Reg 1

Def 1 A commutative ring A is normal if $A_{\mathfrak{p}}$ is an integral domain which is integrally closed in $\mathrm{frac}(A_{\mathfrak{p}})$ $\forall \mathfrak{p} \in \mathrm{Spec}(A)$

Def 2(a) Let $A \subseteq B$ be commutative rings, and let $u \in B$
We say that u is integral over A if $\exists$ a monic polynomial $f(X) \in A[X]$ s.t. $f(u) = 0$, or equivalently, $\exists$ a finite A-submodule $M \subseteq B$ such that $u \cdot M \subseteq M$

Def 2(b) Let A be a commutative ring, and let $B = \mathrm{frac}(A)$ = total ring of fractions of A
An element $u \in B$ is almost integral over A is $\exists$ a non-zero divisor $a \in A$ s.t. $a \cdot A[u] \subseteq A$. In otherwords $(A : A[u])$ contains a non-zero divisor of A

Note: (i) $u \in \mathrm{frac}(A)$ is integral $\Rightarrow$ u is almost integral
(ii) A Noetherian $\Rightarrow$ (almost integral $\Rightarrow$ integral)

Def 2(c) (i) A commutative ring is normal if $A_{\mathfrak{p}}$ is a local domain integrally closed in $\mathrm{frac}(A_{\mathfrak{p}})$ $\forall \mathfrak{p} \in \mathrm{Spec}(A)$.
(ii) A commutative ring A is completely normal if A contains every element $u \in \mathrm{frac}(A)$ which is almost integral over A

Prop 3 (1) A completely normal domain $\Rightarrow$ So are $A[X_1, \dots, X_n]$ and $A[[X_1, \dots, X_n]]$
(2) A normal domain $\Rightarrow$ so is $A[X_1, \dots, X_n]$.

Pf (1) A completely normal domain. Consider $A[X]$. Let $K = \mathrm{frac}(A)$.
$u \in K(x)$ almost integral over $A[x]$ $\Longrightarrow$ $u \in K[x]$, and the leading coeff of u is almost integral over A
$\Rightarrow$ The leading coeff of u is in A, etc.
The same argument works for $A[[X]]$

(2) Suppose that A is a normal domain. Given $u \in K(X)$, integral over $A[X]$

Write $u = \dfrac{f(X)}{g(X)}$ and let $A_0 =$ the subring of A generated by coefficients of $f(X)$ and $g(X)$

$f(X),\ g(X) \in A[X]$ $\quad \leftarrow$ Noetherian

The argument in (1) shows that:

- $u \in K[X]$, and all coefficients b_j of u are almost integral over A_0
  $u = b_d X^d + \cdots + b_0$, $b_j \in K$ $\quad$ A_0 Noetherian $\rightsquigarrow$ b_j is integral over A_0 $\forall j$

$\Rightarrow b_j \in A \quad \forall j = 0, 1, \ldots, d$

QED

Recall / Def^n:

$\forall$ ideal $J \subsetneq A$ (comm. ring), Consider the family $\left\{ I \,\middle|\, \begin{array}{l} I \subseteq A \text{ ideal} \\ I \subseteq Z(A/J) = \text{zero divisors of } A/J \end{array} \right\} =: \mathcal{F}_{A/J}$

$\rightsquigarrow Z(A/J) = \bigcup$ maximal elements P in $\mathcal{F}$

and every maximal element of $\mathcal{F}$ is a prime ideal of A

<u>Prop 4</u> Let A be a commutative ring.

Then $A = \left\{ y \in \operatorname{frac}(A) \,\middle|\, \begin{array}{l} \text{the image of } y \text{ in } \operatorname{frac}(A_\mathfrak{p}) \text{ lies in } A_\mathfrak{p} \\ \forall \text{ max. element } \mathfrak{p} \text{ contained in } Z(A/sA) \\ s = \text{a non-zero divisor in } A \end{array} \right\}$

$\operatorname{frac}(A)$ = total ring of fractions of A

Symbolically

$$A = \bigcap_{\mathfrak{p}} A_\mathfrak{p},$$

$\mathfrak{p}$ runs through all prime ideals s.t. $\exists\, s \in A$, s not a zero divisor and $\mathfrak{p}$ is a maximal element of the family $\mathcal{F}_{A/sA}$

Pf: Suppose that $y = \frac{a}{s} \in \bigcap_{\mathfrak{p}} A_\mathfrak{p}$, $0 \neq a \in A$, s = not a zero divisor of A

Let $I := (sA : aA)$ $\quad$ Suppose that $y \notin A$, i.e. $I \neq A \Leftrightarrow a \notin sA$

$\rightsquigarrow I \subseteq Z(A/sA)$. Let $\mathfrak{p}$ be a maximal element among the family of ideals I' s.t. $I \subseteq I' \subseteq Z(A/sA)$

Know: The image of y in $\operatorname{frac}(A_\mathfrak{p})$ is in $A_\mathfrak{p}$ i.e. $\exists\, t \notin \mathfrak{p}$, $b \in A$

s.t. $ta = sb \Rightarrow t \in \underset{\mathfrak{p}}{\overset{I}{\cap\!\!|}} \Rightarrow t \in \mathfrak{p}$, a contradiction

q.e.d.

Rmk: (a) Prop 4 is often statement in the case when A is a domain (and also Noetherian)

(b) There are two generalizations of the notion of associated primes to non-Noetherian case. In prop 4 we used the maximal prime ideals contained in $Z(A/sA)$, s: non-zero-divisor in A. Nagata's approach/def^n of "prime divisors", denoted by $Ass_d(A/IA)$ in Exercise set 3, is based on this consideration.

(Recall) Prop 5. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then A is a regular local ring iff it is normal

Lemma 6 A Notherian domain, $A \ni a \neq 0$, $\mathfrak{p} \in Ass_A(A/aA)$. Then $\mathfrak{p}^{-1} \overset{def}{=} \{x \in \mathrm{Frac}(A) : x \cdot \mathfrak{p} \subseteq A\} \overset{?}{\neq} A$

pf: $\exists\, b \in A \setminus aA$ s.t. $(aA : bA) = \mathfrak{p} \rightsquigarrow \frac{b}{a} \in \mathfrak{p}^{-1}$ q.e.d.

Lemma 7 $(A, \mathfrak{m})$: Noeth local domain. Assume that $\underset{(0)}{\mathfrak{m} \neq}$ and $\mathfrak{m} \cdot \mathfrak{m}^{-1} = A$

Then A is regular of $\dim = 1$; i.e. A is a DVR

Pf: Pick $a \in \mathfrak{m} \setminus \mathfrak{m}^2$, Consider $a \cdot \mathfrak{p}^{-1} \subseteq A \rightsquigarrow a \cdot \mathfrak{p}^{-1} \nsubseteq \mathfrak{p}$, for otherwise $\underbrace{a \cdot \mathfrak{p}^{-1} \cdot \mathfrak{p}}_{= a \cdot A} \subseteq \mathfrak{p}^2$.

i.e. $a \cdot \mathfrak{p}^{-1} = A \Rightarrow aA = \mathfrak{p}$. q.e.d.

Prop 8 Let A be a Noetherian normal domain. Then (a) every non-zero principal ideal is unmixed, (b) $A = \bigcap_{ht(\mathfrak{p})=1} A_{\mathfrak{p}}$. (c) $A_{\mathfrak{p}}$ is CM $\forall \mathfrak{p}$ with $ht(\mathfrak{p}) \leq 2$

pf: (a) Consider $\mathfrak{p} \in Ass(A/aA)$, $a \neq 0$. May and do assume: A is local

By Lemma 6, $\mathfrak{p}^{-1} \supsetneq A$ Claim: $\mathfrak{p}\mathfrak{p}^{-1} = A$. Otherwise $\mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}$

$\Rightarrow \mathfrak{p}(\mathfrak{p}^{-1})^n = \mathfrak{p} \quad \forall n \quad \Rightarrow$ Every element of $\mathfrak{p}^{-1}$ is integral over A

$\Rightarrow \mathfrak{p}^{-1} \subseteq A$, contradiction. Claim proved

Lemma 7 $\Rightarrow ht(\mathfrak{p}) = 1$, $\forall \mathfrak{p} \in Ass(A/aA)$. Have proved (a)

(b) Follows from (a) and Prop 4

(c) also follows. QED.

Def 9 Conditions (S_i) and (R_i) for a commutative ring A $\quad i \in \mathbb{N}$

$(S_i) = \operatorname{depth}(A_\mathfrak{p}) \geq \inf(i, ht(\mathfrak{p})) \quad \forall \mathfrak{p} \in \operatorname{Spec}(A)$

$(R_i) = A_\mathfrak{p}$ is regular $\forall \mathfrak{p} \in \operatorname{Spec}(A)$ with $ht(\mathfrak{p}) \leq i$

Remark: (S_0) holds always. (S_1) means: $\operatorname{Ass}_A(0) = \{\text{minimal ideals of } A\}$

(S_2) means: $\operatorname{Ass}_A(0)$ has no embedded primes and $\operatorname{Ass}_A(A/aA)$ has no embedded primes for every non-zero-divisor a in A

$(S_0) + (R_0) \Leftrightarrow A$ is reduced

Thm 10 A Noetherian ring A is normal iff it satisfies (S_2) and (R_1)

Pf "only if": Prop 5 $\Rightarrow (R_1)$ $\quad$ Prop 8 $\Rightarrow (S_2)$

"if": Let $\operatorname{frac}(A)$ = total ring of fractions of $A \underset{\uparrow}{=} K_1 \times \cdots \times K_m$ $\quad K_i$ = fields

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal ideals of A $\quad \because A$ is reduced

$(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m \qquad A \hookrightarrow (A/\mathfrak{p}_1) \times \cdots \times (A/\mathfrak{p}_i) \qquad K_i = \operatorname{frac}(A_i)$

Claim: A is integrally closed in $K_1 \times \cdots \times K_m$ $\qquad e_i$ = i-th idempotents

Claim $\Rightarrow e_i \in A \ \forall i$ $\qquad \sum_i e_i = 1, \quad e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\rightsquigarrow A_i := A \cdot e_i \subseteq A \ \forall i$ and $A = A_1 \times \cdots \times A_m$

Suppose $\frac{a}{b} \in \operatorname{frac}(A)$, $a, b \in A$, b is a non-zero-divisor

$$\left(\frac{a}{b}\right)^n + c_{n-1}\left(\frac{a}{b}\right)^{n-1} + \cdots + c_1\left(\frac{a}{b}\right) + c_0 = 0 \qquad c_{n-1}, \ldots, c_0 \in A$$

i.e. $\quad a^n + c_{n-1} a^{n-1} b + \cdots + c_1 a b^{n-1} + c_0 b^n = 0$

A is S_2, $\quad A = \bigcap_{ht(\mathfrak{p})=1} A_\mathfrak{p}$

Suffices to show: $a \in bA_\mathfrak{p} \quad \forall \mathfrak{p} \in \operatorname{Spec}(A)$. But $A_\mathfrak{p}$ is a DVR by (R_2).

QED.

Prop 11 Let A be a Noetherian regular ring, i.e. $A_{\mathfrak{p}}$ is a regular local ring $\forall \mathfrak{p} \in \mathrm{Spec}(A)$. Then $A[X_1, \dots, X_n]$ is regular $\forall n \in \mathbb{N}$

Pf May assume A is local with max ideal $\mathfrak{m}$

P = a prime ideal of $\underbrace{A[X]}_{B}$, $P \cap A = \mathfrak{m}$. $\mathfrak{m} = x_1 A + \dots + x_d A$, $x_i \in \mathfrak{m}$ $\forall i$

Must show: B_P is regular

Two cases (a) $P = \mathfrak{m}B$ (b) $P = \mathfrak{m}B + f(X)B$, $f(X)$ monic in $B = A[X]$

Case (a): $P = x_1 B + \dots + x_d B$ $\quad \dim B_P = d$

Case (b): $P = x_1 B + \dots + x_d B + f \cdot B$ $\quad \dim B_P = d+1$ $\quad$ q.e.d.

Recall: M: A-module

$\mathrm{proj.dim}_A(M)$ = the length of a shortest projective resolution of M

$\mathrm{inj.dim}_A(M)$ = the length of a shortest injective resolution of M

Lemma/Fact 12. (a) M is projective $\iff \mathrm{Ext}^1_A(M, N) = 0$ $\forall$ A-module N

(b) M is injective $\iff \mathrm{Ext}^1_A(A/I, M) = 0$ $\forall$ ideal $I \subseteq A$

Lemma/Fact 13 The following statements for a commutative ring A and an integer $n \in \mathbb{N} \cup \{\infty\}$ are equivalent

(i) $\mathrm{proj.dim}_A(M) \le n$ $\forall$ A-module M

(ii) $\mathrm{proj.dim}_A(M) \le n$ $\forall$ finite A-module M

(iii) $\mathrm{inj.dim}_A(M) \le n$ $\forall$ A-module M

(iv) $\mathrm{Ext}^{n+1}_A(M, N) = 0$ $\forall$ A-modules M, N

Def^n: The global dimension of A is

$$\mathrm{gl.dim}(A) = \sup_{\underset{A\text{-module}}{M}} (\mathrm{proj.dim}(M)) = \sup_{M} (\mathrm{inj.dim}(M))$$

Cor/Fact 14 A finite module M over a Noetherian ring A is projective iff $\mathrm{Ext}^1_A(M, N) = 0$ $\forall$ finite A-module N

Lemma / Fact 15 $(A, \mathfrak{m})$ Noetherian local, $\kappa = A/\mathfrak{m}$

M: finite A-module. Then

$$\mathrm{proj.dim}_A(M) \leq n \iff \mathrm{Tor}^A_{n+1}(M, \kappa) = 0$$

pf. "$\Rightarrow$": obvious

"$\Leftarrow$" Key case: $n=0$. So assume $\mathrm{Tor}^A_1(M, \kappa) = 0$. Consider $0 \to M_1 \to L \xrightarrow{u} M \to 0$

L finite free s.t. $L/\mathfrak{m}L \underset{\sim}{\xrightarrow{\bar{u}}} M/\mathfrak{m}\cdot M$

$\leadsto$ $\underset{\overset{\shortparallel}{0}}{\mathrm{Tor}^A_1(M, \kappa)} \to M_1 \otimes_A \kappa \to L \otimes_A \kappa \underset{\sim}{\xrightarrow{\bar{u}}} M \otimes_A \kappa \to 0$

$\Rightarrow M_1 \otimes_A \kappa = 0 \Rightarrow M_1 = 0$. q.e.d.

Cor / Exer. 16 A Noetherian

(1) $\mathrm{proj.dim}_A(M) \leq n \iff \mathrm{Tor}^A_{n+1}(M, A/\mathfrak{p}) = 0 \quad \forall \mathfrak{p} \in \mathrm{Spec}(A)$. M: A-module

(2) The following are equivalent

(i) $\mathrm{gl.dim}(A) \leq n$

(ii) $\mathrm{proj.dim}_A(N) \leq n$ $\forall$ finite A-module N

(iii) $\mathrm{inj\,dim}_A(N) \leq n$ $\forall$ finite A-module N

(iv) $\mathrm{Ext}^{n+1}_A(N, N') = 0$ $\forall$ finite A-modules N, N'

(v) $\mathrm{Tor}^{n+1}_A(N, N') = 0$ $\forall$ finite A-modules N, N'

(3) $\mathrm{gl.dim}(A) = \sup_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{gl.dim}(A_\mathfrak{p})$

Prop. 17 $(A, \mathfrak{m})$ Noetherian local, $\kappa = A/\mathfrak{m}$. Then $n \in \mathbb{N}$

$$\mathrm{gl.dim}(A) \leq n \iff \mathrm{Tor}^A_{n+1}(\kappa, \kappa) = 0$$

pf. "$\Leftarrow$": $\mathrm{Tor}^A_{n+1}(\kappa, \kappa) = 0 \underset{\text{Lemma 15}}{\Rightarrow} \mathrm{proj.dim}_A(\kappa) \leq n \Rightarrow \mathrm{Tor}^A_{n+1}(M, \kappa) = 0$ $\forall A$-module M

$\Downarrow$ Lemma 15

$\mathrm{gl.dim}(A) \leq n \underset{\text{Cor 16}}{\Leftarrow} \mathrm{proj.dim}_A(M) \leq n$ $\forall$ finite A-module M

q.e.d.

Lemma 18 $(A, \mathfrak{m})$ Noeth. local, $\kappa = A/\mathfrak{m}$, M: finite A-module

If $\mathrm{proj.dim}(M) = m < \infty$, and $x \in \mathfrak{m}$ is M-regular, then $\mathrm{proj.dim}_A(M/xM) = m+1$

pf. $0 \to M \xrightarrow{x} M \to M/xM \to 0 \leadsto \mathrm{Tor}^A_{m+1}(M/xM, \kappa) \cong \mathrm{Tor}^A_m(M, \kappa) \neq 0$

and $\mathrm{Tor}^A_i(M/xM, \kappa) = 0 \quad \forall i \geq m+2$. q.e.d.

Prop. 19 (Auslander-Buchsbaum) $(A, \mathfrak{m})$ Noetherian local ring, $\kappa = A/\mathfrak{m}$

$0 \neq M$: finite A-module. Assume $\text{proj dim}_A(M) < \infty$. Then

$$\text{proj. dim}(M) + \text{depth}(M) = \text{depth}(A)$$

pf: Induction on $h = \text{proj. dim}(M)$

Case $h = 0$ obvious

Key case: $h = 1$ $\Rightarrow$ Have a minimal free resolution $L_1 \cong A^r$, $L_2 \cong A^s$ $r, s \geq 1$

$$0 \to L_1 \xrightarrow{\alpha} L_0 \xrightarrow{u} M \to 0 \quad \text{s.t. } \alpha \in \mathfrak{m} \cdot \text{Hom}_A(L_1, L_0)$$

and $u \otimes_A \kappa : L_0 \otimes_A \kappa \xrightarrow{\sim} L_1 \otimes_A \kappa$

$$\Rightarrow 0 \to \text{Hom}_A(\kappa, L_1) \xrightarrow{0} \text{Hom}_A(\kappa, L_0) \to \text{Hom}_A(\kappa, M)$$
$$\to \text{Ext}^1_A(\kappa, L_1) \xrightarrow{0} \text{Ext}^1_A(\kappa, L_0) \to \text{Ext}^1_A(\kappa, M) \to \text{Ext}^1_A(\kappa, L_0) \to \cdots$$

$\Rightarrow \text{Hom}_A(\kappa, L_1) = 0$, i.e. $\text{Hom}_A(\kappa, A) = 0$,

and $\quad 0 \to \text{Ext}^i_A(\kappa, A)^s \to \text{Ext}^i_A(\kappa, M) \to \text{Ext}^{i+1}_A(\kappa, A)^r \to 0$

$\forall i \geq 0$.

$\Rightarrow \text{depth}(A) = \text{depth}(M) + 1$. This is the asserted equality when $\text{proj. dim}(M) = 1$

Case $h \geq 2$: Take a minimal resolution of M, and reduce to the case when $\text{proj. dim}(N)$ $\quad 0 \to N \to L_0 \to M \to 0$

$= \text{proj. dim}(M) - 1 \geq 1$ $\quad$ (L_0 finite free)

$\Rightarrow \text{depth}(N) = \text{depth}(A) - \text{proj. dim}(N)$

$\Rightarrow \text{depth}(N) = \text{depth}(M) + 1$ $\qquad$ QED.

Prop. 20 $(A, \mathfrak{m}, \kappa)$ Noetherian reg. local ring of dimension d. Then $\mathrm{gl.dim}(A) = d$

Pf. Lemma 18 $\Rightarrow$ $\mathrm{proj\,dim}_A(\kappa) = d \underset{\text{Prop 17}}{\Longrightarrow} \mathrm{gl.dim}(A) = d$ q.e.d

Cor 21 (Hilbert syzygy theorem, homological proof)
k: field $\rightsquigarrow$ $\mathrm{gl.\,dim}\, k[X_1, \dots, X_n] = n$
(Immediate from Prop. 11, Cor 16 and Prop 20)

The converse of Prop. 20 holds (Serre's Theorem): If $(A, \mathfrak{m})$ is a Noeth. local ring with finite global dim, then $(A, \mathfrak{m})$ is regular

Def 22 (Koszul complex) Let A be a commutative ring. $x_1, \dots, x_n \in A$

(i) $K_\bullet(x_i) \overset{\text{def}}{=} \Big(0 \to \underset{\text{deg } 1}{A \cdot e_i} \xrightarrow{x_i} \underset{\text{deg } 0}{A} \to 0 \Big)$ a chain complex of length 1

(ii) $K_\bullet(\underline{x}) \overset{\text{def}}{=} K_\bullet(x_1, \dots, x_n) = K_\bullet(x_1) \otimes_A \cdots \otimes_A K_\bullet(x_n)$ tensor product of chain complexes

(iii) $K_\bullet(\underline{x}, M) \overset{\text{def}}{=} K_\bullet(\underline{x}) \otimes_A M$ ← thought of as a chain complex of length 0 concentrated at degree 0.

(iv) Notation: $L = Ae_1 \oplus \cdots \oplus A \cdot e_n$ free A-module with basis $e_1, \dots, e_n$

$$K_p(x_1, \dots, x_n) = \Lambda^p_A(L) = \bigoplus_{\substack{\alpha = (i_1, \dots, i_p) \\ 1 \le i_1 < i_2 < \cdots < i_p \le n}} A \cdot e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p}$$

$$d(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{\alpha_r} \cdot e_{\alpha_1} \wedge \cdots \wedge \widehat{e_{\alpha_r}} \wedge \cdots \wedge e_{\alpha_p}$$

Lemma 23 $C_\bullet$: a chain complex of A-modules, $x \in A$. $K(x) = (0 \to \underset{\text{deg } 1}{Ae} \xrightarrow{x} \underset{\text{deg } 0}{Ae} \to 0)$

(1) We have a long exact sequence

$$\cdots \to H_{i+1}(C_\bullet) \to H_{i+1}(C_\bullet \otimes_A K(x)) \to H_i(C_\bullet) \xrightarrow{(-1)^i x} H_i(C_\bullet) \to H_i(C_\bullet \otimes_A K(x)) \to$$
$$\cdots \to H_1(C_\bullet) \to H_1(C_\bullet \otimes_A K(x)) \to H_0(C_\bullet) \xrightarrow{x} H_0(C_\bullet) \to H_0(C_\bullet \otimes K(x)) \to 0$$

Lemma 23

(2) If $C_\bullet$ is acyclic, i.e. $H_i(C_\bullet)=0$ $\forall i\geq 1$, then $H_i(C_\bullet \otimes_A K(x))=0$ $\forall i\geq 2$, and we have a 4-term exact sequence

$$0\to H_1(C_\bullet\otimes_A K(x)) \to H_0(C_\bullet) \xrightarrow{x} H_0(C_\bullet) \to H_0(C_\bullet \otimes_A K(x)) \to 0$$

In particular, if x is $H_0(C_\bullet)$-regular, then $C_\bullet \otimes_A K(x)$ is acyclic.

Pf. $0\to C_\bullet \to C_\bullet(x) \to C_\bullet[1]\to 0$

where $(C_\bullet[1])_i = \begin{cases} C_{i-1} & \text{if } i\geq 0 \\ 0 & \text{if } i=0\end{cases}$

and $d_i^{C[1]} = d_{i-1}^{C_\bullet}$ $\forall i\geq 1$ q.e.d.

Prop. 24 If $\underline{x} = (x_1,\cdots,x_n)$ is an M-regular sequence of an A-module M, $x_i\in A$ $\forall i=1,\cdots,n$

Then $H_i(K_\bullet(\underline{x})\otimes_A M)=0$ $\forall i\geq 1$ and $H_0(K_\bullet(\underline{x})\otimes_A M) = M/(x_1M+\cdots+x_nM)$.

Lemma 25 Let $L_\bullet \to M = (\cdots \to L_i \xrightarrow{d_i} L_{i-1} \to \cdots \to L_1 \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M)$ be a minimal resolution of a finite module M over a Noetherian local ring $(A,\mathfrak{m})$ $k=A/\mathfrak{m}$

Let $F_\bullet = (\cdots \to F_i \xrightarrow{d'} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0)$, $F_0 \xrightarrow{\varepsilon'} M$, be an augmented complex

s.t. (a) F_i is finite free $\forall i$

(b) $\varepsilon'\otimes_A k : F_0\otimes_A k \to M\otimes_A k$ is injective

(c) $d_i'(F_i)\subseteq \mathfrak{m}F_{i-1}$ $\forall i\geq 1$, and d_i' induces an injection

$$F_i\otimes_A k \to (\mathfrak{m}/\mathfrak{m}^2)\otimes_A F_{i-1} \quad \forall i\geq 1$$

Then $\exists$ a homomorphism of augmented complexes

$$\begin{array}{ccc} F_\bullet & \xrightarrow{\alpha_\bullet} & L_\bullet \\ \downarrow{\scriptstyle\varepsilon'} & & \downarrow{\scriptstyle\varepsilon} \\ M & = & M \end{array}$$

such that α_i is injective and $\alpha_i(F_i)$ is an A-direct summand of L_i $\forall i\geq 0$. In particular

$$\operatorname{rank}_A(F_i) \leq \operatorname{rank}_A(L_i) = \dim_k(\operatorname{Tor}_i^A(M,k)) \quad \forall i\geq 0$$

Pf of Lemma 25. The existence of $\alpha_\cdot$ follows from the exactness of $F_\cdot \xrightarrow{\varepsilon} M \to 0$.
The assumption (b), (c) + easy induction shows that
$$f_i \otimes_A \kappa : F_i \otimes_A \kappa \to L_i \otimes_A \kappa$$
is injective $\forall i \geq 0$. q.e.d.

Prop. 26 $(A, \mathfrak{m}, \kappa)$ Noetherian local ring. $d = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$
Then $\dim_\kappa \operatorname{Tor}_i^A(\kappa, \kappa) \geq \binom{d}{i}$ $\forall i = 0, 1, \cdots, d$
pf: Apply Lemma 25, with $F_\cdot = K_\cdot(x_1, \cdots, x_d)$,
where $x_1, \cdots, x_d \in \mathfrak{m}$ s.t. $(x_i \bmod \mathfrak{m}^2)_{1 \leq i \leq d}$ form a κ-basis of $\mathfrak{m}/\mathfrak{m}^2$.
q.e.d.

Theorem 27 (Serre) A Noetherian local ring $(A, \mathfrak{m}, \kappa)$ is regular iff $\operatorname{gl.dim}(A) < \infty$. (And if so, $\operatorname{gl.dim}(A) = \dim(A) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$)
pf: Only need to prove "if" Let $d = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$.
Prop 26 $\Rightarrow \operatorname{Tor}_d^A(\kappa, \kappa) \neq 0 \Rightarrow \operatorname{gl.dim}(A) \geq d$
Auslander-Buchsbaum + Prop. 17 $\Rightarrow \operatorname{gl.dim}(A) = \operatorname{proj\,dim}_A(\kappa) = \operatorname{depth}(A)$
$\rightsquigarrow \dim(A) \leq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) \leq \operatorname{gl.dim}(A) = \operatorname{depth}(A) \leq \dim(A)$. QED

Cor. $(A, \mathfrak{m}, \kappa)$ Noeth. regular local $\Rightarrow A_{\mathfrak{p}}$ is a regular local ring $\forall \mathfrak{p} \in \operatorname{Spec}(A)$