Math 626 Exercise Set 3

1. Let k be a field, and let $k(x_1,\ldots,x_m), k(y_1,\ldots,y_n)$ be polynomial algebras over k in m and n variables respectively. Determine the Krull dimension of $k(x_1,\ldots,x_m)\otimes_k k(y_1,\ldots,y_n)$.

2. (This is a lemme for problem 3 below.) Let A be a local domain with fraction field M. Let L be a subfield of M. Let $B := A\cap L$.

(i) Show that A is a local domain.

(ii) Show that for any non-zero element $b\in B$, we have $bA\cap B = bB$.

(iii) Suppose that A is a discrete valuation ring whose maximal ideal is equal to πA for an element $\pi\in L$. Show that $B := A\cap L$ is a discrete valuation ring with maximal ideal πB.

3. (This problem gives a method to construct two-dimensional Noetherian regular local rings.)

Let D be a discrete valuation ring with fraction field K. Let x be a variable, and let L be a subfield of the fraction field $\mathrm{frac}(D[[x]])$ of $D[[x]]$ which contains $K(x)$. Let π be a generator of the maximal ideal of D. Let $B := L\cap D[[x]]$.

(a) Show that B is a local ring whose maximal ideal is generated by π and x.

(b) Suppose that $\wp$ is a prime ideal of B such that $\pi\notin\wp$. Let b be an element of $\wp$ such that $\wp+\pi B = bB+\pi B$.

(b1) Show that $\wp\subseteq bB+\pi^n B$ for every $n\geq\mathbb{N}$.

(b2) Show that $\wp\subset bD[[x]]\cap B = bB$.

(c) Show that B is a Noetherian two-dimensional regular local ring whose formal completion is (isomorphic to) $D[[x]]$.

4. Let k be a field, and let $t=\sum_{i\in\mathbb{N}} c_i x^i$ be an element of $k[[x]]$ which is transcendental over $k(x)$. For each $n\in\mathbb{N}$, define the n-th tail of t by

$$t_n = x^{-n}\big(t-\sum_{i\leq n} c_i x_i\big) = \sum_{i\geq n+1} c_i x^{i-n}.$$

Note that $t_n = c_{n+1}x + x t_{n+1}$ for all $n\geq 0$. Define a k-subalgebra R of $k[[x]]$ by $R := k[x, t_n : n\in\mathbb{N}]$.

(a) Show that R is a discrete valuation ring.

(b) Show that $k(x,t)\cap k[[x]]$ is a discrete valuation ring.

(c) Determine whether R is equal to $k(x,t)\cap k[[x]]$.

5. Let $(R,\mathfrak{m})$ be a local ring such that the maximal idel $\mathfrak{m}$ is finitely generated. Show that R is a Noetherian ring if and only if $\cap_{n\in\mathbb{N}}\mathfrak{m}^n = 0$ and every finitely generated ideal of R is closed in the $\mathfrak{m}$-adic topology.

6. Let k be a field. Let $t = \sum_{i\in\mathbb{N}} c_i x^i$ be an element of $k[[x]]$ which is transcendental over $k(x)$. For each $n \in \mathbb{N}$, let t_n be the n-th tail of t as defined in problem 4. Let $R := k(x,y,t) \cap k[x]_{(x)}[[y]]$, where $k[x]_{(x)}$ is the ring of all power series in y with coefficients in the local ring $k[x]_{xk[x]}$. Note that R is a two-dimensional Noetherian regular local ring according to problem 3. Define a subring B of R as follows.

- For each $n \in \mathbb{N}$, let $f_n := yt_n$. Note that $f_n = c_{n+1}xy + xf_{n+1}$ for all $n \geq 0$.

  Define an increasing sequence of subrings B_n of $R \subseteq k(x,y,t)$ by

  $$B_n := k[x,y,f_n]_{(x,y,f_n)},$$

  the localization of $k[x,y,f_n]$ at the maximal ideal generated by x, y and all f_n's. Note that B_n is a three-dimensional Noetherian regular local ring for each n.

- Define a k-subalgebra B of $k(x,y,f_n) = k(x,y,t_n)$ by

  $$B := \cup_{n\in\mathbb{N}} B_n.$$

(a) Let $\mathfrak{n} = xB + yB$. Show that $\mathfrak{n}$ is a maximal ideal of B and B is a local ring.

(b) Let P be the ideal of B generated by y and all f_n's, i.e.

$$P := yB + \sum_{n\geq 0} f_n B.$$

Show that $P = yA \cap B = \cup_{n\geq 0}(P \cap B_n) = \cup_{n\geq 0}(yB_n + f_n B_n)$.

(c) Show that $Q \subsetneq \mathfrak{n}$, and Q is a prime ideal of B.

(d) Show that yB is a prime ideal of B and $yB \subsetneq Q$.

(e) Conclude from (b)–(d) that B is not Noetherian.

(f) Determine whether the principal ideal yB is closed in the $\mathfrak{n}$-adic topology of B.

(g)* Show that the Krull dimension of B is 3.