Math 626 Exercise Set 6

1. Let R be a commutative ring with 1, and let $R[x]$ be the polynomial ring in one variable over R.

(a) Let $g(x) \in R[x]$ be a zero divisor in $R[x]$. Show that there exists an element $b \neq 0$ in R such that $b \cdot g(x) = 0$.

(Hint: Write $g = \sum_i a_i x^i$, and let $h = \sum_{j \le t} c_j x^j \neq 0$ be an element of $R[x]$ such that $c_t \neq 0$ and $g \cdot h = 0$. Induction on $t = \deg(h)$, and consider the the smallest integer s such that $a_j \cdot c_t = 0$ for all $j \geq s+1$.)

(b) Extend the argument used in (a) to prove the following generalization of (a): Let $g_1, \ldots, g_m$ be element in the polynomial ring $R[x_1, \ldots, x_n]$ such that there exists an element $h \neq 0$ in $R[x_1, \ldots, x_n]$ such that $g_i \cdot h = 0$ for all $i = 1, \ldots, m$. Then there exists an element $b \neq 0$ in $R = [x_1, \ldots, x_n]$ such that $b \cdot g_i = 0$ for all $i = 1, \ldots, m$.

(c) Let $S_{R[x]}$ be the set consisting of all elements $f \in R[x]$ such that the ideal of R generated by all coefficients of f is equal to R. Show that S is multiplicatively closed and does not contain any zero divisor of $R[x]$.

(d) Let $R(x)$ be the localization of $R[x]$ with respect to $S_{R[x]}$.

(i) Show that the natural homomorphism $R[x] \to R(x)$ is injective.

(ii) Show that for every prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}R(x)$ is a prime ideal of $R(x)$. Moreover $\mathfrak{p}R(x)$ is a maximal ideal of $R(x)$ if $\mathfrak{p}$ is a maximal ideal of R.

(iii) Show that every maximal ideal of $R(x)$ is equal to $\mathfrak{m}R(x)$ for some maximal ideal $\mathfrak{m}$ of R.

(e) Iterating the construction of $R(x)$ from R, we get a ring $R(x_1)(x_2)\cdots(x_n)$ from n variables $x_1, \ldots, x_n$. Show that $R(x_1)(x_2)\cdots(x_n)$ is naturally isomorphic to the localization of $R[x_1, \ldots, x_n]$ with respect to the subset $S_{R[x_1,\ldots,x_n]}$ consisting of all elements $f \in R[x_1, \ldots, x_n]$ such that the ideal in R generated by all coefficients of f is R.

2. Let $(A, \mathfrak{n})$ be a one-dimensional Noetherian semi-local domain. Let $\hat{A}$ be the $\mathfrak{n}$-adic completion of A, and let A_{nm} be the normalization of A in the fraction field of A.

(a) Suppose that A_{nm} is a finite A-module. Show that $\hat{A}$ is reduced.

(b) Suppose that $\hat{A}$ is reduced. Show that A_{nm} is a finite A-module.

(Hint: For part (b), you can use the fact that for every complete local domain B, the normalization of B in the fraction field of B is a finite B-module. This fact is usually proved using the structure theorem of separated complete local rings.)

3. Let R be a commutative ring of characteristic p, i.e. $p \cdot 1 = 0$ in R. Let $W(R)$ be the ring of p-adic Witt vectors with entries in R.

(a) Show that $W(R)$ is an integral domain if and only if R is an integral domain.

(b) Show that $W(R)$ is reduced if and only if R is reduced.

(c) Recall that a commutative ring is said to be *perfect* if and only the absolute Frobenius map $a \mapsto a^p$ induces a ring automorphism of R. Prove that R is perfect if and only if $W(R)/pW(R)$ is reduced.

(d) Show that the topology of $W(R)$ defined by the filtration $\left(V^n W(R)\right)_{n \in \mathbb{N}}$ coincides with the p-adic filtration of $W(R)$ if and only if R is perfect.

4. Let R be a commutative ring and let $\xi \in R$ be an element of R satisfying $\sum_{j=0}^{p-1} \xi^i = 0$, where p is a prime number. Let $W(R)$ be the ring of p-adic Witt vectors with entries in R. Show that the equality

$$V(\mathbf{1}) = \sum_{i=0}^{p-1} \tau_R(\xi^i)$$

holds in $W(R)$. Here $\mathbf{1}$ denotes the element $(1,0,0,0,\ldots) \in W(R)$, and $\tau_R : R \to W(R)$ is the map $a \mapsto (a,0,0,0,\ldots)$.

5. Let k be a field of characteristic $p > 0$, and let $W(k)$ be the ring of p-adic Witt vectors with entries in k. Show that $W(R)$ is Noetherian if and only if k is perfect.

6. Let p be a prime number. For every commutative ring R we have two ring endomorphisms of $W(W(R))$, $W(F_R)$ and $F_{W(R)}$. The former is the result of applying the p-adic Witt functor to the ring endomorphism $F_R : W(R) \to W(R)$. The latter is the functorial endomorphism F of the Witt functor evaluated at the ring $W(R)$. Determine whether $W(F_R) = F_{W(R)}$ for every commutative ring R.

7. Let p be a prime number.

(a) Show that for every commutative ring A there exists a unique ring homomorphism

$$\Delta_A : W(A) \to W(W(A)),$$

functorial in A, such that

$$\Delta_A \circ F_A = F_{W(A)} \circ \Delta_A \quad \text{and} \quad \Phi_{0,W(A)} \circ \Delta_A = \mathrm{Id}_{W(A)}.$$

Here $F_A : W(A) \to W(A)$ and $F_{W(A)} : W(W(A)) \to W(W(A))$ are the functorial ring endomorphisms of the p-adic Witt functor evaluated at A and $W(A)$ respectively. Moreover

$$\Phi_{n,W(A)} \circ \Delta_A = F_A^n \quad \forall n \in \mathbb{N}.$$

(Hint: Define first Δ_A for $A = \mathbb{Z}[X_0, X_1, \ldots, X_i, \ldots]$.)

(b) Prove that the two ring homomorphisms

$$W(\Delta_A) \circ \Delta_A,\, \Delta_{W(A)} \circ \Delta_A : W(A) \longrightarrow W(W(W(A)))$$

are equal, for all commutative rings A.

(c) Prove that the two maps

$$\Delta_A \circ \tau_A,\, \tau_{W(A)} \circ \tau_A : A \longrightarrow W(W(A))$$

are equal.

(d) Prove that the two maps

$$\Delta_A \circ V_A,\, V_{W(A)} \circ \Delta_A W(A) \longrightarrow W(W(A))$$

are equal. In particular the map $\Delta_A : W(A) \to W(W(A))$ is continuous with respect to the V-adic topologies on $W(A)$ and $W(W(A))$ respectively.