Abstract. Using geometric methods on the moduli space (in characteristic zero) we try to understand which automorphism groups can appear.

The moduli space of curves can be stratified by locally closed sets on which all fibers have the same automorphism group. (We will give a precise definition.) A curve defining a stratum of dimension zero is said to have many automorphisms.

Theorem. There are infinitely many curves with many automorphisms that are not CM curves.

We try to understand minimal and maximal strata.

(Maximal strata were studied by Cornalba, we will review his results.)

Definition (a new notion). A final curve defines a minimal Hurwitz stratum. Equivalently: in a family of final curves (closed in the moduli space) there is no special fiber with strictly more automorphisms.

Clearly: ”many automorphisms” implies ”final”. We will show there are many final curves that do not have many automorphisms (i.e. defining a positive dimensional closed stratum).

Theorem. For every curve D there exists a Galois cover \( C \to D \) such that \( C \) is a final curve. (In these cases \( D \) can have a large automorphisms group, while the automorphism group of \( C \) is small.)

We will give motivation and examples. We will see that there are (infinitely many) Hurwitz strata that are minimal and maximal (for \( g \to \infty \), and their dimensions are unbounded). We will give complete proofs of the results mentioned above.

Methods: signature, monodromy, descending and lifting automorphisms in a Galois cover.

(1). The Hurwitz stratification. We discuss algebraic curves over an algebraically closed field of characteristic zero. For a curve \( C \) we have:

\[
\varphi : C \to \text{Aut}(C) \setminus C =: D,
\]

with branch locus \( B = B(\varphi) \subset D \), and

\[
\pi_1(D - B) =: \Gamma \to G := \text{Aut}(C).
\]

We write

\[
\sigma = (G, g, t, h, \Gamma \to G), \quad g = \text{genus}(C), \quad h = \text{genus}(D), \quad t = \#(B).
\]

The Hurwitz scheme \( H_\sigma \) representing covers with these properties exists, and we write

\[
H_\sigma \to \mathcal{H}_\sigma \subset \mathcal{M}_g, \quad \mathcal{M}_g = \sqcup_\sigma \mathcal{H}_\sigma,
\]
for the image, the related locus in the moduli space. Note that $\mathcal{H}_\sigma \subset \mathcal{M}_g$ is locally closed and irreducible.

(2). Definition. We say $C$ has many automorphisms (a very classical notion) if the related $\mathcal{H}_\sigma$ has dimension zero. Equivalently: $\text{Def}(C, \text{Aut}(C))$ has dimension zero.

We say a curve of genus $g$ is a CM curve if its Jacobian $J(C)$ is a CM abelian variety, i.e. if $\text{End}_0^0(J(C))$ contains a semi-simple commutative subalgebra of rank $2g$ over $\mathbb{Q}$.

(3). Theorem. There are infinitely many curves with many automorphisms that are not CM curves

(4). Definition (a new notion). We say $C$ is a final curve if the related Hurwitz stratum $\mathcal{H}_\sigma$ is closed in $\mathcal{M}_g$.

Equivalently: in a family of curves with generic fiber a final curve, any special fiber does not have strictly more automorphisms.

We are interested in “going down” in the Hurwitz stratification, and seeing where such a chain ends.

Easy example. For every prime number $p \geq 11$, with $g := (p - 1)/2$ there exists a final curve $C$ that has “many automorphisms” with $\text{Aut}(C) \cong \mathbb{Z}/p$. (In this case the related Hurwitz stratum is zero-dimensional, minimal and also maximal). We see that the notion “many automorphisms” need not imply that the automorphism group is large.

(5). Theorem. For every prime number $p \geq 5$ there exists a final curve $C$ with $\text{Aut}(C) \cong \mathbb{Z}/p$.

Curious. Such a curve not only defines a minimal Hurwitz stratum but also a maximal Hurwitz stratum.

Remark. Cornalba classified maximal strata, and we see that for $g \geq 3$ every maximal stratum is defined by a cyclic group of prime order.

(6). Theorem. For every curve $D$ there exists a Galois covering

$$C \to G\backslash C =: D, \quad G = \text{Aut}(C)$$

(actually with prime order cyclic Galois group $G$) such that $C$ is a final curve. We see that the dimension of minimal strata is unbounded for $g \to \infty$.

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