Depth.

Def 1 A: comm. ring M: A-module $a_1, \dots, a_r \in A$

$(a_1, \dots, a_r)$ is an <u>M-regular sequence</u> if

(1) $M \neq a_1 M + \dots + a_r M$

(2) a_i is not a zero divisor of $M / a_1 M + \dots + a_{i-1} M$

(This notion depends on the order of the elements $a_1, \dots, a_r$

Def 2 M: A-module, $a_1, \dots, a_r \in A$, $I = a_1 A + \dots + a_r A$ $(a_1, \dots, a_r)$ is an <u>M-quasi-regular sequence</u> if the map

$$\psi : \underset{\displaystyle (M/IM)[X_1, \dots, X_r]}{\underbrace{(M/IM) \otimes_A (A/I)[X_1, \dots, X_r]}_{\parallel}} \xrightarrow{\sim} gr_I^{\bullet}(M) \quad \text{as graded } (A/I)\text{-modules}$$

Prop 3 Suppose that $(a_1, \dots, a_r)$ is an M-regular sequence in an A-module M.
Then $(a_1^{n_1}, \dots, a_r^{n_r})$ is an M-regular sequence $\forall n_1, \dots, n_r \in \mathbb{N}_{\geq 1}$

Lemma If $\xi_1, \dots, \xi_r \in M$ and $a_1 \xi_1 + \dots + a_r \xi_r = 0$, then $\xi_i \in (a_1, \dots, a_r) M$ $\forall i = 1, \dots, r$

pf of Lemma: Induction on r. Obvious if $r = 1$

$r \geq 2$: $\exists \eta_1, \dots, \eta_{r-1} \in M$ s.t. $\xi_r = a_1 \eta_1 + \dots + a_{r-1} \eta_{r-1}$

$0 = a_1 \xi_1 + \dots + a_r \xi_r = a_1(\xi_1 + a_r \eta_1) + \dots + a_{r-1}(\xi_{r-1} + a_r \eta_{r-1})$

$\overset{\text{induction}}{\Longrightarrow}$ $\xi_i + a_r \eta_i \in (a_1, \dots, a_{r-1}) M$ for $i = 1, \dots, r-1$. q.e.d.

pf of Thm 3. Suffices to show: $a_1^n, a_2, \dots, a_r$ is M-regular $\forall n \in \mathbb{N}_{\geq 1}$

Induction on $n \geq 2$. $\overset{\text{induction}}{\Longrightarrow}$ $(a_1^{n-1}, a_2, \dots, a_r)$ is M-regular

Show inductively that $(a_1^n, a_2, \dots, a_i)$ is M-regular $\forall i = 2, \dots, r-1$

$\underset{\text{induction}}{\Longrightarrow}$ $(a_1^n, a_2, \dots, a_{i-1})$ is M-regular

Suppose $a_i \cdot x = a_1^n \xi_1 + \dots + a_{i-1} \xi_{i-1}$, $x, \xi_1, \dots, \xi_{i-1} \in M$

$\underset{\text{induction hyp. for } n-1}{\Longrightarrow}$ $x = a_1^{n-1} \eta_1 + a_2 \eta_2 + \dots + a_{i-1} \eta_{i-1}$ $\quad \eta_1, \dots, \eta_{i-1} \in M$

$\Rightarrow a_1^{n-1}(a_1 \xi_1 - a_i \eta_1) + a_2(\xi_2 - a_i \eta_2) + \dots + a_{i-1}(\xi_{i-1} - a_i \eta_{i-1}) = 0$

$\underset{\text{Lemma}}{\Longrightarrow}$ $a_1 \xi_1 - a_i \eta_1 \in (a_1^{n-1}, \dots, a_{i-1}) M \Rightarrow a_i \eta_1 \in (a_1, \dots, a_{i-1}) M \Rightarrow \eta_1 \in (a_1, \dots, a_{i-1}) M$

QED $\Rightarrow x \in (a_1^n, a_2, \dots, a_{i-1}) M$

Depth 2

Prop. 4 Let $(a_1, \cdots, a_n)$ be an M-regular sequence, $a_1, \cdots, a_n \in A$, M: A-module
Then $(a_1, \cdots, a_n)$ is M-quasi-regular

Lemma 5 Let $(b_1, \cdots, b_m)$ be an M-quasi-regular sequence. Let $J = \sum_{j=1}^{m} b_j A$.
Suppose that $x \in A$, and $(JM : x) = JM$ (i.e. $M/JM \xrightarrow{x} M/JM$)
Then $(J^\nu M : x) = J^\nu M \quad \forall \nu \in \mathbb{N}_{\geq 1}$

Pf. Induction on $\nu \geq 1$. Suppose true for $\nu - 1$, and $\xi \in M$, $x \cdot \xi \in J^\nu M$.
Want to show: $\xi \in J^\nu M$. Induction hypoth. $\Rightarrow$ $\xi \in J^{\nu-1} M$
$\rightsquigarrow \exists$ homog. poly $F(Y_1, \cdots, Y_m) \in M[Y_1, \cdots, Y_m]$ of degree $\nu - 1$ s.t. $\xi = F(b_1, \cdots, b_m)$

$x \cdot \xi = \underbrace{(x \cdot F)}_{\in M[Y_1, \cdots, Y_m]}(b_1, \cdots, b_m) \in J^\nu M$
$\Rightarrow x \cdot F(Y_1, \cdots, Y_m) \in JM[Y_1, \cdots, Y_m]$
$\Rightarrow$ all coef. of $x F(Y_1, \cdots, Y_m)$ are in JM q.e.d.

Pf of Prop 4: Induction on n; case $n = 1$ trivial.
Suffices to show: If $F(X_1, \cdots, X_n) \in M[X_1, \cdots, X_n]$ homog. of degree ν
s.t. $F(a_1, \cdots, a_n) = 0$, then $F \in IM[X_1, \cdots, X_n]$, i.e. all coef. of F are in IM.
Prove this statement by induction on ν: suppose statement holds for $\nu - 1$
Write $F(X_1, \cdots, X_n) = G(X_1, \cdots, X_{n-1}) + X_n \cdot H(X_1, \cdots, X_n)$ $\quad G(X_1, \cdots, X_n)$ homog. deg $= \nu$
$\rightsquigarrow a_n \cdot H(a_1, \cdots, a_n) = -G(a_1, \cdots, a_{n-1}) \in (a_1 A + \cdots + a_{n-1} A)^\nu \cdot M$ $\quad H(X_1 \cdots X_n)$ homog. deg $= \nu - 1$
Lemma 5 $\Longrightarrow$ $H(a_1, \cdots, a_n) \in (a_1 A + \cdots + a_{n-1} A)^\nu \cdot M \subseteq (a_1 A + \cdots + a_{n-1} A + a_n A)^\nu A$
$\rightsquigarrow$ (a) (Since H is homog of deg $\nu - 1$) $\overset{\text{induction on } \nu}{\Longrightarrow}$ $H(X_1, \cdots, X_n) \in IM[X_1, \cdots, X_n]$
(b) $\exists f(X_1, \cdots, X_{n-1}) \in M[X_1, \cdots, X_{n-1}]$, homog. of degree ν, s.t. $f(a_1, \cdots, a_{n-1}) = H(a_1, \cdots, a_n)$

Let $g(X_1, \cdots, X_{n-1}) = G(X_1, \cdots, X_{n-1}) + a_n \cdot f(X_1, \cdots, X_{n-1})$ $\in M[X_1, \cdots, X_{n-1}]$, homog. of deg. ν
We have: $g(a_1, \cdots, a_{n-1}) = 0$
$\overset{\text{induction on } n}{\Longrightarrow}$ $g(X_1, \cdots, X_{n-1}) \in IM[X_1, \cdots, X_{n-1}]$
$\Longrightarrow$ $G(X_1, \cdots, X_{n-1}) = g(X_1, \cdots, X_{n-1}) - a_n f(X_1, \cdots, X_{n-1}) \in IM[X_1, \cdots, X_{n-1}]$
$\rightsquigarrow$ $F(X_1, \cdots, X_n) = G + X_n H \in IM[X_1, \cdots, X_n]$.

q.e.d.

$a_i \in A$ $\quad\quad\quad\quad\quad\quad M \neq 0$,

<u>Prop 6</u> Suppose that $(a_1, \cdots, a_n)$ is M-quasiregular, $M/(a_1, \cdots, a_{n-1})M \neq 0$, and M, M/a_1M, $M/(a_1, a_2)M$, $\cdots$, $M/(a_1, \cdots, a_{n-1})M$ are I-adically separated, where $I = a_1A + a_2A + \cdots + a_nA$

Then $(a_1, \cdots, a_n)$ is M-regular

<u>Rmk</u>: The separation condition holds if (a) A is Noetherian, M: f.g A-module, $I \subseteq \mathrm{rad}(A)$, or if (b) A is $\mathbb{N}$-graded, M is $\mathbb{N}$-graded A-module, $a_1, \cdots, a_n$ homog. deg >0

Pf: Induction on n.

Case $n = 1$: Suppose $x \in M$ and $a_1 x = 0$. $\rightsquigarrow$ show inductively that $x \in I^\nu M$ $\forall \nu \in \mathbb{N}$

General n: Elementary argument $\Rightarrow$

$(a_2, \cdots, a_n)$ is $\left(M/a_1M\right)$-quasi-regular

<u>Example</u> k: a field $A = k[X, Y, Z]$, $a_1 = XY$, $a_2 = Y-1$, $a_3 = YZ$

<u>Claim 1</u> (a_1, a_2, a_3) is A-regular

$k[X,Y,Z]/(XY, Y-1) \cong k[X,Z]/(X)$ $\rightsquigarrow$ clear

<u>Claim 2</u> (a_1, a_3, a_2) is not A-regular

In fact, a_1, a_3 is not A-regular: $k[X,Y,Z]/(XY) \ni \overset{\xi}{\overset{\|}{X}} \pmod{XY}$

and $a_3 \xi = 0$ in $k[X,Y,Z]/(XY)$

Conclusion: In favorable situations (such as those in the Rmk after Prop 6), $(a_{\sigma(1)}, a_{\sigma(2)}, \cdots, a_{\sigma(n)})$ is M-regular $\forall \sigma \in S_n$ if $(a_1, \cdots, a_n)$ is M-regular

Exer. Let A be a local ring and M a f.g. A-module. Let $a_1, \cdots, a_n \in A$. Show that if $a_1, \ldots, a_n$ is M-regular, then $a_{\sigma(1)}, \cdots, a_{\sigma(n)}$ is M-regular $\forall \sigma \in S_n$. Explain why localization argument does not prove that this statement hold for a general commutative ring

Depth 4

Theorem 7. Let A be a Noetherian ring, M be a finite A-module, $I \subseteq A$ be an ideal s.t. $IM \neq M$. Let $n \in \mathbb{N}_{\geq 1}$. The following are equivalent

(1) $\operatorname{Ext}^i_A(N, M) = 0$ for $i = 0, 1, \dots, n-1$ $\forall$ finite A-module N with $\operatorname{supp}(N) \subseteq \operatorname{Spec}(A/I)$

(2) $\operatorname{Ext}^i_A(A/I, M) = 0$. for $i = 0, 1, \dots, n-1$

(3) $\exists$ a finite A-module N with $\operatorname{supp}(N) = \operatorname{Spec}(A/I)$ s.t.

$$\operatorname{Ext}^i_A(N, M) = 0 \quad \text{for } i = 0, 1, \dots, n-1$$

(4) $\exists$ an M-regular sequence $a_1, \dots, a_n$ with $a_j \in I$ $\forall j = 1, \dots, n$

Pf. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) obvious

(3) $\Rightarrow$ (4): Show first that $\exists$ a non-zer-divisor in I.

Otherwise, $I \subseteq \bigcup_{\mathfrak{q} \in \operatorname{Ass}_A(M)} \mathfrak{q} \Rightarrow \exists\, \mathfrak{q} \in \operatorname{Ass}_A(M)$ s.t. $I \subseteq \mathfrak{q}$

$A/\mathfrak{q} \hookrightarrow M \Rightarrow 0 \neq \operatorname{Hom}_{A_\mathfrak{q}}(\underbrace{A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q}}_{\text{a field}}, M_\mathfrak{q}) \Rightarrow \operatorname{Hom}_{A_\mathfrak{q}}(N_\mathfrak{q}, M_\mathfrak{q}) \neq 0$,

$\operatorname{Hom}_{A_\mathfrak{q}}(N_\mathfrak{q}, M_\mathfrak{q}) = \operatorname{Hom}_A(N, M) \otimes_A A_\mathfrak{q}$, contradiction

Pick $a_1 \in I$, $0 \to M \xrightarrow{a_1} M \to M_1 \to 0$

$\rightsquigarrow \operatorname{Ext}^i_A(N, M_1) = 0$ for $i = 0, 1, \dots, n-2$ $\xRightarrow{\text{induction}}$ $\exists\, a_2, \dots, a_n \in I$ s.t. $(a_2, \dots, a_n)$ is M_1-regular.

(4) $\Rightarrow$ (1) $\quad 0 \to M \xrightarrow{a_1} M \to M_1 = M/a_1M \to 0$

induction on n $\Longrightarrow$ $\operatorname{Ext}^i_A(N, M_1) = 0$ for $i = 0, 1, \dots, n-2$ $\forall$ finite A-module N with $\operatorname{supp}(N) \subseteq \operatorname{Spec}(A/I)$

long exact sequence: $(*)$ $0 \to \operatorname{Ext}^i_A(N, M) \xrightarrow{a_1} \operatorname{Ext}^i_A(N, M)$ for $i = 0, 1, \dots, n-1$

$I \subseteq \operatorname{rad}(\operatorname{Ann}_A(N))$ i.e. $\exists\, r \in \mathbb{N}$ s.t. $a_1^r \cdot N = 0$

So $(*) \Rightarrow$ $\quad \rightsquigarrow a_1^r \cdot \operatorname{Ext}^i_A(N, M) = 0 \;\; \forall i \in \mathbb{N}$

$\operatorname{Ext}^i_A(N, M) = 0$ for $i = 0, 1, \dots, n-1$. $\forall$ finite A-module N with $\operatorname{supp}(N) \subseteq \operatorname{Spec}(A/I)$

QED

Def^n 8 Let M be a finite A-module, A Noetherian. Let $I \subseteq A$ be an ideal

1) The I-depth of M is
$$\mathrm{depth}_I(M) = \min\{ i \in \mathbb{N} \mid \mathrm{Ext}^i_A(A/I, M) \neq 0\} \in \mathbb{N} \cup \{\infty\}$$
$(\mathrm{depth}_I(M) = \infty \text{ if } M = 0)$

2) The grade of M is
$$\mathrm{grade}(M) = \min\{ i \in \mathbb{N} \mid \mathrm{Ext}^i_A(M, A) \neq 0\} \underset{\text{Thm 7}}{=} \mathrm{depth}_{\mathrm{Ann}_A(M)}(A) \quad \rightsquigarrow \mathrm{grade}(A/I) = \mathrm{depth}_I(A)$$

Lemma 9 A: noetherian, M: finite A-module as before, $\overset{k}{\overset{\parallel}{\mathrm{grade}(M)}} < \infty$
$$\mathrm{Ext}^i_A(M, N) = 0 \quad \forall i = 0, 1, \dots, \mathrm{grade}(M) - \underset{\overset{\parallel}{\ell}}{\mathrm{proj.dim.}(N)} - 1$$
proj dim(N)

pf: Induction on ℓ. Case $\ell = 0$ immediate from def^n of grade(M)

$0 \to N' \to \underset{\text{free}}{F} \to N \to 0$, proj$\cdot$dim$(N') = \ell - 1$

$$i \le k - \ell - 1 \Rightarrow \underset{\overset{\parallel}{0}}{\mathrm{Ext}^i_A(M, F)} \to \mathrm{Ext}^i_A(M, N) \to \underset{\overset{\parallel}{0}}{\mathrm{Ext}^{i+1}_A(M, N')}$$
$\because$ $i+1 \le k - (\ell - 1) - 1$ and induction hypoth.

q.e.d.

Lemma 10 $(A, \mathfrak{m})$ Noeth. local, $\underset{\neq 0}{N}, \underset{\neq 0}{M}$: finite A-modules

Then
$$\mathrm{Ext}^i_A(N, M) = 0 \quad \text{for } i = 0, 1, \dots, \overset{k}{\overset{\parallel}{\mathrm{depth}_{\mathfrak{m}}(M)}} - \overset{\ell}{\overset{\parallel}{\dim(N)}} - 1$$

pf Induction on $\ell = \dim(N)$. Case $\ell = 0$: Clear by Thm 7

Assume $\ell > 0$. Dévissage on $N \rightsquigarrow$ May assume: $N = A/\mathfrak{p}$, $\mathfrak{p}$ = a prime ideal

Pick $x \in \mathfrak{m} \setminus \mathfrak{p} \rightsquigarrow$ have $0 \to N \xrightarrow{x} N \to \underset{A/(\mathfrak{p} + Ax)}{\overset{\parallel}{N/xN}} \to 0$ $\quad \dim(N/xN) = \ell - 1$

$$i \le \mathrm{depth}(M) - \dim(N) - 1 \Rightarrow 0 \to \mathrm{Ext}^i_A(N, M) \xrightarrow{x} \mathrm{Ext}^i_A(N, M) \to \underset{\overset{\parallel}{0}\ \leftarrow \text{induction}}{\mathrm{Ext}^{i+1}_A(N/xN, M)}$$

$\xRightarrow{\text{Nakayama}}$ $\mathrm{Ext}^i_A(N, M) = 0$. q.e.d.

Theorem 11 $(A, \mathfrak{m})$ Noeth. local, $0 \neq M$ finite A-module. Then $\mathrm{depth}(M) \leq \dim(A/\mathfrak{p})$ $\forall \mathfrak{p} \in \mathrm{Ass}_A(M)$

Pf. $\mathfrak{p} \in \mathrm{Ass}_A(M) \Rightarrow \mathrm{Hom}_A(A/\mathfrak{p}, M) \neq 0$

Lemma 10 $\Rightarrow$ $\mathrm{depth}(M) \leq \dim(A/\mathfrak{p})$. q.e.d.

Lemma 12 $(A, \mathfrak{m})$ Noetherian local, M: finite A-module, $(a_1, \cdots, a_r)$: M-regular sequence. Then

$$\dim(M/(a_1, \cdots, a_r)M) = \dim(M) - r$$

pf. $\geq$ obvious

$\leq$: May assume $r = 1$. a_1 is M-reg $\Leftrightarrow$ $a_1 \notin$ any minimal element of $\mathrm{supp}(M)$. q.e.d.

Prop 13 A Noetherian, M: finite A-module, $I \subseteq A$ ideal. Then

$$\mathrm{depth}_I(M) = \inf\{\mathrm{depth}(M_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A/I)\}$$

pf. Either induction on r.h.s. of the above, (using Lemma 12), or use the Thm 7, plus

$$\mathrm{Ext}^i_A(N, M)_\mathfrak{p} \simeq \mathrm{Ext}^i_{A_\mathfrak{p}}(N \otimes_A A_\mathfrak{p}, M \otimes_A A_\mathfrak{p}) \quad \forall \mathfrak{p} \in \mathrm{Spec}(A).$$

q.e.d.

~~(a) A finite module M~~

Def 14 (a) A finite module M over a Noeth. local ring $(A, \mathfrak{m})$ is <u>Macaulay</u>, or <u>Cohen-Macaulay</u>, if either $M = 0$, or if $\mathrm{depth}_\mathfrak{m}(M) = \dim(M)$

(b) A Noeth. local ring $(A, \mathfrak{m})$ is CM if A is a CM A-module

Prop 14 $(A, \mathfrak{m})$ Noeth. local ring, M: finite A-module

(a) If M is CM and $\mathfrak{p} \in \mathrm{Ass}(M)$, then $\mathrm{depth}(M) = \dim(A/\mathfrak{p})$.

(unmixedness) In particular M has no associated primes, and $\mathrm{supp}(M)$ is equi-dim$^{\ell}$

(b) Let $(a_1, \cdots, a_r)$ be an M-reg. sequence, and let $M' \stackrel{=}{\scriptstyle\mathrm{def}} M/(a_1, \cdots, a_r)M$.

Then M is CM $\Longleftrightarrow$ M' is CM

(c) Suppose M is CM. Then $M_\mathfrak{p}$ is a CM $A_\mathfrak{p}$ module and $\mathrm{depth}_\mathfrak{p}(M) = \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ $\forall \mathfrak{p} \in \mathrm{supp}(M)$

Pf (a) Given $\mathfrak{p} \in \mathrm{Ass}(M)$. $\Rightarrow M_{\mathfrak{p}} \neq 0 \rightsquigarrow \mathrm{depth}(M) = \dim(M) \geq \dim(A/\mathfrak{p})$
Combined with the general inequality $\mathrm{depth}(M) \leq \dim(A/\mathfrak{p})$ gives $=$ (Thm 11).

(b) Note first that $M = 0 \Leftrightarrow M' = M/(a_1, \dots, a_r)M$ (Nakayama). Assume $M \neq 0$.
Have $\dim(M') = \dim(M) - r$ and $\mathrm{depth}(M') = \mathrm{depth}(M) - r$ (Thm 7).

(c). May assume $M_{\mathfrak{p}} \neq 0$.
Clearly: $\dim(M_{\mathfrak{p}}) \geq \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \mathrm{depth}_{\mathfrak{p}}(M)$.
$\because \mathrm{Ext}^i_A(A/\mathfrak{p}, M) = 0 \Rightarrow \mathrm{Ext}^i_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$

Prove $\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{depth}_{\mathfrak{p}}(M)$ by induction on $\mathrm{depth}_{\mathfrak{p}}(M)$

Key case: when $\mathrm{depth}_{\mathfrak{p}}(M) = 0$ i.e. $\exists Q \in \mathrm{Ass}_A(M)$ s.t. $\underset{\mathfrak{p} \subseteq Q}{\mathrm{Ann}_A(M) \cap}$ $\Rightarrow \mathfrak{p} = Q$ by (a)
$\Rightarrow \dim M_{\mathfrak{p}} = 0$
case when $\mathrm{depth}_{\mathfrak{p}}(M) > 0$. Pick $x \in \mathfrak{p}$ which is M-regular $\Rightarrow \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$
$\rightsquigarrow \mathrm{depth}_{\mathfrak{p}}(M) = 1 + \mathrm{depth}_{\mathfrak{p}}(M/xM) \underset{\text{induction}}{=} 1 + \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}/xM_{\mathfrak{p}}) = \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$
$\because M/xM$ is CM by (b)

QED

Theorem 15. Let $(A, \mathfrak{m})$ be a CM Noeth local ring, $I \subseteq \mathfrak{m}$ an ideal

(a) $\mathrm{ht}(I) = \mathrm{depth}_I(A) = \mathrm{grade}(A/I)$
$\mathrm{ht}(I) + \dim(A/I) = \dim(A)$

(b) A is <u>catenary</u>, i.e. $\forall \mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(A)$, all maximal chains of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ have the same length

(c) $\forall a_1, \dots, a_r \in \mathfrak{m}$, the following are equivalent
(i) $a_1, \dots, a_r$ is A-regular
(ii) $\mathrm{ht}(a_1, \dots, a_i) = i \quad \forall i = 1, \dots, r$
(iii) $\mathrm{ht}(a_1, \dots, a_r) = r$
(iv) $\exists a_{r+1}, \dots, a_n$ with $n = \dim(A)$ s.t. $(a_1, \dots, a_n)$ is a system of parameters of A

Pf. (c): (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) easy.
(iv) $\Rightarrow$ (i): Must show: every system of parameters is a regular sequence

Depth 8

Given a system of parameters $x_1, \dots, x_n$ of A. $\leadsto$ x_1 is not a zero-divisor

$\Rightarrow$ A/x_1A is CM. Conclude by induction on $\dim(A)$.

(a) Let $a_1, \dots, a_r \in I$, an A-reg sequence $r = \mathrm{depth}_I(A)$

(c) $\Rightarrow$ $\underbrace{ht(a_1, \dots, a_r) = r}_{= \mathrm{depth}_I(A)} \leq ht(I)$ $\because$ $a_1A + \dots + a_rA \subseteq I$

Conversely, pick $b_1, \dots, b_s \in I$ s.t. $ht(b_1, \dots, b_i) = i$ $\forall i = 1, \dots, s$
$s = ht(I)$

(c) $\Rightarrow$ $b_1, \dots, b_s$ is an A-reg sequence $\Rightarrow$ $s \leq \mathrm{depth}_I(A)$
$\quad \| \quad ht(I)$

Have shown:

$$ht(I) = \mathrm{depth}_I(A) = \mathrm{grade}(A/I)$$

Must show $\dim(A/I) + ht(I) = \dim(A)$ (Note "$\leq$" holds always)

Suffices to show:

$$ht(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p}) \quad \text{(Exer.)}$$
$$\forall \mathfrak{p} \in \mathrm{Spec}(A)$$

Given $\mathfrak{p} \in \mathrm{Spec}(A)$.

Know $A_\mathfrak{p}$ is CM (Prop 14) $\leadsto$ $ht(\mathfrak{p}) = \dim(A_\mathfrak{p}) = \mathrm{depth}(A_\mathfrak{p}) = \mathrm{depth}_\mathfrak{p}(A)$

$\Rightarrow$ $\exists$ an Aregular sequence $a_1, \dots, a_r \in \mathfrak{p}$ $r = \mathrm{depth}_\mathfrak{p}(A)$

$\leadsto$ $A/(a_1A + \dots + a_rA)$ is CM, of dim $\dim(A) - r$

and $\mathfrak{p}$ is a minimal element in $\mathrm{Ass}(A/a_1A + \dots + a_rA)$

and $\dim(A/\mathfrak{p}) = \mathrm{depth}(A/a_1A + \dots + a_rA) = \dim(A) - r$
$= \dim(A) - ht(\mathfrak{p})$.

(b) $\mathfrak{p} \supseteq \mathfrak{q}$. $A_\mathfrak{p}$ is CM.

(a) $\Rightarrow$ $\dim(A_\mathfrak{p}) = ht_{A_\mathfrak{p}}(\mathfrak{q}A_\mathfrak{p}) + \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p})$. QED.

<u>Def 15</u> A commutative Noetherian $I \subseteq A$ ideal

(a) We say A is CM if $A_\mathfrak{p}$ is CM $\forall \mathfrak{p} \in \mathrm{Spec}(A)$
(or equiv. $A_\mathfrak{m}$ is CM $\forall \mathfrak{m} \in \mathrm{Spm}(A)$.)

(b) We say I is <u>unmixed</u> if $ht(\mathfrak{p}) = ht(I)$ $\forall \mathfrak{p} \in \mathrm{Ass}_A(A/I)$

Def 6 (c): We say that "the unmixedness theorem holds for A" if every ideal I generated by $ht(I)$ elements is unmixed

Note: If $r = ht(I)$, $a_1, \cdots, a_r \in I$ and $I = a_1A + \cdots + a_rA$. then $ht(\mathfrak{p}) \geq r$ $\forall \mathfrak{p} \in Ass_A(A/I)$. So the condition cond^n for I means: A/I has no embeded prime

Prop 17 A Noetherian commutative ring is Cohen-Macaulay iff the unmixedness theorem holds for A.

Theorem 18 Let A be a CM Noetherian ring. Then $A[X_1, \cdots X_n]$ is CM.

Cor. A: CM $\Rightarrow$ every f.g. A-algebra is catenary.

Pf of Thm 18. May assume $n=1$. $B = A[X]$, $P \in Spec(B)$, $\mathfrak{p} = P \cap A$

Must show: B_P is CM. May assume: A is local $\mathfrak{p}$ = max. ideal of A

$\rightsquigarrow B/\mathfrak{p}B = k[X]$ k: a field

$\rightsquigarrow$ either $P = \mathfrak{p} \cdot B$, or $P = \mathfrak{p}B + fB$, $f \in B[X]$ monic and $\deg(f) > 0$

Let $a_1, \cdots, a_d$ be an A-regular sequence in $\mathfrak{p}$. $d = \dim(A)$

Note: $a_1, \cdots, a_d$ is also a B-regular sequence in B $\because$ B/A is flat

Case $P = \mathfrak{p}B \Rightarrow \dim(B_P) = \dim A$

$\Rightarrow depth_P(B_P) = d = \dim(B_P)$. OK

Case $P = \mathfrak{p}B + fB \Rightarrow \dim(B_P) = \dim(A) + 1$

$\rightsquigarrow (a_1, \cdots, a_d, f)$ is a regular sequence in P

$\because$ f is a non-zero divisor in $A/(a_1A + \cdots + a_dA)\,[X]$

monic.

$\Rightarrow depth_P(B) = d+1$ O.K.

QED.