Math 626 Exercise Set 4

1. Let B be a commutative ring and let A be a subring of B such that B is integral over A. Let J be an ideal of B and let $I := J \cap B$.

(a) Show that $\dim(A/I) = \dim(B/J)$.

(b) Show that $\mathrm{ht}(J) \leq \mathrm{ht}(I)$ and

A note on terminology: $\dim(A/I)$ is also called *coheight* of I. Recall also that the height of an ideal I in a commutative ring R is defined as

$$\mathrm{ht}(I) = \min\{\mathrm{ht}(P) \mid P \in \mathrm{Spec}(A),\ P \supseteq I\}.$$

(c) Either prove that $\mathrm{ht}(J) = \mathrm{ht}(I)$ under the given assumptions, or give an explicit example in which $\mathrm{ht}(J) < \mathrm{ht}(I)$.

(d) Show that $\mathrm{alt}(J) \leq \mathrm{alt}(I)$.

Note: The *altitude* of an ideal I is

$$\mathrm{alt}(I) := \sup\{\mathrm{ht}(P) \mid P \text{ is a minimal element in the family of prime ideals containing } I\}.$$

(e) Either prove that $\mathrm{alt}(J) = \mathrm{alt}(I)$ under the given assumptions, or give an explicit example in which $\mathrm{alt}(J) < \mathrm{alt}(I)$.

2. Let k be a field, let A be a finitely generated k-algebra, and let $I_1 \subseteq \cdots \subseteq I_r$ be a chain of ideals of A. Suppose that $y_1, \ldots, y_d$ are elements of A which are algebraically independent over k with the following properties.

- A is integral over the polynomial ring $k[y_1, \ldots, y_d]$.
- There exist natural numbers $h_1 \leq h_2 \leq \cdots \leq h_r \leq d$ such that $I_i \cap k[y_1, \ldots, y_d] = y_1 k[y_1, \ldots, y_d] + \cdots + y_{h_i} k[y_1, \ldots, y_d]$ for $i = 1, \ldots, r$.

Recall that the existence of elements $y_1, \ldots, y_d$ with the above properties is guaranteed by the Noether normalization theorem.

(a) Suppose that A is an integral domain. Show that $\mathrm{ht}(I_i) = h_i$ for $i = 1, \ldots, d$.

(b) Does the conclusion of (a) hold without the assumption that A is an integral domain? Either prove the more general statement, or produce a counter-example.

Note: Part (b) comes from a minor error in Serre's *Algèbre Local Multiplicités*, III-24, Cor. 1.

3. Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $x_1, \ldots, x_d \in \mathfrak{m}$ be a system of parameters of A, where $d = \dim(A)$. Let $I := x_1 A + \cdots + x_d A$.

(a) Show that we have a natural ring isomorphism

$$\alpha : (A/\mathfrak{m})[X_1, \ldots, X_d] \xrightarrow{\sim} \left(\oplus_{n \geq 0} I^n / I^{n+1}\right) \otimes_{A/I} (A/\mathfrak{m})$$

from the polynomial ring $(A/\mathfrak{m})[X_1, \ldots, X_d]$ to $\oplus_{n \geq 0}(I^n / \mathfrak{m} I^n)$ such that $\alpha(X_i) = x_i + \mathfrak{m} I$ for $i = 1, \ldots, d$.

(b) Suppose that A contains a subring k which is a field. Show that $x_1, \ldots, x_d$ are algebraically independent over k.

4. Let A be an integral domain (not necessarily Noetherian), A contains a subring k which is a field, and the transcendental degree of the fraction field K of A over k is r, $r \in \mathbb{N}$. Show that $\dim(A) \leq r$. (Hint: Use problem 2.)

5. Let $A = \oplus_{i\geq 0} A_i$ be a graded ring, and let $M = \oplus_{j\geq 0} M_j$ be a graded A-module. Let $a = \sum_{0\leq i\leq d} a_i \in A$, $a_i \in A_i$ for all i be an element of A. $x = \sum_{0\leq j\leq t} x_j$ be a *non-zero* element of M, $x_j \in M_j$ for all j. Suppose that $a \cdot x = 0$. Show that there exists a non-zero homogeneous element $y \in \sum_j A x_j$ such that $a_j \cdot y = 0$ for all j.
[Hint: Induction on $h := \mathrm{card}\{j \in \mathbb{N} \mid x_j \neq 0\}$. Let $r := \min\{n \in \mathbb{N} \mid a_i \cdot x = 0 \ \forall i \geq n+1\}$. Consider the element $a_r \cdot x$ of M.]

6. Let $(A, \mathfrak{m})$ be a local ring. Prove that A is Noetherian if and only if the following conditions hold:

(i) The ideal $\mathfrak{m}$ is finitely generated.

(ii) $\cap_{n\in\mathbb{N}} \mathfrak{m}^n = (0)$.

(iii) Every finitely generated ideal I of A is closed in the $\mathfrak{m}$-adic topology of A.

(Hint: Show first that the $\mathfrak{m}$-adic completion $\hat{A}$ of A is finitely generated. Given any ideal I of A, let $J \subseteq I$ be a finitely generated ideal of A such that $J\hat{A} = I\hat{A}$, and apply condition (iii).)

7. Let k be a field, and let L be an algebraic extension field of k such that $[L:k] = \infty$. Let $k[[x_1, \ldots, x_m]]$ be the formal power series ring over k in n-variables $x_1, \ldots, x_m$.

(1) Show that for any subextension K of L/k with $[K:k] < \infty$, $K \otimes_k k[[x_1, \ldots, x_m]]$ is isomorphic to $K[[x_1, \ldots, x_m]]$.

(2) Show that the natural ring homomorphism $L \otimes_k k[[x_1, \ldots, x_m]] \to L[[x_1, \ldots, x_m]]$ is injective and identifies $L \otimes_k k[[x_1, \ldots, x_m]]$ with the subring consisting of all formal power series

$$f(\underline{x}) = \sum_{\underline{n}\in\mathbb{N}^m} a_{\underline{n}} \underline{x}^{\underline{n}}$$

in $L[[x_1, \ldots, x_m]]$ such that the subfield $k\big(a_{\underline{n}} : \underline{n} \in \mathbb{N}^m\big)$ generated by all coefficients of $f(\underline{x})$ is a finite extension field of k. Here $\underline{x}^{\underline{n}}$ denotes the monomial $x_1^{n_1} \cdots x_m^{n_m}$ for each element $\underline{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$.

(3) Show that $L \otimes_k k[[x_1, \ldots, x_m]]$ is a local ring.

(4) Show that $L \otimes_k k[[x_1, \ldots, x_m]]$ is a Noetherian local ring. (Hint: Use problem 6.)

8. Let k be a field, and let K and L be finitely generated extension field of k.

(a) Show that $K \otimes_k L$ is a Noetherian ring.

(b) Show that $\dim(K \otimes_k L) = \min\big(\mathrm{tr.deg}(K/k), \mathrm{tr.deg}(L/k)\big)$.

(c) Show that $K \otimes_k L$ is not Noetherian if $\mathrm{tr.deg}(K/k) = \infty$ and $\mathrm{tr.deg}(L/k) = \infty$.

9. Let k be a field, L be an extension field of k. Consider the power series ring $k[[x_1, \ldots, x_m]]$ over k in m-variables, $m \geq 1$.

(i) If L is a finitely generated extension field of k, then $L \otimes_k k[[x_1, \ldots, x_m]]$ is Noetherian.

(ii) Give an example of a field extension L/k in which the ring $L \otimes_k k[[x_1, \ldots, x_m]]$ is not Noetherian.