Math 626 Exercise Set 1

1. Let R be a non-zero commutative ring such that every ideal is a prime ideal. Show that R is a field.

2. Let $I, P_1, P_2, P_3, \ldots, P_m$ be ideals of a commutative ring R, $m \geq 2$ such that $I \subseteq \cup_{i=1}^{m}$ and P_i is a prime ideal for each $i \geq 3$. Show that there exists an i such that $I \subseteq P_i$.

3. (a) Give an explicit example of a homomorphism of commutative rings $A \to B$ such that B is a finite A-module and the going down property does not hold.

(b) Give an explicit example of a homomorphism of commutative rings $A \to B$ such that B is faithfully flat over A and the going up property does not hold.

4. (a) Show that for every prime ideal Q of a commutative ring R, there exists a minimal prime ideal of R contained in Q.

(b) Let $\phi : A \to B$ be a homomorphism of commutative rings. Show that going down holds for ϕ if and only if the following condition holds:

> *For every prime ideal $\wp$ in A and every prime ideal P of B containing $\wp$ whose image in $B/\wp B$ is a minimal prime ideal of $B/\wp B$, we have $\phi^{-1}(P) = \wp$.*

5. Let I be a finitely generated ideal in a commutative ring R. Suppose that $(0 : I) = (0)$. Show that the annihilator

$$\mathrm{Ann}_R(I/I^2) := \{x \in R \mid x \cdot I \subseteq I^2\}$$

of the R-module I/I^2 is contained in the radical

$$\mathrm{rad}(I) := \{y \in R \mid \exists n \in \mathbb{N} \text{ s.t. } y^n \in I\}$$

of I.

DEFINITION. Let X be a topological space.

(a) A subset U of X is *retro-compact* if for every quasi-compact open subset $V \subseteq X$, the intersection $U \cap V$ is quasi-compact.

(b) Let $\mathfrak{F}$ be the family of subsets of the power set of X, consists of all subsets $\mathcal{G}$ of 2^X with the following properties: (a) Every retro-compact open subset of X belongs to $\mathcal{G}$, (b) The complement of every element of $\mathcal{G}$ is an element of $\mathcal{G}$.

(c) All finite union of elements of $\mathcal{G}$ is an element of $\mathcal{G}$. The family $\mathfrak{F}$ has a unique minimal element $\mathcal{C} := \cap_{\mathcal{G} \in \mathfrak{F}} \mathcal{G}$. Elements of $\mathcal{C}$ are called *constructible subsets* of X.

6. (i) Suppose that U, V are retro-compact open subsets of X. Show that $U \cap V$ and $U \cup V$ are both retro-compact.

(ii) Show that a subset of X is constructible if and only if it is a finite union of subsets of the form $U \cap (X \setminus V)$, where U, V are retro-compact open subsets of X.

7. Let R be a commutative ring. Show that an open subset U of $\mathrm{Spec}(R)$ is retro-compact if and only if there exists a finitely generated ideal I of R such that $\mathrm{Spec}(R) \setminus U = \mathrm{Spec}(R/I)$.

8. Let $\phi : A \to B$ be a homomorphism of commutative rings, and let ${}^a\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the map between the spectra attached to ϕ. Show that if Z is a constructible subset of $\mathrm{Spec}(A)$, then ${}^a\phi^{-1}(Z)$ is a constructible subset of $\mathrm{Spec}(B)$.

9. Let R be a commutative ring, let M be a finitely generated R-module, and let $\alpha : R^{\oplus n} \to M$ be an R-linear surjection. Suppose that M is a finitely presented R-module. Show that $\mathrm{Ker}(\alpha)$ is a finitely generated R-module.

10. Let A be a commutative ring, let B be a finitely generated A-algebra, and let $\phi : A[x_1, \ldots, x_n] \to B$ be a surjective R-linear ring homomorphism, where $A[x_1, \ldots, x_n]$ is the polynomial ring over A in variables $x_1, \ldots, x_n$, $n \in \mathbb{N}$. Suppose that B is an R-algebra of finite presentation. Show that $\mathrm{Ker}(\phi)$ is a finitely generated ideal of $A[x_1, \ldots, x_n]$.

11. Let A be a commutative ring, and let B be a commutative A-algebra which is a finite A-module. Suppose that the B is of finite presentation as an A-algebra. Prove that the A-module underlying B is of finite presentation. (Hint: Show first that there exists a commutative A-algebra B' which is a finite free A-module and a surjective A-algebra homomorphism $B' \to B$.)

12. Let Z be a constructible subset of the spectrum $\mathrm{Spec}(A)$ of a commutative ring A. Show that there exists an A-algebra B of finite presentation over A such that Z is equal to the image of $\mathrm{Spec}(B)$ in $\mathrm{Spec}(A)$.