2$ is indeed α. □

Step. VII: Proof of ($\mathrm{a_i}$): viz. construction of the obstruction to deforming X_1 over R.

PROOF. Starting with X_1, take a special affine covering $\{U_{i,1}\}$ of X_1. By Step I, $U_{i,1}$ deforms to a special affine $U_{i,2}$ over R. This gives us two deformations of the affine scheme $U_{i,1} \cap U_{j,1}$ over R, viz. the open subschemes

$$\begin{aligned} {}_jU_{i,2} &\subset U_{i,2} \\ {}_iU_{j,2} &\subset U_{j,2}. \end{aligned}$$

By Step VI, these must be isomorphic so choose

$$\phi_{ij} \colon {}_jU_{i,2} \xrightarrow{\approx} {}_iU_{j,2}.$$

If we try to glue the schemes $U_{i,2}$ together by these isomorphisms, consistency requires that the following commutes:

$$\begin{tikzcd} & {}_iU_{j,2} \cap {}_kU_{j,2} \arrow[dr, "\operatorname{res}\phi_{jk}"] & \\ {}_jU_{i,2} \cap {}_kU_{i,2} \arrow[ur, "\operatorname{res}\phi_{ij}"] \arrow[rr, "\operatorname{res}\phi_{ik}"'] & & {}_jU_{k,2} \cap {}_iU_{k,2}. \end{tikzcd}$$

But, in general, $(\operatorname{res}\phi_{ij}) \circ (\operatorname{res}\phi_{ik})^{-1} \circ (\operatorname{res}\phi_{jk})$ will be an automorphism of ${}_iU_{j,2} \cap {}_kU_{j,2}$ given by a derivation $D_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \Theta_{X_0} \otimes_k I)$. One checks easily (1) that D_{ijk} is a 2-cocycle, (2) that altering the ϕ_{ij}'s adds to the D_{ijk} a 2-coboundary, and conversely that any D'_{ijk} cohomologous to D_{ijk} in $H^2(\{U_i\}, \Theta_{X_0} \otimes_k I)$ is obtained by altering the ϕ_{ij}'s, and (3) that refining the covering $\{U_i\}$ replaces D_{ijk} by the refined 2-cocycle. Thus $\{D_{ijk}\}$ defines an element $\alpha \in H^2(X_0, \Theta_{X_0} \otimes_k I)$ depending only on X_1, and $\alpha = 0$ if and only if X_2 exists. □

STEP. VIII Proof of (a_{iii}).

PROOF. Given X_1, Y_1 and f, take special affine coverings $\{U_{i,1}\}$, $\{V_{i,1}\}$ of X_1 and Y_1 such that $f(U_{i,1}) \subset V_{i,1}$. Deform $U_{i,1}$ (resp. $V_{i,1}$) to $U_{i,2}$ (resp. $V_{i,2}$) over R. By Step II, lift f to $f_i \colon U_{i,2} \to V_{i,2}$. Consider the diagram:

$$\begin{tikzcd} {}_jU_{i,2} \arrow[r, "\operatorname{res} f_i"] \arrow[d, "\phi_{ij}"'] & {}_jV_{i,2} \arrow[d, "\psi_{ij}"'] \\ {}_iU_{j,2} \arrow[r, "\operatorname{res} f_j"] & {}_iV_{j,2}. \end{tikzcd}$$

It need not commute, so let

$$(\operatorname{res} f_j) \circ \phi_{ij} = \psi_{ij} \circ (\operatorname{res} f_i) + F_{ij}$$

where $F_{ij} \in \Gamma(U_i \cap U_j, f_0^*\Theta_{Y_0} \otimes_k I)$. It is a simple calculation to check now that if the ϕ_{ij}'s define a 2-cocycle D_{ijk} representing $\operatorname{obstr}(X_1)$ and the ψ_{ij}'s similarly define E_{ijk}, then

$$df_0(D_{ijk}) - f_0^* E_{ijk} = F_{ij} - F_{ik} + F_{jk}.$$

□

This completes the proof of the main results of infinitesimal deformation theory. We get some important corollaries:

COROLLARY 5.2. *Let R be an artin local ring with maximal ideal M and residue field k and let $I \subset R$ be* any *ideal contained in M. If X_1 is a scheme smooth of finite type over* $\operatorname{Spec} R/I$ *such that $H^2(X_0, \Theta_{X_0}) = (0)$ — e.g., if* $\dim X_0 = 1$ *— then a deformation X_2 of X_1 over R exists.*

Proof. Filter I as follows: $I \supset MI \supset M^2 I \supset \cdots \supset M^\nu I = (0)$. Then deform X_1 successively as follows:

$$\begin{array}{ccccccccc}
X_1 & \subset & X_1^{(1)} & \subset & \cdots & \subset & X_1^{(l)} & & & \\
\downarrow & & \downarrow & & & & \downarrow & & & \\
\operatorname{Spec} R/I & \subset & \operatorname{Spec} R/MI & \subset & \cdots & \subset & \operatorname{Spec} R/M^l I & \subset & \cdots & \subset & \operatorname{Spec} R
\end{array}$$

using case ($\mathrm{a_i}$) of each stage to show that $X_1^{(l)}$ can be deformed to a $X_1^{(l+1)}$. Set $X_2 = X_1^{(\nu)}$. □

Corollary 5.3. *Let R be an artin local ring with residue field k and let X be a scheme smooth*[17] *of finite type over R. Let $X_0 = X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ and let*

$$f_0 \colon Y_0 \longrightarrow X_0$$

be an étale morphism. Then there exists a unique deformation Y of Y_0 over R such that f_0 lifts to $f \colon Y \to X$.

Proof. Let M be the maximal ideal of R. Deform Y_0 successively as follows:

$$\begin{array}{ccccccccccc}
Y_0 & \subset & Y_1 & \subset & \cdots & \subset & Y_{l-1} & & & & \\
{\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & & & {\scriptstyle f_{l-1}}\downarrow & & & & \\
X_0 & \subset & X_1 & \subset & \cdots & \subset & X_{l-1} & \subset & \cdots & \subset & X \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
\operatorname{Spec} k & \subset & \operatorname{Spec}(R/M^2) & \subset & \cdots & \subset & \operatorname{Spec}(R/M^l) & \subset & \cdots & \subset & \operatorname{Spec} R
\end{array}$$

where $X_{l-1} = X \times_{\operatorname{Spec} R} \operatorname{Spec}(R/M^l)$. Because f_0 is étale, $df_0 \colon \Theta_{Y_0} \to f_0^* \Theta_{X_0}$ is an isomorphism. Therefore at each stage, the existence of the deformation X_l of X_{l-1} gurantees by ($\mathrm{a_{iii}}$) the existence of a deformation Y_l' of Y_{l-1}. Then choose any Y_l' and ask whether f_{l-1} lifts. We get an obstruction α:

$$H^1(Y_0, \Theta_{Y_0}) \otimes_k (M^l/M^{l+1}) \xrightarrow[df_0]{\approx} H^1(Y_0, f_0^* \Theta_{X_0}) \otimes_k (M^l/M^{l+1}).$$

Then alter the deformation Y_l' by $df_0^{-1}(\alpha)$, giving a new deformation Y_l. By functoriality ($\mathrm{b_{iv}}$), f_{l-1} lifts to $f_l \colon Y_l \to X_l$ and by injectivity of df_0 this is the only deformation for which this is so. □

The most exciting applications of deformation theory, however, are those cases when one can construct deformations not only over artin local rings, but over complete local rings. If the ring R is actually an integral domain, then one has constructed, by taking fibre product, a scheme over the quotient field K of R as well. A powerful tool for extending constructions to this case is Grothendieck's GFGA Theorem (Theorem 2.17). This is applied as follows:

Definition 5.4. Let R be a complete local noetherian ring with maximal ideal M. Then a *formal scheme* $\mathcal{X}$ over R is a system of schemes and morphisms:

$$\begin{array}{ccccccccccc}
X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & \cdots\cdots & & \\
\downarrow & & \downarrow & & & & \downarrow & & & & \\
\operatorname{Spec}(R/M) & \to & \operatorname{Spec}(R/M^2) & \to & \cdots & \to & \operatorname{Spec}(R/M^{n+1}) & \to & \cdots\cdots & & \operatorname{Spec} R
\end{array}$$

[17] A more careful proof shows that the smoothness of X is not really needed here and that Corollary 5.3 is true for any X of finite type over R. It is even true for comparing étale coverings of X and X_{red}, any noetherian scheme X (cf. SGA1 [**4**, Exposé I, Théorème 8.3, p. 14]).

where $X_{n-1} \cong X_n \times_{\mathrm{Spec}(R/M^{n+1})} \mathrm{Spec}(R/M^n)$. $\mathcal{X}$ is *flat* over R if each X_n is flat over $\mathrm{Spec}(R/M^{n+1})$. If X is a scheme over $\mathrm{Spec}\, R$, the associated formal scheme $\widehat{X}$ is the system of schemes $X_n = X \times_{\mathrm{Spec}\, R} \mathrm{Spec}(R/M^{n+1})$ together with the obvious morphisms $X_n \to X_{n+1}$.

THEOREM 5.5. *Let R be a complete local noetherian ring and let $\mathcal{X} = \{X_n\}$ be a formal scheme flat over R. If X_0 is smooth and projective over $k = R/M$, and if $H^2(X_0, \mathcal{O}_{X_0}) = (0)$, then there exists a scheme X smooth and proper over R such that:*

$$\mathcal{X} = \widehat{X}.$$

PROOF. Since X_0 is projective over k, there exists a very ample invertible sheaf $\mathcal{L}_0$ on X_0. By Ex. ????? Veronese…, $\mathcal{L}_0^n$ is very ample for all $n \geq 1$; by Theorem VII.8.1, $H^1(\mathcal{L}_0^n) = (0)$ if $n \gg 0$. So we may replace $\mathcal{L}_0$ by $\mathcal{L}_0^n$ and assume that $H^1(\mathcal{L}_0) = (0)$ too. The first step is to "lift" $\mathcal{L}_0$ to a sequence of invertible sheaves $\mathcal{L}_n$ on X_n such that

$$\mathcal{L}_n \cong \mathcal{L}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}, \qquad \text{all } n \geq 0.$$

To do this, recall that the isomorphism classes of invertible sheaves on any scheme X are classified by $H^1(X, \mathcal{O}_X^*)$. Therefore to construct the $\mathcal{L}_n$'s inductively, it will suffice to show that the natural map:

$$H^1(X_n, \mathcal{O}_{X_n}^*) \longrightarrow H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}^*)$$

(given by restriction of functions from X_n to X_{n-1}) is surjective. But consider the map of sheaves:

$$\begin{aligned} \exp\colon M^n \cdot \mathcal{O}_{X_n} &\longrightarrow \mathcal{O}_{X_n}^* \\ a &\longmapsto 1 + a. \end{aligned}$$

Since $M^n \cdot M^n \equiv 0$ in $\mathcal{O}_{X_n}$, this map is a homomorphism from $M^n \cdot \mathcal{O}_{X_n}$ in its additive structure to $\mathcal{O}_{X_n}^*$ in its multiplicative structure, and the image is obviously $\mathrm{Ker}\left(\mathcal{O}_{X_n}^* \to \mathcal{O}_{X_{n-1}}^*\right)$, i.e., we get an exact sequence:

$$0 \longrightarrow M^n \cdot \mathcal{O}_{X_n} \xrightarrow{\exp} \mathcal{O}_{X_n}^* \longrightarrow \mathcal{O}_{X_{n-1}}^* \longrightarrow 1.$$

But now the flatness of X_n over R/M^{n+1} implies that for any ideal $\mathfrak{a} \subset R/M^{n+1}$,

$$\mathfrak{a} \otimes_{R/M^{n+1}} \mathcal{O}_{X_n} \longrightarrow \mathfrak{a} \cdot \mathcal{O}_{X_n}$$

is an isomorphism. Apply this with $\mathfrak{a} = M^n/M^{n+1}$:

$$\begin{aligned} M^n \cdot \mathcal{O}_{X_n} &\cong (M^n/M^{n+1}) \otimes_{R/M^{n+1}} \mathcal{O}_{X_n} \\ &\cong (M^n/M^{n+1}) \otimes_{R/M} \left((R/M) \otimes_{R/M^{n+1}} \mathcal{O}_{X_n}\right) \\ &\cong (M^n/M^{n+1}) \otimes_k \mathcal{O}_{X_0} \\ &\cong \mathcal{O}_{X_0}^{\nu_n}, \text{ if } \nu_n = \dim_k M^n/M^{n+1}. \end{aligned}$$

Therefore we get an exact sequence[18]:

$$0 \longrightarrow \mathcal{O}_{X_0}^{\nu_n} \longrightarrow \mathcal{O}_{X_n}^* \longrightarrow \mathcal{O}_{X_{n-1}}^* \longrightarrow 1$$

hence an exact cohomology sequence:

$$\begin{array}{ccccc} H^1(\mathcal{O}_{X_n}^*) & \longrightarrow & H^1(\mathcal{O}_{X_{n-1}}^*) & \xrightarrow{\delta} & H^2(\mathcal{O}_{X_0})^{\nu_n}. \\ & & & & \| \\ & & & & (0) \end{array}$$

[18]Note that all X_n are topologically the same space, hence this exact sequence makes sense as a sequence of sheaves of abelian groups on X_0.

This proves that the sheaves $\mathcal{L}_n$ exist.

The second step is to lift the projective embedding of X_0. Let $s_0, \ldots, s_N$ be a basis of $\Gamma(X_0, \mathcal{L}_0)$, so that $(\mathcal{L}_0, s_0, \ldots, s_N)$ defines a projective embedding of X_0 in $\mathbb{P}_k^N$, as in the theory of ?????. I claim that for each n there are sections $s_0^{(n)}, \ldots, s_N^{(n)}$ of $\mathcal{L}_n$ such that via the isomorphism:

$$\mathcal{L}_{n-1} \cong \mathcal{L}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \cong \mathcal{L}_n / M^n \cdot \mathcal{L}_n,$$

$s_i^{(n-1)} =$ image of $s_i^{(n)}$. To see this, use the exact sequence:

$$0 \longrightarrow M^n \cdot \mathcal{L}_n \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1} \longrightarrow 0$$

and note that because $\mathcal{L}_n$ is flat over R/M^{n+1} too,

$$\begin{aligned} M^n \cdot \mathcal{L}_n &\cong (M^n/M^{n+1}) \otimes_{R/M^{n+1}} \mathcal{L}_n \\ &\cong (M^n/M^{n+1}) \otimes_{R/M} \big((R/M) \otimes_{R/M^{n+1}} \mathcal{L}_n\big) \\ &\cong (M^n/M^{n+1}) \otimes_k \mathcal{L}_0, \end{aligned}$$

hence we get an exact cohomology sequence:

$$\begin{tikzcd} H^0(\mathcal{L}_n) \arrow[r] & H^0(\mathcal{L}_{n-1}) \arrow[r, "\delta"] & H^1(\mathcal{L}_0)^{\nu_n}. \arrow[d, equal] \\ & & (0) \end{tikzcd}$$

This allows us to define $s_i^{(n)}$ inductively on n. Now for each n, $(\mathcal{L}_n, s_0^{(n)}, \ldots, s_N^{(n)})$ defines a morphism

$$\phi_n \colon X_n \longrightarrow \mathbb{P}^N_{(R/M^{n+1})}$$

such that the diagram:

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d] & \mathbb{P}^N_{(R/M^n)} \arrow[d] \\ X_n \arrow[r] & \mathbb{P}^N_{(R/M^{n+1})} \end{tikzcd}$$

commutes. I claim that ϕ_n is a closed immersion for each n. Topologically it is closed and injective because topologically $\phi_n = \phi_0$ and ϕ_0 is by assumption a closed immersion. As for structure sheaves, ϕ_n^* lies in a diagram:

$$\begin{tikzcd} 0 \arrow[r] & (M^n/M^{n+1}) \otimes_k \mathcal{O}_{\mathbb{P}_k^N} \arrow[r] \arrow[d, "1_{M^n} \otimes \phi_0^*"] & \mathcal{O}_{\mathbb{P}^N_{(R/M^{n+1})}} \arrow[r] \arrow[d, "\phi_n^*"] & \mathcal{O}_{\mathbb{P}^N_{(R/M^n)}} \arrow[r] \arrow[d, "\phi_{n-1}^*"] & 0 \\ 0 \arrow[r] & (M^n/M^{n+1}) \otimes_k \mathcal{O}_{X_0} \arrow[r] & \mathcal{O}_{X_n} \arrow[r] & \mathcal{O}_{X_{n-1}} \arrow[r] & 0. \end{tikzcd}$$

Since ϕ_0^* is surjective, this shows ϕ_{n-1}^* surjective $\implies$ ϕ_n^* surjective. So by induction, all the ϕ_n are closed immersions.

Finally, let ϕ_n induce an isomorphism of X_n with the closed subscheme $Y_n \subset \mathbb{P}^N_{(R/M^{n+1})}$. Then the sequence of coherent sheaves $\{\mathcal{O}_{Y_n}\}$ is a formal coherent sheaf on $\mathbb{P}_R^N$ in the sense of §2 above. By the GFGA theorem (Theorem 2.17), there is a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_R^N$ such that

$$\mathcal{O}_{Y_n} \cong \mathcal{F} \otimes_R (R/M^{n+1})$$

for every n. Moreover since $\{\mathcal{O}_{Y_n}\}$ is a quotient of the formal sheaf $\left\{\mathcal{O}_{\mathbb{P}^N_{(R/M^{n+1})}}\right\}$, $\mathcal{F}$ is quotient of $\mathcal{O}_{\mathbb{P}^N}$, i.e., $\mathcal{F} = \mathcal{O}_Y$ for some closed subscheme $Y \subset \mathbb{P}^N$. Therefore since:

$$X_n \cong Y_n \cong Y \times_{\operatorname{Spec} R} \operatorname{Spec}(R/M^{n+1}),$$

it follows that $\widehat{Y} \cong \mathcal{X}$. □

Corollary 5.6. *Let R be a complete local noetherian ring and let $\mathcal{X} = \{X_n\}$ be a formal scheme flat over R. If X_0 is a smooth complete curve, then $\mathcal{X} = \widehat{X}$ for some X smooth and proper over R.*

Proof. By Exercise 2, §4D, X_0 is projective over k and because $\dim X_0 = 1$, $H^2(\mathcal{O}_{X_0}) = (0)$. □

Corollary 5.7. (Severi[19] -Grothendieck) *Let R be a complete local noetherian ring and let X_0 be a smooth complete curve over $k = R/M$. Then X_0 has a deformation over R, i.e., there exists a scheme X, smooth and proper over R such that $X_0 \cong X \times_{\operatorname{Spec} R} \operatorname{Spec} k$.*

Proof. Corollaries 5.2 and 5.6. □

An important supplementary remark here is that if for simplicity X_0 is geometrically irreducible (also said to be absolutely irreducible), i.e., $X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is irreducible ($\overline{k} =$ algebraic closure of k), then $H^1(\mathcal{O}_X)$ is a free R-module such that

$$H^1(\mathcal{O}_X) \otimes_R k \cong H^1(\mathcal{O}_{X_0})$$
$$H^1(\mathcal{O}_X) \otimes_R K \cong H^1(\mathcal{O}_{X_\eta})$$
$$(K = \text{ quotient field of } R,\ X_\eta = X \times_{\operatorname{Spec} R} \operatorname{Spec} K).$$

Since the *genus* of a curve Y over k is nothing but $\dim_k H^1(\mathcal{O}_Y)$, this shows that $\operatorname{genus}(X_\eta) = \operatorname{genus}(X_0)$. The proof in outline is this:

a) $X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ irreducible and X_0 smooth over k implies k algebraically closed in $\mathbb{R}(X)$, hence k algebraically closed in $H^0(\mathcal{O}_{X_0})$. Thus
$$k \xrightarrow{\approx} H^0(\mathcal{O}_{X_0}).$$
b) Show that there are exact sequences
$$\begin{tikzcd} 0 \arrow[r] & M^n \cdot \mathcal{O}_{X_n} \arrow[r] \arrow[d, equal] & \mathcal{O}_{X_n} \arrow[r] & \mathcal{O}_{X_{n-1}} \arrow[r] & 0. \\ & \mathcal{O}_{X_0}^{\nu_n} & & & \end{tikzcd}$$
c) Show by induction on n that if $g = \dim_k H^1(\mathcal{O}_{X_0})$, then $R/M^{n+1} \xrightarrow{\approx} H^0(\mathcal{O}_{X_n})$ and $H^1(\mathcal{O}_{X_n})$ is a free (R/M^{n+1})-module of rank g such that
$$H^1(\mathcal{O}_{X_{n-1}}) \cong H^1(\mathcal{O}_{X_n})/M^n \cdot H^1(\mathcal{O}_{X_n}).$$
d) Apply GFGA (Theorem 2.17) to prove that $H^1(\mathcal{O}_X)$ is a free R-module of rank g such that $H^1(\mathcal{O}_{X_n}) \cong H^1(\mathcal{O}_X)/M^{n+1} \cdot H^1(\mathcal{O}_X)$ for all n.
e) Apply (4.??) using the flatness of K over R to prove that
$$H^1(\mathcal{O}_{X_\eta}) \cong H^1(\mathcal{O}_X) \otimes_R K.$$

Corollary 5.7 is especially interesting when k is a perfect field of characteristic p and R is the Witt vectors over k (see, for instance, Mumford [**73**]), in which case one summarizes Corollary 5.7 by saying: "non-singular curves can be lifted from characteristic p to characteristic 0". On the other hand, Serre [**92**][20] has found non-singular projective varieties X_0 over algebraically

[19] Modulo translating Italian style geometry into the theory of schemes, a rigorous proof of this is contained in Severi [**95**, Anhang]. This approach was worked out by Popp [**80**].

[20] (Added in Publication) See also Illusie's account in FAG [**3**, Chapter 8].

closed fields k of characteristic p such that for every complete local characteristic 0 domain R with $R/M = k$, no such X exists: such an X_0 is called a non-liftable variety!

One can strengthen the application of deformation theory to coverings in the same way:

THEOREM 5.8. *Let R be a complete local noetherian ring with residue field k and let X be a scheme smooth*[21] *and proper over R. Let $X_0 = X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ and let*

$$f_0 \colon Y_0 \longrightarrow X_0$$

be a finite étale morphism. Then there exists a unique finite étale morphism

$$f \colon Y \longrightarrow X$$

such that f_0 is obtained from f by fibre product $\times_{\operatorname{Spec} R} \operatorname{Spec} k$.

PROOF. By Corollary 5.3 we can lift $f_0 \colon Y_0 \to X_0$ to a unique formal finite étale scheme $F \colon \mathcal{Y} \to \widehat{X}$, i.e., $\mathcal{Y} = \{Y_n\}$, $F = \{f_n\}$ where $f_n \colon Y_n \to X_n$ is finite and étale, where $X_n = X \times_{\operatorname{Spec} R} \operatorname{Spec} R/M^{n+1}$ and the diagram:

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d, "f_n"'] & Y_{n+1} \arrow[d, "f_{n+1}"] \\ X_n \arrow[r] & X_{n+1} \end{tikzcd}$$

commutes (the inclusion $Y_n \to Y_{n+1}$ being part of the definition of a formal scheme $\mathcal{Y}$). If $\mathcal{A}_n = f_{n,*}(\mathcal{O}_{Y_n})$, then

$$\begin{aligned} \mathcal{S}pec_{X_n}(\mathcal{A}_n) &\cong Y_n \\ &\cong Y_{n+1} \times_{X_{n+1}} X_n \\ &\cong \mathcal{S}pec_{X_n}\left(\mathcal{A}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}\right) \end{aligned}$$

hence $\mathcal{A}_n \cong \mathcal{A}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}$. Therefore $\{\mathcal{A}_n\}$ is a coherent formal sheaf on X, hence by Theorem 2.17 there is a unique coherent sheaf $\mathcal{A}$ on X such that $\mathcal{A}_n \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n. Using the fact that

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}, \mathcal{A}) \cong \text{Formal } \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}_{\text{for}} \otimes_{\mathcal{O}_X} \mathcal{A}_{\text{for}}, \mathcal{A}_{\text{for}})$$

and similar facts with $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, we see immediately that $\mathcal{A}$ is a sheaf of commutative algebras. Let $Y = \mathcal{S}pec_X(\mathcal{A})$, and let $f \colon Y \to X$ be the canonical morphism. f is obviously proper and finite to one. Moreover since for all $x \in X_0$, $\mathcal{A}_{n,x}$ is a free $\mathcal{O}_{x,X_n}$-module, it follows immediately that $\mathcal{A}_x$ is a free $\mathcal{O}_{x,X}$-module, i.e., f is flat at x. And f_0 étale implies $\Omega_{Y/X} \otimes_{\mathcal{O}_X} \Bbbk(x) = (0)$, hence $(\Omega_{Y/X})_x = (0)$ by Nakayama's lemma. Therefore by Criterion V.4.1, f is étale at x. Since this holds for all x, f is étale in an open set $U \subset X$, with $U \supset X_0$. But X proper over $\operatorname{Spec} R$ implies that every such open set U equals X. Thus f is étale. Finally if $f' \colon \mathcal{S}pec_X \mathcal{A}' \to X$ is another such lifting with $\mathcal{A}'_0 = \mathcal{A}_0$, then by Theorem 2.17 there is a unique isomorphism

$$\phi_n \colon \mathcal{A}_n \xrightarrow{\approx} \mathcal{A}'_n$$

of $\mathcal{O}_{X_n}$-algebras inducing the identity $\mathcal{A}_0 \xrightarrow{\approx} \mathcal{A}'_0$. Then these ϕ_n patch together into a *formal* isomorphism $\mathcal{A}_{\text{for}} \xrightarrow{\approx} \mathcal{A}'_{\text{for}}$, which comes by Theorem 2.17 from a unique algebraic isomorphism $\phi \colon \mathcal{A} \xrightarrow{\approx} \mathcal{A}'$. □

[21] As mentioned above, Corollary 5.3 is actually true without assuming smoothness and hence since smoothness is not used in proving this result from Corollary 5.3, it is unnecessary here too.

Corollary 5.9. *In the situation of Theorem 5.8, there is an isomorphism of pro-finite groups:*

$$\pi_1^{\text{alg}}(X_0) \xrightarrow{\approx} \pi_1^{\text{alg}}(X)$$

canonical up to inner automorphism.

Proof. If $f\colon Y \to X$ is any connected covering, i.e., f finite and étale, then by fibre product $\times_X X_0$, we get a covering $f_0\colon Y_0 \to X_0$. Note that Y_0 is connected (if not, we could lift its connected components separately by Theorem 5.8, hence find a disconnected covering $f'\colon Y' \to X$ lifting f_0, thus contradicting the uniqueness in the theorem). By Theorem 5.8 every connected covering $f_0\colon Y_0 \to X_0$, up to isomorphism, arises in this way. Moreover, if $\mathbb{R}(Y)$ is Galois over $\mathbb{R}(X)$, then we get a homomorphism:

$$\mathrm{Gal}(\mathbb{R}(Y)/\mathbb{R}(X)) \xrightarrow{\approx} \mathrm{Aut}(Y/X) \longrightarrow \mathrm{Aut}(Y_0/X_0) \xrightarrow{\approx} \mathrm{Gal}(\mathbb{R}(Y_0)/\mathbb{R}(X_0))$$

which is easily seen to be an isomorphism, e.g., by Ex. 1, §6A????. Now fix separable algebraic closures $\mathbb{R}(X)^*$ of $\mathbb{R}(X)$ and $\mathbb{R}(X_0)^*$ of $\mathbb{R}(X_0)$, and let $\widetilde{\mathbb{R}(X)} \subset \mathbb{R}(X)^*$, $\widetilde{\mathbb{R}(X_0)} \subset \mathbb{R}(X_0)^*$ be the maximal subfields such that normalization in finitely generated subfields of these is étale over X or over X_0. Now write $\widetilde{\mathbb{R}(X)}$ as an increasing union of finite Galois extensions L_n of $\mathbb{R}(X)$; we get a tower of coverings $Y_{L_n} =$ normalization of X in L_n; let $Z_n = Y_{L_n} \times_X X_0$ (this is a tower of connected coverings of X_0); choose inductively in n isomorphisms:

$$\mathbb{R}(Z_n) \xrightarrow{\approx} K_n \subset \widetilde{\mathbb{R}(X_0)}.$$

It follows readily that $\bigcup K_n = \widetilde{\mathbb{R}(X_0)}$, and that

$$\begin{aligned}\pi_1^{\text{alg}}(X) &\cong \varprojlim \mathrm{Gal}(L_n/\mathbb{R}(X))\\ &\cong \varprojlim \mathrm{Gal}(K_n/\mathbb{R}(X_0))\\ &\cong \pi_1^{\text{alg}}(X_0).\end{aligned}$$

□

This result can be used to partially compute π_1 of liftable characteristic p varieties in terms of π_1 of varieties over $\mathbb{C}$, hence in terms of classical topology. This method is due to Grothendieck and illustrates very beautifully the Kroneckerian idea of §IV.1: Let

$$\begin{aligned} k &= \text{algebraically closed field of characteristic } p,\\ R &= \text{complete local domain of characteristic 0 with } R/M = k,\\ K &= \text{quotient field of } R,\\ \overline{K} &= \text{algebraic closure of } K.\end{aligned}$$

Choose an isomorphism (embedding?):

$$\overline{K} \cong \mathbb{C}.$$

Let

$$\begin{aligned} X &= \text{scheme proper and smooth over } R,\\ X_0 &= X \times_{\operatorname{Spec} R} \operatorname{Spec} k: \ \text{we assume this is irreducible},\\ X_\eta &= X \times_{\operatorname{Spec} R} \operatorname{Spec} K,\\ \overline{X}_\eta &= X \times_{\operatorname{Spec} R} \operatorname{Spec} \overline{K}: \ \text{we assume this is irreducible},\\ \widetilde{X}_\eta &= X \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}.\end{aligned}$$

THEOREM 5.10. *There is a surjective homomorphism*

$$\pi_1^{\mathrm{alg}}(\overline{X}_\eta) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X_0)$$

canonical up to inner automorphism, and hence, fixing an isomorphism $\overline{K} \approx \mathbb{C}$, *a surjective homomorphism:*

$$\widehat{\pi}_1^{\mathrm{top}}(\widetilde{X}_\eta) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X_0).$$

PROOF. By Theorem 2.16 and Corollary 5.9, it suffices to compare $\pi_1^{\mathrm{alg}}(\overline{X}_\eta)$ and $\pi_1^{\mathrm{alg}}(X)$. Let $\Omega \supset \mathbb{R}(\overline{X}_\eta)$ be an algebraic closure. Note that

i) $\mathbb{R}(X) = \mathbb{R}(X_\eta)$
ii) $\mathbb{R}(\overline{X}_\eta) = \mathbb{R}(X_\eta) \otimes_K \overline{K}$ is algebraic over $\mathbb{R}(X_\eta)$, hence Ω is an algebraic closure of $\mathbb{R}(X_\eta)$ too.

Thus we may consider the maximal subfields of Ω such that the normalization of any of the schemes X, X_η and $\overline{X}_\eta$ is étale. Note that:

iii) $L \subset \Omega$ finite over $\mathbb{R}(X)$, normalization of X in L étale over $X \implies$ normalization of X_η in L étale over X_η,
iv) $K_0 \subset \overline{K}$ finite over K, normalization of X_η in $\mathbb{R}(X_\eta) \otimes_K K_0$ is $X_\eta \times_{\operatorname{Spec} K} \operatorname{Spec} K_0$ which is étale over X_η.

Thus we get a diagram

$$\begin{tikzcd}
 & & \Omega \arrow[d, dash] & \\
 & & \Omega_1 \arrow[dl, dash] \arrow[dr, dash] & \\
 & \overline{K} \cdot \mathbb{R}(X_\eta) = \mathbb{R}(\overline{X}_\eta) \arrow[dl, dash] \arrow[dr, dash] & & \Omega_2 \arrow[dl, dash] \\
\overline{K} \arrow[dr, dash] & & \mathbb{R}(X_\eta) = \mathbb{R}(X) \arrow[dl, dash] & \\
 & K & &
\end{tikzcd}$$

where

$$\Omega_1/\mathbb{R}(\overline{X}_\eta) = \text{maximal extension étale over } \overline{X}_\eta$$
$$\Omega_1/\mathbb{R}(X_\eta) = \text{maximal extension étale over } X_\eta$$
$$\Omega_2/\mathbb{R}(X_\eta) = \text{maximal extension étale over } X$$

i.e.,

$$\pi_1^{\mathrm{alg}}(\overline{X}_\eta) \cong \mathrm{Gal}(\Omega_1/\mathbb{R}(\overline{X}_\eta))$$
$$\pi_1^{\mathrm{alg}}(X_\eta) \cong \mathrm{Gal}(\Omega_1/\mathbb{R}(X_\eta))$$
$$\pi_1^{\mathrm{alg}}(X) \cong \mathrm{Gal}(\Omega_2/\mathbb{R}(X_\eta)).$$

Since

v) $\mathrm{Gal}(\mathbb{R}(\overline{X}_\eta)/\mathbb{R}(X_\eta)) \cong \mathrm{Gal}(\overline{K}/K)$,

we get homomorphisms:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{\mathrm{alg}}(\overline{X}_\eta) \arrow[r] \arrow[dr, "\phi"'] & \pi_1^{\mathrm{alg}}(X_\eta) \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{K}/K) \arrow[r] & 1 \\ & & \pi_1^{\mathrm{alg}}(X) & & \end{tikzcd}$$

To finish the proof of Theorem 5.10, we must show that ϕ is surjective. A small consideration of this diagram of fields shows that this amounts to saying:

vi) $\Omega_2 \otimes_K \overline{K} \to \Omega$ is injective; or equivalently (cf. §IV.2) K is algebraically closed in Ω_2.

If this is not true, then suppose $L \subset \overline{K}$ is finite over K and $L \subset \Omega_2$. Let S be the integral closure of R in L. Then $X \times_{\operatorname{Spec} R} \operatorname{Spec} S$ is smooth over $\operatorname{Spec} S$, hence is normal; since $\mathbb{R}(X \times_{\operatorname{Spec} R} \operatorname{Spec} S) = \mathbb{R}(X) \otimes_K L$, $X \times_{\operatorname{Spec} R} \operatorname{Spec} S$ is the normalization of X in $\mathbb{R}(X) \otimes_K L$. Now $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is certainly *not* étale unless $R = S$: because if $[M] \in \operatorname{Spec} R$ is the closed point, then (a) by Hensel's lemma (Lemma IV.6.1), $f^{-1}([M]) =$ one point, so (b) if f is étale, the closed subscheme $f^{-1}([M])$ is isomorphic to $\operatorname{Spec} k$, hence (c) by Ex. §4D?????, f is a closed immersion, i.e., $R \to S$ is surjective. But then neither can $g\colon X \times_{\operatorname{Spec} R} \operatorname{Spec} S \to X$ be étale because I claim there is a section s:

$$\begin{tikzcd} X \arrow[d, shift right] \\ \operatorname{Spec} R \arrow[u, shift right, bend right, "s"'] \end{tikzcd}$$

hence g étale implies by base change via s that f is étale. To construct s, just take a closed point $x \in X_0$, let $\overline{a}_1, \ldots, \overline{a}_n$ be generators of $\mathfrak{m}_{x,X_0}$ in the regular local ring $\mathcal{O}_{x,X_0}$, lift these to $a_1, \ldots, a_n \in \mathfrak{m}_{x,X}$ and set $Z = \operatorname{Spec}(\mathcal{O}_{x,X}/(a_1, \ldots, a_n))$. By Hensel's lemma (Lemma IV.6.1), Z is finite over R, hence is isomorphic to a closed subscheme of X. Since the projection $Z \to \operatorname{Spec} R$ is immediately seen to be étale, $Z \to \operatorname{Spec} R$ is an isomorphism, hence there is a unique s with $Z = \operatorname{Image}(s)$. $\square$

At this point we can put together Parts I and II to deduce the following famous result of Grothendieck.

Corollary 5.11. *Let k be an algebraically closed field of characteristic p and let X be a non-singular complete curve over k. Let $g = \dim_k H^1(\mathcal{O}_X)$, the genus of X. Then $\exists a_1, \ldots, a_g, b_1, \ldots, b_g \in \pi_1^{\mathrm{alg}}(X)$ satisfying:*

$$(*) \qquad a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = e$$

and generating a dense subgroup: equivalently $\pi_1^{\mathrm{alg}}(X)$ is a quotient of the pro-finite completion of the free group on the a_i's and b_i's modulo a normal subgroup containing at least $(*)$.

Proof. Lift X to a scheme Y over the ring of Witt vectors $W(k)$, and via an isomorphism of $\mathbb{C}$ with (embedding into $\mathbb{C}$ of?) the algebraic closure of the quotient field of $W(k)$ (see, for instance, Mumford [**73**]), let Y induce a curve Z over $\mathbb{C}$. Note that $g = \dim_{\mathbb{C}} H^1(\mathcal{O}_Z)$ by the remark following Corollary 5.7 and by (4.?????). By Part I [**76**, §7B], we know that topologically Z is a compact orientable surface with g handles. It is a standard result in elementary topology (cf. ?????) that π_1^{top} of such a surface is free on a_i's and b_i's modulo the one relation $(*)$. Thus everything follows from Theorem 5.10. $\square$

What is the kernel of $\pi_1^{\mathrm{alg}}(\overline{X}_\eta) \to \pi_1^{\mathrm{alg}}(X_0)$? A complete structure theorem is not known, even for curves, but the following two things have been discovered:

a) Grothendieck has shown that the kernel is contained in the closed normal subgroup generated by the p-Sylow subgroups: i.e., if H is finite such that $p \nmid \#H$ and $\pi_1^{\text{alg}}(\overline{X}_\eta) \to H$ is a continuous map, then this map factors through $\pi_1^{\text{alg}}(X_0)$.
b) If you abelianize the situation, and look at the p-part of these groups, the kernel tends to be quite large. In fact
$$\pi_1^{\text{alg}}(\overline{X}_\eta)/\left[\pi_1^{\text{alg}}, \pi_1^{\text{alg}}\right] \cong \prod_{\text{primes } l} \mathbb{Z}_l^{2g} \times T_0$$
while
$$\pi_1^{\text{alg}}(X_0)/\left[\pi_1^{\text{alg}}, \pi_1^{\text{alg}}\right] \cong \prod_{l \neq p} \mathbb{Z}_l^{2g} \times \mathbb{Z}_p^r \times T_p$$
where $0 \leq r \leq g$ and T_0, T_p are finite groups, (0) in the case of curves (compare (6.?????) (a)). In fact, Shafarevitch [**97**] has shown for curves that the maximal pro-p-nilpotent quotient of $\pi_1^{\text{alg}}(X_0)$ is a free pro-p-group on r generators.

Going back now to general deformation theory, it is clear that the really powerful applications are in situations where one can apply the basic set-up: $R \to R/I$ $(I \cdot M = (0))$ inductively and get statements over general artin rings and via GFGA (Theorem 2.17) to complete local ring. In the two cases examined above, we could do this by proving that there were no obstructions. However even if obstructions may be present, one can seek to build up inductively a *maximal* deformation of the original variety X_0/k. This is the point of view of moduli, which we want to sketch briefly.

Start with an arbitrary scheme X_0 over k. Then for all artin local rings R with residue field k, define

$$\begin{array}{l} \mathrm{Def}(X_0/R) \\ \text{or} \\ \text{the deformations} \\ \text{of } X_0 \text{ over } R \end{array} = \left\{ \begin{array}{l} \text{the set of triples } (X, \phi, \pi), \text{ where} \\ \pi\colon X \longrightarrow \operatorname{Spec} R \text{ is a flat morphism} \\ \text{and } \phi\colon X \times_{\operatorname{Spec} R} \operatorname{Spec} k \xrightarrow{\approx} X_0 \text{ is a} \\ k\text{-isomorphism, modulo } (X, \phi, \pi) \sim (X', \phi', \pi') \\ \text{if } \exists \text{ an } R\text{-isomorphism } \psi\colon X \xrightarrow{\approx} X' \text{ such that} \\ \begin{array}{ccc} X \times_{\operatorname{Spec} R} \operatorname{Spec} k & \xrightarrow{\psi \times 1_k} & X' \times_{\operatorname{Spec} R} \operatorname{Spec} k \\ & {\scriptstyle\phi} \searrow \quad X_0 \quad \swarrow {\scriptstyle\phi'} & \end{array} \\ \text{commutes.} \end{array} \right\}$$

Note that
$$R \longmapsto \mathrm{Def}(X_0/R)$$
is a covariant functor for all homomorphisms $f\colon R \to R'$ inducing the identities on the residue fields. In fact, if $(X, \phi, \pi) \in \mathrm{Def}(X_0/R)$, let

$X' = X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$

$\pi' =$ projection of X' onto $\operatorname{Spec} R'$

$\phi' =$ the composition:

$$X' \times_{\operatorname{Spec} R'} \operatorname{Spec} k = (X \times_{\operatorname{Spec} R} \operatorname{Spec} R') \times_{\operatorname{Spec} R'} \operatorname{Spec} k \cong X \times_{\operatorname{Spec} R} \operatorname{Spec} k \xrightarrow{\phi} X_0.$$

Then $(X', \phi', \pi') \in \mathrm{Def}(X_0/R')$ depends only on the equivalence class of (X, ϕ, π) and on f. One says that X'/R' is the deformation obtained from X/R by the base change f. What one wants to do next is to *build up inductively the biggest possible deformation of* X_0 so that any other is obtained from it by base change! More precisely, suppose that $\mathfrak{R}$ is a complete noetherian local

ring with maximal ideal $\mathfrak{M}$ and residue field $\mathfrak{R}/\mathfrak{M} = k$. Then we get a sequence of artin local rings $R_n = \mathfrak{R}/\mathfrak{M}^{n+1}$. Then, by definition, a *formal deformation* $\mathcal{X}$ *of* X_0 *over* $\mathfrak{R}$ is a sequence of deformations X_n and closed immersions ϕ_n:

$$\begin{array}{ccccccccc} & & \cdots \longleftarrow & X_n & \xleftarrow[\phi_n]{} & X_{n-1} & \longleftarrow \cdots \xleftarrow[\phi_1]{} & X_0 \\ & & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\mathfrak{R} & \longleftarrow \cdots \longleftarrow & & \operatorname{Spec} R_n & \longleftarrow & \operatorname{Spec} R_{n-1} & \longleftarrow \cdots \longleftarrow & \operatorname{Spec} k \end{array}$$

where ϕ_n induces an isomorphism:

$$X_{n-1} \xrightarrow{\approx} X_n \times_{\operatorname{Spec} R_n} \operatorname{Spec} R_{n-1}.$$

Note that if S is artin local with residue field k, $\mathfrak{R} \to S$ is a homomorphism inducing identity on residue fields and $\mathcal{X}/\mathfrak{R}$ is a formal deformation, we can define a real deformation $\mathcal{X} \times_{\operatorname{Spec}\mathfrak{R}} \operatorname{Spec} S$ by base change, since $\mathfrak{R} \to S$ factors through R_n if n is large enough. Then a formal deformation $\mathcal{X}/\mathfrak{R}$ is said to be *versal* or *semi-universal* if:

(1) every deformation Y of X_0 over S is isomorphic to the one obtained by base change $\mathcal{X} \times_{\operatorname{Spec}\mathfrak{R}} \operatorname{Spec} S$ for a suitable $\alpha\colon \mathfrak{R} \to S$, and
(2) if the maximal ideal $N \subset S$ satisfies $N^2 = (0)$, then one asks that there be only one α for which (1) holds.
(2′) $\mathcal{X}/\mathfrak{R}$ is *universal* if α is always unique. It is clear that a universal deformation is unique if it exists, and it is not hard to prove that a versal one is also unique, but only up to a non-canonical isomorphism.

A theorem of Grothendieck and Schlessinger [**85**] asserts the following:

a) If X_0 is smooth and proper over k, then a versal deformation $\mathcal{X}/\mathfrak{R}$ exists and there is a canonical isomorphism:

$$\left.\begin{array}{ll} \operatorname{char} k = 0: & \operatorname{Hom}_k(\mathfrak{M}/\mathfrak{M}^2, k) \\ \operatorname{char} k = p: & \operatorname{Hom}_k(\mathfrak{M}/(\mathfrak{M}^2 + (p)), k) \end{array}\right\} \cong H^1(X_0, \Theta_{X_0}).$$

b) If $H^0(X_0, \Theta_{X_0}) = (0)$, then $\mathcal{X}/\mathfrak{R}$ is universal.
c) If $H^2(X_0, \Theta_{X_0}) = (0)$, then

$$\left\{\begin{array}{lll} \operatorname{char} k = 0: & \mathfrak{R} \cong k[[t_1, \ldots, t_n]], & n = \dim H^1(X_0, \Theta_{X_0}) \\ \operatorname{char} k = p,\ k \text{ perfect}: & \mathfrak{R} \cong W(k)[[t_1, \ldots, t_n]], & n = \dim H^1(X_0, \Theta_{X_0}). \end{array}\right.$$

A further development of these ideas leads us to the global problem of moduli. Starting with any X_0 smooth and proper over k, suppose you drop the restriction that R be an artin local ring and for any pair $(R, \mathfrak{m})$, R a ring, $\mathfrak{m} \subset R$ a maximal ideal such that $R/\mathfrak{m} = k$ you define $\operatorname{Def}(X_0/R)$ to be the pairs (X, ϕ) as before, but now X is assumed smooth and proper over $\operatorname{Spec} R$. If moreover you isolate the main qualitative properties that X_0 and its deformations have, it is natural to cut loose from the base point $[\mathfrak{m}] \in \operatorname{Spec} R$ and consider instead functors like:

$$\mathfrak{M}_{\mathbf{P}}(S) = \left\{\begin{array}{l} \text{set of smooth proper morphisms } f\colon X \to S \text{ such that} \\ \text{all the fibres } f^{-1}(s) \text{ of } f \text{ have property } \mathbf{P}, \\ \text{modulo } f \sim f' \text{ if } \exists \text{ an } S\text{-isomorphism } g\colon X \xrightarrow{\approx} X' \end{array}\right\},$$

where S is any scheme and $\mathbf{P}$ is some property of schemes X over fields k. Provided that $\mathbf{P}$ satisfies: [if X/k has $\mathbf{P}$ and $k' \supset k$, then $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ has $\mathbf{P}$], then $\mathfrak{M}_{\mathbf{P}}$ is a functor in S, i.e., given $g\colon S' \to S$ and $X/S \in \mathfrak{M}_{\mathbf{P}}(s)$, then $X \times_S S' \in \mathfrak{M}_{\mathbf{P}}(S')$.

For instance, take $\mathbf{P}(X/k)$ to mean

$$\begin{aligned}&\dim X = 1\\&H^0(X, \mathcal{O}_X) \xrightarrow{\approx} k\\&\dim_k H^1(X, \mathcal{O}_X) = g;\end{aligned}$$

then $\mathfrak{M}_{\mathbf{P}}$ is the usual moduli functor for curves of genus g. The "problem of moduli" is just the question of describing $\mathfrak{M}_{\mathbf{P}}$ as explicitly as possible and in particular asking how far it deviates from a representable functor. The best case, in other words, would be that as functors in S, $\mathfrak{M}_{\mathbf{P}}(S) \cong \mathrm{Hom}(S, M_{\mathbf{P}})$ for some scheme $M_{\mathbf{P}}$ which would then be called the moduli space. For an introduction to these questions, see Mumford et al. [**72**].

CHAPTER IX

Elementary treatment of Hilb, ?? Macaulay $\boxed{\textbf{No manuscript}}$

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