Math 626 Exercise Set 2

1. Give an example of a commutative ring and a non-trivial A-module M of finite presentation such that $\mathrm{Ass}_A(M) = \emptyset$.

2. Let M be a module over a commutative ring R. We define two variants of the definition of *associated primes*.

(a) Define $\mathrm{Ass}_d(M)$ to be the set consisting of all elements $\wp \in \mathrm{Spec}(R)$ such that there exists a multiplicatively closed subset $S \subseteq R$ with the following properties: (i) $S \cap \wp = \emptyset$, (ii) $\wp \cdot S^{-1}R$ is a maximal element of the family of all ideals $J \subseteq S^{-1}R$ of $S^{-1}R$ such that for each $a \in J$, there exists a non-zero element $y \in S^{-1}M$ with $a \cdot y = 0$.

(When $M = R/I$ for an ideal I in R, $\mathrm{Ass}_d(R/I)$ is classically known as the set of *prime divisors* of I, as defined by Nagata.)

(b) Define $\mathrm{Ass}_w(M)$ to be the set consisting of all elements $\wp \in \mathrm{Spec}(R)$ such that there exists an element $x \in M$ such that $\wp$ is a minimal element in the family of all prime ideals containing $\mathrm{Ann}_R(x)$.

(The definition of $\mathrm{Ass}_w(M)$ is due to Bourbaki.)

(1) Show that $M = (0)$ is equivalent to $\mathrm{Ass}_w(M) = \emptyset$ and also equivalent to $\mathrm{Ass}_d(M) = \emptyset$.

(2) Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. Do the inclusions

$$\mathrm{Ass}_w(M') \subseteq \mathrm{Ass}_w(M) \subseteq \mathrm{Ass}_w(M') \cup \mathrm{Ass}_w(M'')$$

hold? Do the inclusions

$$\mathrm{Ass}_d(M') \subseteq \mathrm{Ass}_d(M) \subseteq \mathrm{Ass}_d(M') \cup \mathrm{Ass}_d(M'')$$

hold?

(3) Show that $\mathrm{Ass}_w(M) \subseteq \mathrm{Ass}_d(M)$.

(4)* Does the equality $\mathrm{Ass}_w(M) \subseteq \mathrm{Ass}_d(M)$ hold for every module M over a commutative ring R?

(5) Suppose that R is Noetherian. Show that

$$\mathrm{Ass}_w(M) = \mathrm{Ass}(M) = \mathrm{Ass}_d(M).$$

3. Let R be a commutative ring. Let $a \in R$ be an element of R which a not a zero-divisor of R. Let b be an element of R, so that we have an element $\frac{b}{a}$ in the total ring of fractions $\mathrm{frac}(R)$ of R. Suppose that $\frac{b}{a} \notin R$. Show that either there exist an embedded prime in $\mathrm{Ass}(R/aR)$, or there exists a minimal element $\wp$ in the family of all prime ideals containing I such that $a \cdot R_\wp \not\subseteq b \cdot R_\wp$.

4. Let P be a prime ideal of a commutative ring R. Is it true that P^2 is a P-primary ideal in R? Either give a proof or give a counter-example.

5. (An example of a Noetherian ring with infinite Krull dimension)
Let k be a field. Let $\left(J_i\right)_{i\in\mathbb{N}}$ be a family of finite subsets. We assume that the cardinalities of the sets J_i are unbounded.

For each $i \in \mathbb{N}$, introduce a finite set of variables $\left(x_{i,j}\right)_{j\in J_i}$, and let $R' := k[x_{i,j}]_{i\in\mathbb{N},\, j\in J_i}$ be the polynomial ring in the infinitely many variables $x_{i,j}$. For each $i \in \mathbb{N}$, let $P_i = \sum_{j\in J_i} x_{i,j}R'$ be the ideal of R' generated by the elements $(x_{i,j})_{j\in J_i}$. Clearly P_i is a prime ideal for each i.

Let $S := R' \smallsetminus \cup_{i\in\mathbb{N}} P_i$ be the complement of the union of the prime ideals P_i. Let $R := S^{-1}R$ be the localization of R' with respect to S. For each $i \in \mathbb{N}$, let $\mathfrak{m}_i := P_iR$ be the maximal ideal of R corresponding to the prime ideal P_i. Clearly the height of $\mathfrak{m}_i$ is equal to $\mathrm{card}(J_i)$ for each i, hence the Krull dimension of R is ∞.

(a) Define subrings R'_n of R' in finitely variables by

$$R'_n := k[x_{i,j}]_{i\le n,\, j\in J_i},$$

so that we have $R'_0 \subseteq R'_1 \subseteq R'_2 \subseteq \cdots$ and $R = \cup_{n\in\mathbb{N}} R'_n$. Show that $P_i \cap R'_n = 0$ if $i > n$. In particular for every non-zero element $a \in R'$, there exists a natural number N such that $a \notin P_i$ for all $i \ge N$.

(b) Show that if an ideal I of R' is contained in $\cup_{i\in\mathbb{N}} P_i$, then there exists $i_0 \in \mathbb{N}$ such that $I \subseteq P_{i_0}$. Deduce that every maximal ideal of R is equal to $\mathfrak{m}_i$ for some $i \in \mathbb{N}$.

(Hint: The finiteness statement (a) may be useful.)

(c) Show that the local ring $R_{\mathfrak{m}_i}$ is Noetherian for every $i \in \mathbb{N}$.

(d) Show that the Krull dimension of R is equal to $\max\bigl(\mathrm{card}(J_i) \mid i \in \mathbb{N}\bigr)$.

(e) Show that R is a Noetherian ring.

6. Let k be a field. In this problem we discuss how to show that the transcendance degree of $k((x))$ over k is infinite.

(a) Let κ be the prime subfield of k. Thus $\kappa \cong \mathbb{Q}$ if $\mathrm{char}(k) = 0$, and $\kappa \cong \mathbb{F}_p$ if $\mathrm{char}(k) = p > 0$.

(a1) Show that the transcendence degree of $\kappa((x))$ over κ is equal to $\mathrm{card}(\mathbb{R})$.
(Hint: What is the cardinality of $\kappa((x))$? Relate the cardinality of $\kappa((x))$ to the cardinality of the algebraic closure of $\kappa((x))$ and also to the transcendance degree of $\kappa((x))$ over κ.)

(a2) Show that the natural ring homomorphism $k \otimes_\kappa \kappa((x)) \to k((x))$ is injective.

(a2) Show that the transcendence degree of $k \otimes_\kappa \kappa((x))$ over k equal to $\mathrm{card}(\mathbb{R})$. In particular the transcendance degree of $k((x))$ over k is infinite.

(b) Let p be a prime number. Recall that we have a $\mathbb{Z}$-valued p-adic valuation on the fraction field of $\mathbb{Z}_{(p)}[[x]]$.

(b1) Suppose that and $f(x)$ is an element of $\mathbb{Q}[[x]]$ which is integral over $\mathbb{Z}_{(p)}[[x]]$. Show that $f(x) \in \mathbb{Z}_{(p)}[x]$.

(b2) Suppose that $f(x) = \sum_{n\in\mathbb{N}} a_n x^n$ is an element of $\mathbb{Q}((x))$ which is algebraic over the fraction field of $\mathbb{Z}_{(p)}[[x]]$. Show that there exist $c, d \in \mathbb{N}$ such that $\mathrm{ord}_p(a_n) \ge -cn - d$ for all $n \in \mathbb{N}$.

(b3) Construct an infinite sequence $f_i(x)$, $i \in \mathbb{N}$ of elements of $\mathbb{Q}[[x]]$ which are algebraically independent over $\mathbb{Q}$.

7. Let k be a field, and let $k[[x,y,z]]$ be the ring of formal power series in three variables over k. Let $\mathfrak{P}$ be the prime ideal $xk[[x,y,z]] + yk[[x,y,z]]$ of $k[[x,y,z]]$, let R be the localization of $k[[x,y,z]]$ at $\mathfrak{P}$, and let $\hat{R}$ be the formal completion of the Noetherian local ring R. Note that R is a two-dimensional regular local ring, therefore it is a unique factorization domain. Let $\mathfrak{p}$ be the principal ideal of $\hat{R}$ generated by the element

$$x - \sum_{n\geq 1} \tfrac{1}{z^{n!}} y^n \in \hat{R}.$$

(a) Show that $\hat{R}$ is isomorphic to $k((z))[[x,y]]$.

(b) Show that $\mathfrak{p}$ is a prime ideal of $\hat{R}$ of height 1.

(c) Determine whether $\mathfrak{p} \cap R$ is equal to (0) or is a non-zero prime ideal of R.

(Note that $\mathfrak{p} \cap R$ has height at most 1, because $R \to \hat{R}$ is faithfully flat. So if $\mathfrak{p} \cap R \neq (0)$, then it is a principle ideal generated by a non-zero irreducible element of R.)