NOTES

CHAPTERS I – III

1. Coherent sheaves

In Chap I, Definition 5.3, a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *coherent* if for every $x \in X$, there is an affine open $U \subset X$ containing x and a short exact sequence

$$\mathscr{O}_U{}^m \longrightarrow \mathscr{O}_U{}^n \longrightarrow \mathscr{F}|_U \longrightarrow 0;$$

The above finiteness property is usually referred to as *locally of finite presentation*. A footnote says that Definition 5.3 is a good definition (so that it coincides with the usual definition) only when X is noetherian.

The standard definition of *coherent* $\mathscr{O}_X$-module is slightly different: one requires

- $\mathscr{F}$ is locally of finite type over $\mathscr{O}_X$, i.e. locally we have a $\mathscr{O}_X$-linear surjection $\mathscr{O}_U{}^n \twoheadrightarrow \mathscr{F}$, and
- for every affine open $U \subset X$ and every $\mathscr{O}_U$-linear homomorphism $h\colon \mathscr{O}_U{}^n \longrightarrow \mathscr{F}$, the kernel of h is of finite type, i.e. there exists an $\mathscr{O}_X$-linear short exact sequence (exact at $\mathscr{O}_U{}^n$)

$$\mathscr{O}_U{}^m \longrightarrow \mathscr{O}_U{}^n \xrightarrow{h} \mathscr{F}|_U\,.$$

Note that if X is covered by a finite number of affine opens U_i such that the above property holds for each $(U_i, \mathscr{F}|_{U_i})$, then $\mathscr{F}$ is quasi-coherent.

BASIC PROPERTIES

(1) If $\mathscr{H}$ is an $\mathscr{O}_X$-module locally of finite presentation, then for every $\mathscr{O}_X$-module $\mathscr{G}$ and every $x \in X$ $\mathscr{G}$ the natural map

$$\left(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{H}, \mathscr{G})\right)_x \longrightarrow \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{H}_x, \mathscr{G}_x)$$

is an isomorphism.

(2) If $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is an $\mathscr{O}_X$-module homomorphism between coherent $\mathscr{O}_X$-modules, then $\mathrm{Ker}(\phi)$, $\mathrm{Coker}(\phi)$, $\mathrm{Image}(\phi)$ and $\mathrm{Coimage}(\phi)$ are all coherent $\mathscr{O}_X$-modules.

(3) If $0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$ is a short exact sequence of quasi-coherent $\mathscr{O}_X$-modules, and $\mathscr{F}_1$ and $\mathscr{F}_3$ are coherent, then $\mathscr{F}_2$ is coherent. [Actually the statement remains valid if we weaken the assumption that $\mathscr{F}_2$ is a quasi-coherent $\mathscr{O}_X$-module to that $\mathscr{F}_2$ is an $\mathscr{O}_X$-module.]

(4) If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ are coherent $\mathscr{O}_X$-modules.

(5) If $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module, then a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is coherent if and only if it is locally of finite type over $\mathscr{O}_X$. Note that $\mathscr{O}_X$ is coherent if and only if X is locally noetherian.

Remark. The notion of quasi-coherent $\mathscr{O}_X$-modules generalizes to any ringed space $(X, \mathscr{O}_X)$: it is a sheaf of (left) $\mathscr{O}_X$-module $\mathscr{F}$ such that for every $x \in X$ there exist an open neighborhood $U \ni x$ and an exact sequence

$$\mathscr{O}_U{}^J \longrightarrow \mathscr{O}_U{}^I \longrightarrow \mathscr{F} \longrightarrow 0\,.$$

AN UNPLEASANT EXAMPLE. Let k be a field. For each integer $n \geq 1$, let

$$R_n := k[x_0, x_1, \ldots, x_n]/(x_0^2, x_0x_1, x_0x_2, \ldots, x_0x_n)\,.$$

Let $R := \prod_{n\geq 1} R_n$, and let u be the element in R whose n-th component is the image of x_0 in R_n. Let $X = \operatorname{Spec} R$, and let $\phi\colon \mathscr{O}_X \longrightarrow \mathscr{O}_X$ be the $\mathscr{O}_X$-linear map given by the element $u \in R$. Then $\operatorname{Ker}(\phi)$ is the quasi-coherent $\mathscr{O}_X$-ideal associated to the ideal $I := \prod_{n\geq 1} I_n$ of R, where I_n is the ideal of R_n generated by the image of $x_0, x_1, \ldots, x_n$ in R_n. It is easy to see that I is not a finitely generated ideal of R, so $\mathscr{O}_X$ is not a coherent $\mathscr{O}_X$-module.

2. Quasi-compact and quasi-separated morphisms

As noted in a footnote to Chap. II, Prop. 3.11, if $f\colon X \to Y$ is a quasi-compact morphism of schemes, then $\operatorname{Ker}(\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X)$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_Y$. This ideal defines a closed subscheme of Y, which will be called the scheme-theoretic closure of Y.
[Sketch of proof. We may assume that Y is affine. Let $\{U_i : i \in I\}$ be an open affine cover of X indexed by a finite set I. Let $\iota_i\colon U_i \hookrightarrow X$ be the inclusion morphism. Applyingf_* to the injection $\mathscr{O}_X \longrightarrow \prod_{i\in I} \iota_{i*}\iota_i^*\mathscr{O}_X$, we get an injection $f_*\mathscr{O}_X \longrightarrow \prod_{i\in I}(f\circ\iota_i)_*\mathscr{O}_{U_i}$. So $\operatorname{Ker}(\mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X) = \operatorname{Ker}(\mathscr{O}_Y \longrightarrow \prod_{i\in I}(f\circ\iota)_*\mathscr{O}_{U_i})$. Note that $\prod_{i\in I}(f\circ\iota)_*\mathscr{O}_{U_i}$ is a quasi-coherent $\mathscr{O}_Y$-module because each U_i is affine, so the kernel of the above map is quasi-coherent.]

A morphism $f\colon X \to Y$ of schemes is said to be *quasi-separated* if the diagonal morphism

$$\Delta_{X/Y}\colon X \longrightarrow X \times_Y X$$

is quasi-compact. In Chap. II, Prop. 4.10, the morphism $f\colon X \to Y$ is assumed to be quasi-compact and separated. The conclusion that the direct image $f_*\mathscr{F}$ of a quasi-coherent $\mathscr{O}_X$-module is a quasi-coherent $\mathscr{O}_Y$-module remains valid if “separated” is replaced by “quasi-separated”. The proof is essentially the same. From this point on, a blanket assumption is made that all schemes are separated over $\operatorname{Spec}(\mathbb{Z})$, which implies that all morphisms are separated. Without this blanket assumption, some adjustments may be needed in subsequent materials. Some examples are noted below.

In Chap. III, Lemma 4.1 (i) holds for any quasi-compact scheme X, but one needs to assume in Lemma 4.1 (ii) that X is quasi-separated (over $\operatorname{Spec}(\mathbb{Z})$).

CHAPTER V. SINGULAR VS. NON-SINGULAR

- It is easy to give an example of a complex algebraic variety X and a point x of X which is unibranch but not normal: Take $X = \operatorname{Spec}(R)$ with $R = \mathbb{C} + t^2\mathbb{C}[t]$), and x corresponds to the quotient of R by the maximal ideal $t^2\mathbb{C}[t]$ of R. The normalization of R is the polynomial ring $\mathbb{C}[t]$, and $\mathbb{A}^1 \to X$ is a homeomorphism.

- In the statement $\widetilde{U5}$) of Zariski's connectedness theorem for (X,x), that if Z is integral and $f\colon Z \to X$ is a proper birational dominant morphism with connected geometric generic fiber, then $f^{-1}(x)$ is connected, one cannot strengthen the conclusion to “$f^{-1}(x)$ is geometrically connected”. Here is an example.
  Let $Z = \operatorname{Spec}(\mathbb{C}[t]$, the affine line over $\mathbb{C}$. Let $R = \mathbb{R} + t\mathbb{C}[t]$ be the ring of all polynomials $g(t) \in \mathbb{C}[t]$ such that $g(0) \in \mathbb{R}$. We have an isomorphism

  $$\alpha\colon \mathbb{R}[u,v]/(u^2+v^2) \xrightarrow{\sim} R\,, \qquad u \mapsto t\,,\ \ v \mapsto \sqrt{-1}t\,.$$

  Let $X = \operatorname{Spec}(R)$. It is easy to see that $\mathbb{C}[x]$ is the integral closure of R in the fraction field $\mathbb{C}(x)$ of R, $f\colon Z \to X$ is a homeomorphism, and f is an isomorphism outside the closed point $x :=$

$\mathrm{Spec}(R/t\mathbb{C}[t]) \cong \mathrm{Spec}(\mathbb{R})$. But $f^{-1}(x) \cong \mathrm{Spec}(\mathbb{C})$, which is connected but not geometrically connected over $x \cong \mathrm{Spec}(\mathbb{R})$.

Chapter VIII Appendix. Residues of differentials on curves

(1) In the comment "ϕ and ψ induce mutually inverse isomorphisms" for property (T5), "mutually inverse" should be deleted: ϕ is an isomorphism from W' to W, and ψ is an isomorphism from W to W', but they are not necessarily inverse to each other.

(2) The classical style of treating algebraic curves via valuations and adeles (as in the book by Chevalley, following a 1938 paper by Weil written in German) amounts to, in modern language, considering one-dimension irreducible regular schemes X of finite type proper over a field k. Let K be the function field of X, denoted $\mathbb{R}(X)$ in Mumford-Oda. We may and do assume that k is algebraically closed in K. Then X can be recovered from K by considering discrete valuations on K which are trivial on k.

In general the scheme X may not be smooth over k. There are two potential problems. First the field K may not be separable over k, i.e. $K/k(x)$ is not a finite separable extension for every $x \in K$; equivalently $\mathrm{Spec}(K)$ may not be smooth over $\mathrm{Spec}(k)$. An example: $k = F(u,v)$, u,v transcendental over a field $F \supset \mathbb{F}_p$, and K is the fraction field of $k[x,y]/(x^p - uy^p - v)$.

Even when K/k is a regular extension (i.e. K/k is separable and k is algebraically closed in K), the morphism $X \to \mathrm{Spec}(k)$ may still be non-smooth. An example: $k \supset \mathbb{F}_p$, $a \notin k^p$, p odd, and K is the fraction field of $k[x,y]/(y^2 - x^p + a)$. Then $X \supset \mathrm{Spec}(k[x,y]/(y^2 - x^p + a))$, $\mathrm{Spec}(k[x,y]/(y^2 - x^p + a))$ is regular but not smooth over k. Note that the genus of K is $(p-1)/2$, while the genus of $K \cdot k^{1/p}$ is 0. (Whenever X is not smooth over k, the phenomenon "genus change under constant field extension" occurs—see Chapter 15 of Artin's *Algebraic Numbers and Algebraic Functions*.

(3) Tate's definition of residue give a K-linear map $\sigma\colon \Omega^1_{K/k} \to J_{K/k}$, This map is an isomorphism if K/k is separable (hence regular) as shown in the text, while is identically 0 if K is not separable over k. The last assertion follows from Theorem 5 (iii) $K/k(x)$: for any $x \in K$ transcendental over k, K is inseparable over $k(x)$, so $\mathrm{Tr}_{K/k(x)}$ is identically zero. Sheafifying the map σ give an $\mathscr{O}_X$-linear map

$$\sigma_X\colon \Omega^1_{X/k} \longrightarrow J_{X/k}$$

where $J_{X/k}$ is an invertible $\mathscr{O}_X$-module, while $\Omega^1_{X/k}$ is an invertible $\mathscr{O}_X$-module if and only if X is smooth over k.

We have seen that map σ_X is identically zero when K/k is not separable. When X is not smooth over k, the map σ_X is never injective (because $\Omega_{X/k}$ has torsion, and it may not be surjective either. Consider the case K being the fraction field of $k[x,y]/(y^2 - x^p + a)$, $a \notin k^p$, p odd. Let P_0 be the closed point of X corresponding to the principal ideal $yk[x,y]/(y^2 - x^p + a)$ of $k[x,y]/(y^2 - x^p + a)$. Then the image of σ_X is $J_{X/k}(-P_0)$ as can be checked by an easy computation using Theorem 5 (iii).

[Proof: Let $\mathscr{O}'$ be the completed local ring at P_0, and let $\mathscr{O}$ be the completion of the localization of $k[x]$ at the principle ideal generated by $x^p - a$; the maximal ideal of these two discrete valuation rings are generated by y and $x^p - a$ respectively. Then the sheaf of continuous differentials $\Omega^1_{\mathscr{O}'/k}$ is generated by dx and dy, with the relation $ydy = 0$. So $\sigma_{X,P_0}(dy) = 0$, and the image of

σ_{X,P_0} is determined by Theorem 5 (iii). An easy computation shows that $\mathrm{Tr}_{\mathscr{O}'/\mathscr{O}}(y^{-1}\mathscr{O}') = \mathscr{O}$, while $\mathrm{Tr}_{\mathscr{O}'/\mathscr{O}}(y^{-2}\mathscr{O}') = (x^p - a)^{-1}\mathscr{O}$.]

CHAPTER VII §2 GAGA AND GAGF

1. In §2 on GAGA and GFGA, Theorem 2.8 is stated and proved for $\mathbb{P}^n$, or equivalently, all projective varieties over $\mathbb{C}$. It might be useful to add, in the form of a remark or a footnote, that the statement of Theorem 2.8 holds also when the underlying variety is replaced by a complete variety over $\mathbb{C}$, i.e. a scheme proper over $\mathbb{C}$, by an argument using Chow's Lemma and Noetherian induction similar to the proof of Theorem 6.5. It can be left to the reader as an exercise.

   Similarly, Theorem 2.17 is sketched in the case where X is proper over a complete noetherian local ring R and I is the maximal ideal of R. A footnote/remark that an argument using Chow's Lemma and Noetherian induction as in the proof of Theorem 6.5 shows that for any fixed complete noetherian ring (R,I), the general case with X proper over R follows from the case X is projective over R.

2. The proof of Theorem 2.17 is restricted to the case when I is the maximal ideal. This assumptions make the Mittag-Leffler condition automatically satisfied and simplifies the proof.

   Grothendieck's original proof of GAFA does not seem to have been published. The folk lore is that the original proof uses downward induction on the degree of cohomology, as in Serre's proof of GAGA. The published proof in EGA III 4.1, the degree is fixed, and the Artin-Rees/Mittag-Leffler type conditions is deduced from the finiteness theorem for proper morphisms, applied to the base change of $f\colon X \longrightarrow \mathrm{Spec}R$ to the spectrum Artin-Rees algebra $\widetilde{R} := \bigoplus_{n=0^\infty} I^n$. Mumford's proof of 2.8 does not use downward induction on the degree of cohomology, instead using downward induction on the integers m and d for $\mathbb{P}^d$ and $\mathcal{F}(m)$. In the proof of 2.17, the proof of the required uniform vanishing is by applying the usual vanishing theorem to the associated graded ring $\mathrm{gr}(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$.

   It might be of some interest to add a remark/footnote that the trick of passing to the base change to the associated graded ring $\mathrm{gr}(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ also gives a proof that projective system

   $$\big(H^i(\mathcal{F}_n)\big)_{n\in\mathbb{N}}$$

   attached to any compatible family of coherent $\mathscr{O}_{X_n}$-modules $(\mathcal{F}_n)_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition, and the topology induced on the projective limit

   $$\varprojlim_n H^n(\mathcal{F}_n)$$

   is equal to the I-adic topology on $\varprojlim_n H^n(\mathcal{F}_n)$, *without* the assumption that I is the maximal ideal. With this additional ingredient, the proof of Theorem 2.17 becomes valid for a general open ideal I in a complete noetherian local ring R, and not only for case I is maximal. The detailed proof

   (∗) finiteness of $\bigoplus_{n=0}^{\infty} H^i(I^n\mathcal{F}_n)$ as an $\bigoplus_{n=0}^{\infty} I^n/I^{n+1}$-module for $i = i_0$ and i_0+1
   $\Longrightarrow$ uniform Mittag-Leffler/Artin-Rees for $\big(H^i(\mathcal{F}_n)\big)_{n\in\mathbb{N}}$

   is in EGA 0_{III}13.7.7 (with correction).

SKETCH OF THE SPECTRAL SEQUENCE ARGUMENT FOR THE PROOF OF $(*)$. For any triple of natural numbers p,i,n, define

$$\begin{aligned} Z_r^{p,i-p}(H^\bullet(\mathcal{F}_n)) &:= \operatorname{Im}\big(H^i(I^p\mathcal{F}_n/I^{p+r}\mathcal{F}_n) \longrightarrow H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n)\big) \\ B_r^{p,i-p}(H^\bullet(\mathcal{F}_n)) &:= \operatorname{Im}\big(H^{i-1}(I^{p-r+1}\mathcal{F}_n/I^p\mathcal{F}_n) \longrightarrow H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n)\big) \end{aligned}$$

Note that $H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n) = H^i(I^p\mathcal{F}_p)\ \forall n \geq p$. For each fixed n we have natural isomorphisms

$$(\dagger) \qquad d_r^{p,i-p} : \frac{Z_r^{p,i-p}(H^\bullet(\mathcal{F}_n))}{Z_{r+1}^{p,i-p}(H^\bullet(\mathcal{F}_n))} \xrightarrow{\sim} \underline{B_{r+1}^{p+r,i-p-r+1}(H^\bullet(\mathcal{F}_n))/B_r^{p+r,i-p-r+1}(H^\bullet(\mathcal{F}_n))}$$

inclusion relations

$$\begin{aligned} (0) = B_1^{p,i-p}(H^\bullet(\mathcal{F}_n)) \subseteq B_2^{p,i-p}(H^\bullet(\mathcal{F}_n)) \subseteq \cdots \subseteq B_{p+1}^{p,i-p}(H^\bullet(\mathcal{F}_n)) = B_{p+2}^{p,i-p}(H^\bullet(\mathcal{F}_n)) = \cdots \\ = B_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n)) \ \subseteq\ Z_\infty^{p,i-p}(H^\bullet(\mathcal{F}_\infty)) = \cdots = Z_{n-p+2}^{p,i-p}(H^\bullet(\mathcal{F}_n)) = Z_{n-p+1}^{p,i-p}(H^\bullet(\mathcal{F}_n)) \\ \subseteq Z_{n-p}^{p,i-p}(H^\bullet(\mathcal{F}_n)) \subseteq \cdots \subseteq Z_1^{p,i-p}(H^\bullet(\mathcal{F}_n)) = H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n) \end{aligned}$$

and isomorphisms

$$\frac{\operatorname{Im}\big(H^i(I^p\mathcal{F}_n) \longrightarrow H^i(\mathcal{F}_n)\big)}{\operatorname{Im}\big(H^i(I^{p+1}\mathcal{F}_n) \longrightarrow H^i(\mathcal{F}_n)\big)} \xrightarrow{\sim} \frac{Z_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n))}{B_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n))}$$

which comes from th map between long exact sequences attached to the map

$$\begin{tikzcd} 0 \arrow[r] & I^p\mathcal{F}_n \arrow[r] \arrow[d] & \mathcal{F}_n \arrow[r] \arrow[d] & \mathcal{F}_n/I^p\mathcal{F}_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n \arrow[r] & \mathcal{F}_n/I^{p+1}\mathcal{F}_n \arrow[r] & \mathcal{F}_n/I^p\mathcal{F}_n \arrow[r] & 0 \end{tikzcd}$$

between short exact sequences. If we fix p and r fixed, the we have

$$\cdots \xrightarrow{\sim} B_r^{p,i-p}(H^\bullet(\mathcal{F}_{p+2})) \xrightarrow{\sim} B_r^{p,i-p}(H^i(\mathcal{F}_{p+1})) \xrightarrow{\sim} B_r^{p,i-p}(H^i(\mathcal{F}_p))$$

and

$$\cdots \xrightarrow{\sim} Z_r^{p,i-p}(H^\bullet(\mathcal{F}_{p+r+1})) \xrightarrow{\sim} Z_r^{p,i-p}(H^\bullet(\mathcal{F}_{p+r})) \xrightarrow{\sim} Z_r^{p,i-p}(H^\bullet(\mathcal{F}_{p+r-1}))\,;$$

denote by $B_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty))$ and $Z_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty))$ the respective projective limits, and let

$$E_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty)) := Z_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty))/B_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty))\,.$$

We record here that

$$(\ddagger) \qquad \begin{array}{lll} Z_r^{p,q}(H^\bullet(\mathcal{F}_m)) & \text{stabilizes} & \begin{cases} \text{in the } r\text{-direction for } r \geq n-p+1 \\ \text{in the } n\text{-direction for } n \geq p+r-1 \end{cases} \\ B_r^{p,q}(H^\bullet(\mathcal{F}_m)) & \text{stabilizes} & \begin{cases} \text{in the } r\text{-direction for } r \geq p+1 \\ \text{in the } n\text{-direction for } n \geq p \end{cases} \end{array}$$

For each i and r, the direct sum

$$\bigoplus_{p\geq 0} B_r^{p,i-p}(H^i(\mathcal{F}_\infty))$$

has a natural structure as a graded $(\bigoplus_{p\geq 0} I^p/I^{p+1})$-submodule of $\bigoplus_{p\geq 0} H^i(I^p\mathcal{F}_p)$ and increases with r. Since $\bigoplus_{p\geq 0} H^i(I^p\mathcal{F}_p)$ is finitely generated over $\bigoplus_{p\geq 0}(I^p/I^{p+1})$, the increasing chain of submodules $\bigoplus_{p\geq 0} B_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty))$ stabilizes, for $r \geq r(i)$, where $r(i)$ is a positive integer depending on i. (This is where the properness assumption is used.) So the differentials $d_r^{p,q} = 0$ for all p,q with $p+q = i_0 - 1$ and all $r \geq r(i-1)$. So the decreasing chain of submodules $\bigoplus_{p\geq 0} Z_r^{p,i-p}(H^\bullet(\mathcal{F}_\infty))$ stabilizes for $r \geq r(i-1)$. Let r_0 be the maximum of $r(0), r(1), \ldots, r(d)$, where $d = \dim(X_0/\mathrm{Spec}(R/I))$. Then we have

(1) $B_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \xrightarrow{\sim} B_{r_0+1}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \varinjlim_r B_r^{p,q}(H^\bullet(\mathcal{F}_\infty)) \qquad \forall p,q$

(2) $Z_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \xleftarrow{\sim} Z_{r_0+1}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \xleftarrow{\sim} \cdots \xleftarrow{\sim} \varprojlim_r Z_r^{p,q}(H^\bullet(\mathcal{F}_\infty)) \qquad \forall p,q$

(3) There exists a positive integer p_0 such that

$$\begin{aligned} I\cdot Z_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) &= Z_{r_0}^{p+1,q}(H^\bullet(\mathcal{F}_\infty)), \qquad \forall p \geq p_0 \\ I\cdot B_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) &= B_{r_0}^{p+1,q}(H^\bullet(\mathcal{F}_\infty)), \qquad \forall p \geq p_0 \text{ and} \\ I\cdot E_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) &= E_{r_0}^{p+1,q}(H^\bullet(\mathcal{F}_\infty)) \qquad \forall p \geq p_0. \end{aligned}$$

Tis follows from fact that the graded $(\bigoplus_{p\geq 0} I^p/I^{p+1})$-modules

$$\bigoplus_{p\geq 0} B_{r_0}^{p,i-p}(H^\bullet(\mathcal{F}_\infty)), \quad \bigoplus_{p\geq 0} Z_{r_0}^{p,i-p}(H^\bullet(\mathcal{F}_\infty)), \quad \text{and} \quad \bigoplus_{p\geq 0} E_{r_0}^{p,i-p}(H^\bullet(\mathcal{F}_\infty))$$

are finitely generated.

From (1) and the stabilization range for $Z_r^{p,q}(H^\bullet(\mathcal{F}_n))$, we see that $Z_r^{p,q}(H^\bullet(\mathcal{F}_n)) = Z_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty))$ if $p \leq p_0$, $n \geq p + r_0 - 1$ and $r \geq r_0$. Note that the family $\left(Z_r^{p,q}(H^\bullet(\mathcal{F}_n))\right)_{r,n\in\mathbb{N}}$ is projective in both directions r and n, and the transition maps are injective for $n \geq p$. So we see from (3) and (2) and (†) that

(4) $Z_r^{p,q}(H^\bullet(\mathcal{F}_n)) \xrightarrow{\sim} Z_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \quad$ if $n \geq p + r_0 - 1)$ and $r \geq r_0$

(5) $B_r^{p,q}(H^\bullet(\mathcal{F}_n)) \cong B_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \quad$ if $n \geq p$ and $r \geq r_0$.

It is not difficult to deduce from these that

(6) (uniform Mittag-Leffler) $\mathrm{Im}(H^i(\mathcal{F}_{n+r_0-1} \longrightarrow H^i(\mathcal{F}_n))) = \mathrm{Im}(H^i(\mathcal{F}_m \longrightarrow H^i(\mathcal{F}_n))$ for all $m \geq n + r_0 - 1$.

(7) For each i, the topology on $H^i(\mathcal{F}_\infty := \varprojlim_n H^i(\mathcal{F}_n)$ induced by the projective limit coincides with the I-adic topology on $H^i(\mathcal{F}_\infty$.