Math 626 Exercise Set 7

The goal of this set of problems is to offer another look of Cohen p-rings from a slightly different perspective which is closer to the spirit of Witt vectors, i.e. doing arithmetic on p-rings using rings of characteristic p.

In all problems below, p denotes a prime number.

1. Let X, Y be two variables.

(a) Show that there exists a unique sequence of polynomials $R_n(X,Y) \in \mathbb{Z}[X,Y]$, $n \in \mathbb{N}$ such that

$$X^{p^n} + Y^{p^n} = \sum_{i=0}^{n} p^i R_i(X,Y)^{p^{n-i}} \quad \forall i \in \mathbb{N}.$$

(b) If $p > 2$, show that

$$R_n(X,-X) = 0 \quad \forall n \in \mathbb{N}.$$

If $p = 2$, show that

$$R_1(X,Y) = -XY$$

and

$$R_n(X,Y) \equiv 0 \pmod{2} \quad \forall n \geq 2.$$

2. Let A be a commutative ring and let $\big(J_n\big)_{\mathbb{N}}$ be a decreasing sequence of ideals in A such that $J_0 = A$ and $pJ_n + J_n^p \subseteq J_{n+1}$ for all $n \in \mathbb{N}$. Let $m,n \in \mathbb{N}$, $m \geq 1$, and $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \in A$.

(a) Suppose that $a_i \equiv b_i \pmod{J_m}$ for all $i = 0,1,\ldots,n$. Show that

$$\Phi_i(a_0,\ldots,a_i) \equiv \Phi(b_0,\ldots,b_i) \pmod{J_{m+i}} \quad \forall 0 \leq i \leq n.$$

(b) Suppose that multiplication by p induces an injective additive endomorphism of $\oplus_n J_n/J_{n+1}$. Suppose that $\Phi_i(a_0,\ldots,a_i) \equiv \Phi(b_0,\ldots,b_i) \pmod{J_{m+i}}$ for $i = 0,1,\ldots,n$. Show that

$$a_i \equiv b_i \pmod{J_m} \quad \text{for } i = 0,1,\ldots,n.$$

REMARKS.

(i) The statements (a) and (b) above have been shown in class in the case when $J_n = p^n A$ for all $n \in \mathbb{N}$.

(ii) Note that $p \cdot 1_A \in J_1$, but we are not assuming that $J_n \cdot J_m \subseteq J_{n+m}$ for all $m,n \in \mathbb{N}$.

(iii) In the literature sometimes the normalization $J_{-1} = A$ is used.

3. Let A be a commutative ring and let $\left(J_n\right)_{\mathbb{N}}$ be a decreasing sequence of ideals in A such that $J_0 = A$ and $pJ_n + J_n^p \subseteq J_{n+1}$ for all $n \in \mathbb{N}$ as in problem 2 above.

(a) Let $i, n \in \mathbb{N}$. Show that the map $x \mapsto p^i x^{p^{n-i}}$ from A to A induces a map

$$\rho^A_{n,i} : A/J_1 \to A/J_{n+1}.$$

if $i \leq n$. Define $\rho^A_{n,i}$ to be the zero map from A/J_1 to A_{n+1} if $i \geq n+1$.

(b) Let $i, j \in \mathbb{N}$. Show that

$$\rho^A_{n,i}(\bar{x}a) = x^{p-i}\rho^A_{n,i}(a) \quad \forall x \in A/J_{n+1},\ \forall a \in A/J_1$$

and

$$\rho^A_{n,i}(a) \cdot \rho^A_{n,j}(b) = \rho^A_{n,i+j}(a^{p^j} b^{p^i}) \quad \forall a, b \in A/J_1.$$

(c) Let $\left(R_n(X,Y)\right)_{n\in\mathbb{N}}$ be as in problem 1. Show that

$$\rho^A_{n,i}(a) + \rho^A_{n,i}(b) = \sum_{m=0}^{n-i} \rho^A_{n,i+m}(R_m(a,b)) \quad \forall a, b \in A/J_1.$$

4. Let C be a Cohen p-ring with residue field k and length $n+1$. Let $\pi : C \to k$ be the canonical surjection. Let $J_m = p^m C$ for each $m \in \mathbb{N}$. Let $\left(s_\lambda\right)_{\lambda\in\Lambda}$ be a family of elements of A such that $\left(\pi(s_\lambda)\right)_{\lambda\in\Lambda}$ form a p-basis of k. Let M_n be the set of all multi-indices $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$ with finite support such that $0 \leq m_\lambda < p^n$ for each $\lambda \in \Lambda$.

(a) Show that for every element $x \in C$, there exists a uniqe family $\left(a_{i,\mathbf{m}}\right)_{0\leq i\leq n, \mathbf{m}\in M_n}$ of elements in $k = C/J_1$ such that

$$x = \sum_{i=0}^{n} \sum_{\mathbf{m}\in M_n} \rho^C_{n,i}(a_{i,\mathbf{m}})\, \mathbf{s}^{\mathbf{m}},$$

where $\mathbf{s}^{\mathbf{m}} := \prod_{\lambda\in\Lambda} s_\lambda^{m_\lambda}$.

(b) Use (a) to give a presentation of the ring C in terms of the residue field k and a family of variables $\left(X_\lambda\right)_{\lambda\in\Lambda}$.