Math 626 Exercise Set 8

1. Let n is a positive integer, and let $S \subset \mathbb{N}^n$ be a subset of $\mathbb{N}^n$ such that $S + \mathbb{N}^n = S$.

(i) Show that there exists a *finite* subset $V \subseteq S$ such that $S = V + \mathbb{N}^n := \cup_{v \in V}(v + \mathbb{N}^n)$. Such a subset S is said to be a set of generators of S

(ii) A subset $T \subseteq S$ is said to be a *minimal* set of generators of S if $S = T + \mathbb{N}^n$ and $(T \setminus \{t\}) + \mathbb{N}^n \neq S$ for every $t \in T$. Show that S has a unique minimal set of generators, i.e. there exist a minimal set of generators T, and every set of generators of S contains T.

(iii) Let k be a field. Formulate two statements on $\mathbb{N}^n$-graded ideals in $k[x_1, \ldots, x_n]$ which correspond to statements (i) and (ii). (These statements are equivalent to (i) and (ii).)

2. Let $<_1, <_2$ be two term orders for $k[x_1, \ldots, x_n]$, where k is a field. Let $(g_1, \ldots, g_t)$ be a Gröbner basis of an ideal $I \subset k[x_1, \ldots, x_n]$ with respect to $<_1$. Suppose that $lm_{<_1}(g_i) = lm_{<_2}(g_i)$ for $i = 1, \ldots, t$. Show that $(g_1, \ldots, g_t)$ is a Gröbner basis of I with respect to $<_1$.

3. Explain how to compute a Gröbner basis of an ideal of $\mathbb{Q}[x_1, \ldots, x_n]$ with generators $f_1, \ldots, f_s$ by "reduction modulo p_i" for a finite number of prime numbers $p_1, p_2, \ldots, p_r$ and passing to $\mathbb{F}_{p_i}[x_1, \ldots, x_n]$.

4. Let $\alpha : \mathbb{Q}[u, v] \to \mathbb{Q}[x]$ be the $\mathbb{Q}$-algebra homomorphism such that

$$\alpha(u) = x^4 + x^2 + x \quad \text{and} \quad \alpha(v) = x^3 - x.$$

Show that α is not surjective.

5. Let $\beta : \mathbb{Q}[u, v] \to \mathbb{Q}[x]$ be the $\mathbb{Q}$-algebra homomorphism such that

$$\beta(u) = x^3 \quad \text{and} \quad \beta(v) = x^5.$$

Determine $\mathrm{Ker}(\beta)$ by giving an explicit set of generators of the ideal $\mathrm{Ker}(\beta)$.

6. Recall that for polynomials $f(x) = a_0x^l + a_1x^{l-1} + \cdots + a_l$ and $g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m$ of degrees l and m respectively with $l, m > 0$, the Sylvester matrix $\mathrm{Syl}(f, g)$ is the $(l+m) \times (l \times m)$ matrix $\left(c_{i,j}\right)_{1 \leq i,j \leq l+m}$ with

$$c_{ij} = \begin{cases} a_{i-j} & \text{if} 1 \leq j \leq m,\ j \leq i \leq j+l, \\ b_{m+i-j} & \text{if} m+1 \leq j \leq l+m,\ j-m \leq i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

The *resultant* $\mathrm{Res}(f, g)$ of f, g is defined by

$$\mathrm{Res}(f, g) = \det(\mathrm{Syl}(f, g)).$$

(a) Suppose that k is a field and $f(x), g(x)$ are polynomials in $k[x]$ of degrees $l, m > 0$. Show that there exist polynomials $s(x), t(x) \in k[x]$ such that $sf + tg = \mathrm{Res}(f, g)$.

(b) More generally, let $\mathbb{Z}[y_1, \ldots, y_r]$ be a polynomial ring in r variables $y_1, \ldots, y_r$, and let $f, g \in \mathbb{Z}[x, y_1, \ldots, y_r]$ with $\deg_x(f) = l > 0$, $\deg_x(g) = m > 0$. Show that there exist $s, t \in \mathbb{Z}[y_1, \ldots, y_r]$ such that $sf + tg = \mathrm{Res}(f, g) \in \mathbb{Z}[y_1, \ldots, y_r]$. In particular

$$\text{f,g} \in (f\mathbb{Z}[x, y_1, \ldots, y_r] + g\mathbb{Z}[x, y_1, \ldots, y_r]) \cap \mathbb{Z}[y_1, \ldots, y_r].$$