Math 626 Exercise Set 5

1. (Nagata–Schmidt, an example of a one-dimensional Noetherian local domain A whose normalization B is not a finite A-module)

Let p be a prime number and let $K \supseteq \mathbb{F}_p$ be a field such that $[K : K^p] = \infty$. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of elements in K such that $[K^p(a_1, a_1, \ldots, a_m) : K^p] = p^m$ for each $m \in \mathbb{N}$. Let $K^p[[x]]$ and $K[[x]]$ be the ring of formal power series with coefficients in K^p and K respectively. Let $b := \sum_{i\in\mathbb{N}} a_i x^i \in K[[x]]$. Let $R := K^p[[x]] \otimes_{K^p} K$ be the smallest subring of $K[[x]]$ which contains K and $K^p[[x]]$.

(a) Show that R is a one-dimensional local domain whose maximal ideal is xR. Conclude that R is a discrete valuation ring; in particular R is Noetherian.

(Note: The fact that R is Noetherian is a special case of problem 7 of Exercise Set 4. This part asks for an simler and more direct argument.)

(b) Show that $A := R[b]$ is a Noetherian local domain.

(c) Let B be the integral closure of A in the fraction field $\mathrm{frac}(A)$ of A. Show that $B = \mathrm{frac}(A) \cap K[[x]]$.

(d) Show that $B = xB + A$. Deduce that B is not a finite A-module.

2. Let k be a field, let A be the power series ring $A = k[[x,y,z]]$ in three variables, and let $B := A/(xz, yz)$. Let $H_A, H_B \in \mathbb{Z}[T, (1-T)^{-1}]$ be the Poincaré series of the Noetherian local rings A, B with respect to their maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$. B

(a) Show that the Poincaré series $H_B \in \mathbb{Z}[T, (1-T)^{-1}]$ of B is

$$H_B = (1 - 2T^2 + T^3) \cdot H_A = (1 - 2T^2 + T^3) \cdot (1-T)^{-3}.$$

(Hint: Find a finite free resolution of $\mathrm{gr}(B)$ as a graded module over $\mathrm{gr}(A)$, where $\mathrm{gr}(A)$ is the graded ring attached to the $\mathfrak{m}_A$-adic filtration of A, and $\mathrm{gr}(B)$ the graded ring attached to the $\mathfrak{m}_B$-adic filtration of B regarded as a module over $\mathrm{gr}(A)$.)

(b) Deduce from (a) that $\dim_k(\mathfrak{m}_B/\mathfrak{m}_B^2) = 3$, $\dim(B) = 2$, and $e_{\mathfrak{m}_B}(B) = 1$.

(So B is a two-dimensional complete local ring which is not regular, and its multiplicity is 1.)

(c) Find a system of parameters x_1, x_2 in $\mathfrak{m}_B$ such that $e_{(x_1B + x_2B)}(B) = 1$.

3. Let A be a graded algebra over a field k generated by homogeneous elements $u_1, \ldots, u_m$, and the degree of u_i is δ_i for $i = 1, \ldots, m$. Suppose that the Poincare series of A is $P_A = \prod_{j=1}^{n}(1 - T^{d_j})^{-1}$, where $d_1, \ldots, d_n$ are positive integers, $n \geq 1$.

(a) Show that for each $j \in \{1, \ldots, n\}$, there exists an $i \in \{1, \ldots, m\}$ such that $d_j | \delta_i$.

(b) Show that $\min(\delta_1, \ldots, \delta_m) \leq \min(d_1, \ldots, d_n)$.

(c) Show that if $\delta_1 = \ldots = \delta_m$, then A is isomorphic to a polynomial ring in n-variables as a graded algebra.

(d) Give an example where A is not isomorphic to polynomial algebra.

4. Let k be a field, and let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, regarded as a graded ring and each of the variables $x_1, \ldots, x_n$ is homogeneous of degree one. Let $\mathfrak{C}$ be the abelian category of all finitely generated graded A-modules, and let $K(\mathfrak{C})$ be the Grothendieck group of $\mathfrak{C}$.

(i) Suppose that $n \geq 3$. Show that $[A/x_1^3 A] = [A/x_2^2 x_3 A]$ in $K(\mathfrak{C})$.

(ii) A finitely generated homogenous A-module P is said to be *projective* in $\mathfrak{C}$ if for every pair of graded A-linear maps $\alpha : N_1 \to N_0$ and $\pi : P \to N_0$ with π surjective, there exists a homogeneous A-linear map $\beta : P \to N_1$ such that $\alpha \circ \beta = \pi$. Show that every projective A-module in $\mathfrak{C}$ is free.

(iii)* Let M be a finitely generated graded A-module. Let

$$0 \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \to M \to 0$$

be a exact sequence in $\mathfrak{C}$ such that $L_0, L_1, \ldots, L_{n-1}$ are finite free homogenous A-modules. Show that L_n is projective, hence free.

(This is a basic result in homological algebra going back to Hilbert, known as Hilbert's syzygy theorem. You may use it in the rest of this problem.)

(iv) For any $i \in \mathbb{Z}$, denote by η^i the element in $K(\mathfrak{C})$ attached to the free rank-one A module with a free generator in degree i. Define a homomorphism c of abelian groups by

$$c : \mathbb{Z}[X, X^{-1}] \to K(\mathfrak{C}), \quad c : \sum_{i \in \mathbb{Z}} a_i X^i \mapsto \sum_{i \in \mathbb{Z}} a_i \xi^i,$$

where $\mathbb{Z}[X, X^{-1}]$ is the ring of Laurent polynomials in the variable X with coefficients in $\mathbb{Z}$. Show that c is an injection.

(Note: More canonically, the homomorphism c should be identified with the map

$$\tilde{c} : K(k) \otimes_{\mathbb{Z}} \mathbb{Z}[X, X^{-1}] \to K(\mathfrak{C}),$$

where $K(k)$ is the K-group of the category $\mathfrak{V}_k$ of finite dimensional vector spaces over k to $K(\mathfrak{C})$. The restriction of $\tilde{c}$ to $K(k) \subseteq K(k) \otimes_{\mathbb{Z}} \mathbb{Z}[X, X^{-1}]$ arises from the exact functor $\Phi : \mathfrak{V}_k \to \mathfrak{C}$, such that Φ sends each k-module V to the A-module $A \otimes_k V$.)

(v) Show that c is an isomorphism from $\mathbb{Z}[X, X^{-1}]$ to $K(\mathfrak{C})$. Moreover the inverse $\chi : K(\mathfrak{C}) \to \mathbb{Z}[X, X^{-1}]$ of c is given explicitly as follows: For any finitely generated homogeneous A-module M,

$$\chi([M]) = \sum_{j \in \mathbb{N}} \sum_{n \in \mathbb{Z}} (-1)^j \dim_k \big(\mathrm{Tor}_j^A(k, M)_i\big) X^i,$$

where $\mathrm{Tor}_i^A(k, M)_i$ is the degree-n component of the graded A-module $\mathrm{Tor}_i^A(k, M)$.

(You can skip the second part if you don't know about the Tor functor, and come back to it after learning basic homological algebra for polynomial rings.)

(vi) The Poincaré series defines a group homomorphism

$$P : K(\mathfrak{C}) \to \mathbb{Z}[T, T^{-1}, (1-T)^{-1}], \quad P([M]) = \sum_{i \in \mathbb{Z}} \dim_k(M_i) T^i.$$

Compute explicitly the map

$$c \circ P : Z[X, X^{-1}] \to \mathbb{Z}[T, T^{-1}, (1-T)^{-1}]$$

and show that $c \circ P$ is injective. What is the image of $c \circ P$ in $\mathbb{Z}[T, T^{-1}, (1-T)^{-1}]$?

(vii) Let B_0 be an Artinian ring, and let $B = B_0[u_1, \ldots, u_m]$ be the graded polynomial ring with $\deg(u_i) = 1$ for all i. We have a group homomorphism $\tilde{c} : K(B_0) \otimes_{\mathbb{Z}} \to K(\mathfrak{C}_B)$ defined similarly. Is $\tilde{c}$ an isomorphism? Give an explicit form of its inverse if $\tilde{c}$ is an isomorphism. Otherwise give a counter-example.