Moduli of abelian varieties: symmetry and rigidity

Ching-Li Chai

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Goal: survey Hecke symmetry on the moduli spaces of abelian varieties in positive characteristic $p>0$

- history: elliptic curves $\rightsquigarrow$ curves of higher genera, abelian varieties

  $\rightsquigarrow$ moduli spaces $\circlearrowleft$ Hecke symmetry

- phenomena structures in characteristic $p>0$, conjectures

- new tools / results

§1 From elliptic curves to abelian varieties and moduli

1.1. several approaches to elliptic curves

(algebra) $E: \; y^2 = 4x^3 - g_2 x - g_3$, $\quad \Delta = g_2^3 - 27 g_3^2$, $\quad j = 1728\, g_2^3 / \Delta$

(geometry) $E(\mathbb{C}) \xleftarrow{\sim} \mathrm{Lie}(E) / H_1(E(\mathbb{C}), \mathbb{Z})$, $\quad P \longmapsto \int_\infty^P \frac{dx}{y}$

$\cong \mathbb{C} / \mathbb{Z}\tau + \mathbb{Z}$ $\quad \tau \in \mathfrak{h} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$

(analysis) $\wp(z;\tau) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda_\tau}' \left[\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right]$

$$\left(\frac{d}{dz} \wp \right)^2 = 4\wp^3 - g_2 \wp - g_3$$

$$g_2 = 60 \sum_{\gamma \in \Lambda_\tau}' \frac{1}{\gamma^4}, \quad g_3 = 140 \sum_{\gamma \in \Lambda_\tau}' \frac{1}{\gamma^6}$$

1.2 origin of elliptic curves

Fermat to Huygens via Carcavi, 1659

A. (Diophantine equation) Fermat : (E) $x^4 - y^4 = z^2$ has no non-trivial rational solution

infinite descent:

(E') $s^4 + 4t^4 = u^2$

(x_1, y_1, z_1) primitive solution to equation (E) $\dashrightarrow$ $x_1 + y_1 = 4t^2$, $x_1 - y_1 = 2s^2$, $z_1 = 4ust$ (s, t, u) primitive sol^n to eq. (E'), $u > 0$ $\dashrightarrow$ (x_2, y_2, z_2) prim. sol^n to (E)

$$s^2 = x_2^4 - y_2^4$$
$$t^2 = x_2^2 y_2^2$$
$$u = x_2^4 + y_2^4$$
$$s = z_2$$

<u>Note</u>: This is 2-descent via $2 = (1+\sqrt{-1})\cdot(1-\sqrt{-1})$:

$z^2 = x^4 - 1$ defines an elliptic curve over $\mathbb{Q}$ with CM by $\mathbb{Z}[\sqrt{-1}]$.

B. (elliptic integral)

Fagnano
Euler
(December, 1751, paper by Fagnano reached Euler in Berlin)

Fagnano : $\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{dy}{\sqrt{1-y^4}}$ has rational solutions

i.e. $\displaystyle\int_0^x \frac{d\rho}{\sqrt{1-x^4}} = \int_0^y \frac{d\psi}{\sqrt{1-\psi^4}}$ has sol^n
$y =$ a rational function of x

Euler : $$\frac{m\,dx}{\sqrt{1-x^4}} = \frac{n\,dy}{\sqrt{1-y^4}}$$

$$\int_0^r \frac{d\rho}{\sqrt{1-\rho^4}} = \sqrt{2}\int_0^t \frac{d\xi}{\sqrt{1+\xi^4}}, \quad \int_0^t \frac{d\xi}{\sqrt{1+\xi^4}} = \sqrt{2}\cdot\int_0^u \frac{d\eta}{\sqrt{1-\eta^4}}$$

$$r^2 = \frac{2t^2}{1+t^4} \qquad t^2 = \frac{2u^2}{1-u^4}$$

$$\int_0^r \frac{d\rho}{\sqrt{1-\rho^4}} = (1\pm\sqrt{-1})\cdot\int_0^v \frac{d\eta}{\sqrt{1-\eta^4}} \qquad r = \pm\frac{2\sqrt{-1}\,v^2}{1-v^4}$$

<u>inversion</u> of <u>abelian</u> <u>integrals</u> elliptic
(for $y^2 = f(x)$, i.e. hyperelliptic curves)

Abel 1827, Jacobi 1828.

Jacobi 1829. Fundamenta Nova Theoriae Functionum Ellipticarum
defined Jacobi theta functions

1.3. Curves and their Jacobians

Riemann 1857, Theorie der Abel'sche Functionen

$S = C(\mathbb{C})$ compact Riemann surface i.e. C = smooth projective connected algebraic curve over $\mathbb{C}$

$\gamma_1, \dots, \gamma_{2g}$: $\mathbb{Z}$-basis of $H_1(S, \mathbb{Z})$ $\qquad \Delta = (\Delta_{ij})_{2g \times 2g} \quad \Delta_{ij} = \gamma_i \cdot \gamma_j$

$\omega_1, \dots, \omega_g$: $\mathbb{C}$-basis of $\Gamma(S, \Omega^1_S)$

$P = P(\omega_1, \dots, \omega_g; \gamma_1, \dots, \gamma_{2g}) = (P_{ri})_{\substack{1 \le r \le g \\ 1 \le i \le 2g}}$ $\quad g \times 2g \qquad P_{ri} = \int_{\gamma_i} \omega_r$

$$\boxed{\begin{array}{c} P \cdot \Delta^{-1} \cdot {}^t P = 0 \\ -\sqrt{-1}\; P \cdot \Delta^{-1} \cdot {}^t \overline{P} \gg 0_g \end{array}}$$

Riemann bilinear relations

$$C \longrightarrow \mathrm{Pic}^1(C) \circlearrowleft \mathrm{Pic}^0(C) = \mathrm{Jac}(C)$$

principal homog space $\qquad \mathrm{Jac}(C)(\mathbb{C}) \cong \Gamma(C, \Omega^1_C)^\vee / H_1(C(\mathbb{C}); \mathbb{Z})$

1.4. Abelian varieties

Def. A $g \times 2g$ matrix $Q \in M_{g\times 2g}(\mathbb{C})$ is a Riemann matrix if $\exists$ a skew symmetrix $2g \times 2g$ matrix $E \in M_{2g}(\mathbb{Z})$ with $\det(E) \neq 0$ satisfying

$$\begin{cases} Q \cdot E^{-1} \cdot {}^t Q = 0_g \\ \sqrt{-1}\, Q \cdot E^{-1} \cdot {}^t \overline{Q} \gg 0_g \end{cases}$$

(E is called the principle part of Q)

Def. (abelian varieties)

(i) (over $\mathbb{C}$): a compact complex torus $\mathbb{C}^g / Q \cdot \mathbb{Z}^g$ is a complex abelian variety if Q is a Riemann matrix

(i)' (over $\mathbb{C}$): a compact complex torus is an abelian variety if it admits a holomorphic embedding to $\mathbb{P}^N(\mathbb{C})$ for some N

(ii) An irreducible algebraic group variety A over a field k is an abelian variety if A is complete (i.e. proper over k)

Rmk (a) The classical proof of (i) $\Longleftrightarrow$ (ii) uses Riemann's theta function

(b) The algebraic theory of abelian varieties was due to André Weil (1948)

Def. (polarization of abelian varieties)

(i) A polarization of an abelian variety A is an ample divisor on A up to algebraic equivalence (equivalently, up to translation)

(ii) The polarization of A given by an ample divisor D on A is principal if $D^g = g!$

Rmk. (a) The polarization induced by D is uniquely determined by

$$\varphi_{[D]} : A \longrightarrow A^t = \mathrm{Pic}^\circ(A) = \text{dual abelian variety}$$
$$x \longmapsto \mathcal{O}_A(D-x)$$

(b) over $\mathbb{C}$ The fundamental class $c([D]) \in H^2(A(\mathbb{C}), \mathbb{Z}(1))$ corresponds to a Riemann form

Recap: Over $\mathbb{C}$

(i) Every principally polarized abelian variety of dimension g is of the form $A_\Omega := \mathbb{C}^g / \Omega\cdot\mathbb{Z}^g + \mathbb{Z}^g$ with principal part $\begin{bmatrix} 0_g & 1_g \\ -1_g & 0_g \end{bmatrix}$
for some
$$\Omega \in \mathfrak{h}_g = \{ \Omega \in M_g(\mathbb{C}) : {}^t\Omega = \Omega,\ \mathrm{Im}(\Omega) \gg 0_g \}$$ = Siegel's upper space

(ii) $(A_{\Omega_1}, \lambda_{\Omega_1}) \underset{\uparrow}{\cong} (A_{\Omega_2}, \lambda_{\Omega_2}) \iff \exists \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$ such that
preserves polarization
$$(A\Omega_1 + B)\cdot(C\Omega_1 + D)^{-1} = \Omega_2$$

1.5 Moduli spaces (or, classifying spaces)

Idea / phenomenon:

The set of all isomorphism classes of all algebraic varieties with fixed discrete invariants often has a natural structure as an algebraic variety (or an algebraic structure close to an algebraic variety)

Ex. $\mathcal{M}_g$ = the moduli stack / scheme / space classifying all smooth proper curves of genus $g \geq 2$.

$\mathcal{A}_g$ = the moduli stack / scheme / space classifying g-dimensional principally polarized abelian varieties

(First existence proof by Mumford, 1965)

Over $\mathbb{C}$:

$$\mathcal{M}_g(\mathbb{C}) = \Gamma_g \backslash \mathcal{T}_g, \quad \text{where } \mathcal{T}_g = \text{Teichmüller space of genus } g$$

Γ_g = mapping class group for an oriented connected smooth closed surface of genus g

$$\mathcal{A}_g(\mathbb{C}) = Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g$$

Rmk. T_g:
$$\begin{array}{ccc} \mathcal{M}_g & \xrightarrow{\text{Torelli map}} & \mathcal{A}_g \\ \ni & & \ni \\ [C] & \longmapsto & [Jac(C)] \end{array}$$

Torelli theorem:

$T_g(k): \mathcal{M}_g(k) \hookrightarrow \mathcal{A}_g(k)$

k: algebraically closed field

/$\mathbb{C}$: Torelli 1914

general k Weil 1957

Over an arbitrary algebraically closed field k

- $\mathcal{M}_g/_k$ is irreducible

  char$(k) = 0$: from uniformization + Lefschetz principle

  char$(k) = p > 0$: Deligne-Mumford, 1969

- $\mathcal{A}_g/_k$ is irreducible

  char$(k) = 0$: from uniformization

  Char$(k) = p > 0$: Faltings-C. 1984

§2 Hecke symmetry on $\mathcal{A}_g$

a feature share with all Shimura varieties

2.1 Definitions

<u>Over $\mathbb{C}$</u>: $\forall \gamma \in Sp_{2g}(\mathbb{Q})$, the double coset $Sp_{2g}(\mathbb{Z}) \cdot \gamma \cdot Sp_{2g}(\mathbb{Z})$ induces an algebraic correspondence on $Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{h}_g = \Gamma_g(1) \backslash \mathfrak{h}_g$

$$\begin{tikzcd}
\Gamma_g(1) \backslash \mathfrak{h}_g & \mathfrak{h}_g \arrow[l, "pr"'] \arrow[r, "\gamma\cdot"] \arrow[dl, "pr"] & \mathfrak{h}_g \arrow[d, "pr"] \\
(\gamma^{-1} \cdot \Gamma_g(1) \cdot \gamma \cap \Gamma_g(1)) \backslash \mathfrak{h}_g \arrow[u] \arrow[rr, "\gamma\cdot"'] & & \Gamma_g(1) \backslash \mathfrak{h}_g
\end{tikzcd}$$

These are remnants of the transitive action of $Sp_{2g}(\mathbb{R})$ on $\mathfrak{h}_g$, after quotient by $\Gamma_g(1) = Sp_{2g}(\mathbb{Z})$.

algebraic version :

Def. $[(A_1, \lambda_1)]$, $[(A_2, \lambda_2)] \in \mathcal{A}_g(k)$ $\qquad \lambda_i : A_i \xrightarrow{\sim} A_i^t$ principal pol

are in the same Hecke orbit if $\exists$ isogeny $\alpha : A_1 \longrightarrow A_2$

s.t. $\alpha^*(\lambda_2) = n \cdot \lambda_1$ for some $n \in \mathbb{N}$ $\qquad A_1^t \xleftarrow{\alpha^t} A_2^t$

$\alpha^t \circ \lambda_2 \circ \alpha$

Note: Suppose $\mathrm{char}(k) = p > 0$, say $[(A_1, \lambda_1)]$ and $[(A_2, \lambda_2)]$ are in the same prime-to-p Hecke orbit if one can choose α to be an isogeny s.t rank($\mathrm{Ker}(\alpha)$) is prime to p;

equiv. $\gcd(n, p) = 1$ in Def above.

prime-to-p finite adeles

Adelic picture: $\mathbb{A}_f^{(p)} = \prod'_{\substack{\ell:\text{prime} \\ \ell \neq p}} \mathbb{Q}_\ell$ (restrict product)

$k \supseteq \mathbb{F}_p$, alg. closed

$$GSp_{2g}(\mathbb{A}_f^{(p)}) \curvearrowright \widetilde{\mathcal{A}}_{g/k}^{(p)} = \varprojlim_{\gcd(n,p)=1} \mathcal{A}_{g,n/k}$$

$\mathcal{A}_{g,n/k}$ = moduli stack of triples $\left(A, \lambda, A[n] \xleftarrow[\text{symplectic}]{\sim} (\mathbb{Z}/n\mathbb{Z})^{2g}\right)$

$$\begin{array}{c} \widetilde{\mathcal{A}}_{g/k}^{(p)} \\ \downarrow \\ \mathcal{A}_{g/k} \end{array}$$

prime-to-p Hecke orbits on $\mathcal{A}_{g/k}$ $\longleftrightarrow$ orbits of $GSp_{2g}(\mathbb{A}_f^{(p)})$ on $\widetilde{\mathcal{A}}_{g/k}^{(p)}$

Remark. The Hecke correspondences also act on automorphic vector bundles (and automorphic local systems) on $\mathcal{A}_g$, such as

$$e^* \Omega^1_{A/\mathcal{A}_g}, \quad \det(e^* \Omega^1_{A/\mathcal{A}_g}) \quad (\text{and} \quad R^1\pi_* \mathbb{Q}_\ell)$$

therefore induce linear action (via trace) of Hecke algebra(s) on automorphic forms on $\mathcal{A}_g$. (Hecke: case $g=1$)

$$\begin{tikzcd} A_g \arrow[r, "\pi"] & \mathcal{A}_g \arrow[l, bend left, "e"] \end{tikzcd}$$

universal abelian scheme

2.2 p-adic invariants

$k = \bar{k} \supseteq \overline{\mathbb{F}}_p$
base field

(*) Every prime-to-p symplectic isogeny between principally polarized abelian varieties over k preserves p-adic invariants

Example of p-adic invariants

(a) slopes / Newton polygon of an abelian variety $A_{/k}$

$\rightsquigarrow \quad \mathrm{Fr}_A^{(p)} : A \to A^{(p)}$ and its iterates $\mathrm{Fr}_A^{(p^n)} : A \longrightarrow A^{(p^n)}$

slopes (with multiplicity) $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2g}$

$\lambda_i \in \mathbb{Q} \ \forall i, \quad \lambda_i + \lambda_{g+1-i} = 1 \ \forall i$

$\mathrm{denom}(\lambda_i) \mid \mathrm{multiplicity}(\lambda_i) \ \forall i$

properties of slopes / Newton polygon

- They measure asymptotic divisibility properties (by powers of p) of the action of $Fr^{(p^n)}$ on $H^m_{cris}(A)$ as $n \to \infty$
- A is ordinary $\underset{def}{\Longleftrightarrow}$ slopes are 0 and 1 $\Longleftrightarrow$ Hasse invariant of A does not vanish

  When $A/\mathbb{F}_q$, ordinary $\Longleftrightarrow$ half of the eigenvalues of $Fr_{A/\mathbb{F}_q} : A \to A$ are p-adic units
- A is supersingular $\underset{def}{\Longleftrightarrow}$ all slopes $= \frac{1}{2}$

(b) isomorphism class of $(A[p], \lambda[p])$ $\longleftrightarrow$ $\left(H_{1,dR}(A) + \text{symplectic pairing}\right)$

In both (a) and (b), the discrete invariants have natural partial ordering, compatible with specialization

$\rightsquigarrow$ stratification of $\mathcal{A}_{g}/k$ by locally closed ~~subsets~~ subschemes

In each of the above two stratifications of $\mathcal{A}_g$, there is a dense open stratum : the open subscheme $\mathcal{A}_g^{\mathrm{ord}} \subseteq \mathcal{A}_g$ corresponding to ordinary principally polarized abelian varieties

Thm (CLC, 1995) $\forall\, x = [(A,\lambda)] \in \mathcal{A}_{g/k}$ with A ordinary, the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{A}_{g/k}$

⚠ Trivial fact: every prime-to-p Hecke orbit of $\mathcal{A}_{g/\bar{\mathbb{Q}}}$ is Zariski dense in $\mathcal{A}_{g/\bar{\mathbb{Q}}}$. However the above theorem does not follow from this fact and the Serre-Tate canonical lifting of ordinary abelian varieties (to characteristic 0)

Q. How to generalize this result on Hecke symmetry orbits to non-ordinary abelian varieties?

2.3. p-divisible groups (or, Barsotti-Tate groups) Tate 1967, Grothendieck 1970

Def^n: A p-divisible group $X \to S$ is an inductive system of commutative finite locally free group schemes

$$\left((X_n \to S)_{n \in \mathbb{N}},\quad i_{n+1,n} : X_n \hookrightarrow X_{n+1},\quad \pi_{n,n+1} : X_{n+1} \xrightarrow{\text{faithfully flat}} X_n \right)$$

such that $i_{n+1\,n} \circ \pi_{n,n+1} = [p]_{X_{n+1}} \quad \forall n$

Fact: $\exists\, \underset{\text{height}}{h} : S \to \mathbb{N}$, locally constant, such that $rk(X_n) = p^{nh} \quad \forall n$

Primary example: $A \to S$ abelian scheme $\rightsquigarrow$ $\underset{\underset{A[p^\infty]}{\|}}{(A[p^n])_{n \in \mathbb{N}}}$ is a p-divisible group

* $A[p^\infty]$: substitute for Lie algebra in characteristic $p>0$ (or mixed characteristics $(0,p)$)
* $A[p^\infty]$ "gives all p-adic invariants" of A.

2.4 Leaves (or central leaves) in moduli spaces

Definition (Oort, 1999) $k = \bar{k} \supseteq \mathbb{F}_p$, $x \in \mathcal{A}_g(k)$, $x = [(A, \lambda)]$

The leaf $\mathcal{C}(x)$ through x is the locally closed subvariety of $\mathcal{A}_g$ such that $\mathcal{C}(x)(k) = \{ (B, \mu) \in \mathcal{A}_g(k) \mid (B[p^\infty], \mu[p^\infty]) \cong (A[p^\infty], \lambda[p^\infty]) \}$

Fact: Every leaf in $\mathcal{A}_g$ is smooth, and stable under all prime-to-p Hecke correspondences

Conjecture (Oort) Let $\mathcal{C}$ be a leaf in $\mathcal{A}_g$. For any $x \in \mathcal{C}(k)$, the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{C}$.

Note: $\mathcal{A}_g^{\mathrm{ord}}$ is a leaf; have seen that the conjecture holds for $\mathcal{A}_g^{\mathrm{ord}}$

Remark (i) This Hecke orbit conjecture can be formulated for moduli spaces of PEL type (in characteristic p, classify moduli spaces with fixed type of Polarization, Endomorphism and Level structure)

(ii) This conjecture holds for $\mathcal{A}_g$ (Oort + CLC);
proof uses a special property of $\mathcal{A}_g$.
Open for PEL type A and D

phenomena

§3 New tools, structures and conjectures related to Hecke symmetry

3.1. Irreducibility $\qquad k = \bar{k} \supseteq \mathbb{F}_p$

__Proposition A__ Let $Z \subset \mathcal{A}_{g/k}$ be a positive dimensional locally closed subvariety which is stable under all prime-to-p Hecke correspondences. If Hecke operates transitively on $\pi_0(Z)$, then Z is irreducible

__Proposition B__ Let $C \subseteq \mathcal{A}_g$ be a positive dimensional leaf on $\mathcal{A}_g$, then the naive p-adic monodromy for C is maximal

$\uparrow$

$\mathrm{Aut}\,(A_x[p^\infty], \lambda_x[p^\infty]) \qquad x \in C$

Note: Prop. B is useful in Iwasawa theory: "irreducibility of Igusa towers"

Prop C (Oort + C) Every non-supersingular Newton stratum in $\mathcal{A}_g$ is irreducible

Note: (i) supersingular = all slopes $= \frac{1}{2}$

(ii) The supersingular stratum has dimension $\left\lfloor \frac{g^2}{4} \right\rfloor$

Prop D (Oort + C) Every non-supersingular leaf in $\mathcal{A}_g$ is irreducible

Note: a leaf C in $\mathcal{A}_g$ is supersingular $\Longleftrightarrow$ $\dim(C) = 0$

3.2 Local structure of leaves

2-slope case: $\mathcal{C} \ni x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g(k)$. $k = \bar{k} \supseteq \mathbb{F}_p$

slopes of $A_0 = \{\lambda, 1-\lambda\}$, $\lambda < \frac{1}{2}$

Prop. $\mathcal{C}^{/x_0}$ (:= formal completion of $\mathcal{C}$ at x_0) has a natural structure as a (neutral torsor for a) isoclinic p-divisible group with slope $1-2\lambda$ and height $g(g+1)/2$

3.3 Local stabilizer principle

$k = \bar{k} \supseteq \mathbb{F}_p$

<u>Prop.</u> Let $Z \subseteq \mathcal{A}_g$ be a locally closed subvariety, stable under all prime-to-p Hecke correspondences, $x_0 = [(A_0, \lambda_0)] \in Z(k)$.

Then $Z^{/x_0} \subseteq \mathcal{A}_g^{/x_0}$ is stable under the natural action of an open subgroup of $U(\mathrm{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p)$ on $\mathcal{A}_g^{/x_0}$

$$\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty]) \circlearrowright \mathcal{A}_g^{/x_0} = \mathrm{Def}(A_0, \lambda_0) \xrightarrow{\sim} \mathrm{Def}(A_0[p^\infty], \lambda_0[p^\infty])$$

Serre-Tate theorem

$$\cup$$

$$U(\mathrm{End}(A_0), *_{\lambda_0})$$

(prime-top) Hecke correspondences with x_0 as a fixed point

3.4. Rigidity

Theorem (Local rigidity) X : p-divisible formal group over $k = \bar{k} \supseteq \mathbb{F}_p$, $Z \subseteq$ irreducible formal subvariety. Suppose $\exists$ connected p-adic Lie subgroup $G \subseteq \mathrm{Aut}(X_0)$ such that X^G = trivial and Z is stable under G.

fixed points of G

Then Z is a p-divisible formal subgroup of X

"Exer." Case $X = \mathbb{G}_m^h$, $G = (1 + p^2 \cdot \mathbb{Z}_p) \cdot \mathrm{Id}_X$

Example / Cor. E_0: an ordinary elliptic curve$/k=\bar{k}\supseteq\mathbb{F}_p$ $\quad A_0 = \underbrace{E_0\times\cdots\times E_0}_{g \text{ times}}$

$x_0 := [(A_0,\lambda_0)]$ $\quad\lambda_0 =$ product polarization on A_0

Then the prime-to-p Hecke orbit of x_0 is dense in $\mathcal{A}_g$

Proof: $\mathcal{A}_g^{/x_0} \underset{\substack{\uparrow\\ \text{Serre-Tate}}}{\cong} \hat{\mathbb{G}}_m^{\,g(g+1)/2}$, $\quad U(\mathrm{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p) \cong GL_g(\mathbb{Z}_p)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \wr\Vert$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty])$

$M_g(\mathcal{O})$

$\mathcal{O}$ = order in an imaginary quadratic field K

$\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p \cong \mathbb{Z}_p\times\mathbb{Z}_p$

Action of $GL_g(\mathbb{Z}_p)$ on $X^*\left(\hat{\mathbb{G}}_m^{\,g(g+1)/2}\right)$

$\cong S^2$ (standard representation of $GL_g(\mathbb{Z}_p)$ on $\mathbb{Z}_p^{\,g}$)

irreducible !

QED

global rigidity

Conjecture: Suppose $Z \subset \mathcal{A}_g^{ord}$ locally closed irreducible, $x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g^{ord}(k)$. $k = \bar{k} \supseteq \mathbb{F}_p$

and $Z^{/x_0} \subset \mathcal{A}_g^{/x_0}$ is a formal subtorus of $\mathcal{A}_g^{/x_0}$ = Serre-Tate formal torus

Then Z = (reduction of) a Shimura subvariety of $\mathcal{A}_g$

Remark: Known if $Z \subsetneq$ a Hilbert modular (sub) variety

This case has application in Iwasawa theory

(geometric input in Hida, non-vanishing of the μ-invariant, Ann. of Math. 2012 "density of CM points")

<u>Conjecture</u> (Local rigidity special case)

G_0 = 1-dimensional smooth formal group over $\overline{\mathbb{F}}_p$, $ht(G_0)=h$

$\mathcal{M}$ = equi-char. deformation space of G_0 Lubin-Tate, 1966

$\cong \mathrm{Spf}(\overline{\mathbb{F}}_p[[x_1,\dots,x_{h-1}]])$

$Z \subseteq \mathcal{M}$ irreducible formal subscheme

Assume (i) Z is stable under the natural action of an open subgroup of $\mathrm{Aut}(G_0)$ ← units in a central division algebra over $\mathbb{Q}_p$ with Brauer invariant $\frac{1}{h}$

(ii) Z is generically ordinary (i.e. $Z \nsubseteq$ zero locus of Hasse invariant)

Then $Z = \mathcal{M}$

Rmk: A proof will have application in chromatic homotopy theory.

3.6 Sustaind p-divisible groups

Motivation: Find a good (scheme-theoretic) definition of leaves

One benefit: can study local structure of leaves using deformation theory

<u>Definition</u> (Oort + C.) κ = a field of char. $p>0$ (not nec. alg. closed)

X_0/κ : p-divisible group. S/κ : scheme over κ

A p-divisible group $X \to S$ is strongly κ-sustained modeled on X_0

if $\underline{\mathrm{Isom}}_S\,(X[p^n],\ X_0[p^n] \times_{\operatorname{Spec}\kappa} S) \to S$ is faithfully flat $\forall n \in \mathbb{N}$

3.7 "Internal Hom" of p-divisible groups

(actually these are "internal Ext")

κ : field of char. $p>0$, X_0, Y_0 : p-divisible groups over κ

$$G_n := \text{Image}\left(\underline{\mathrm{Hom}}(X_0[p^{n+m}], Y_0[p^{n+m}]) \longrightarrow \underline{\mathrm{Hom}}(X_0[p^n], Y_0[p^n]) \right)$$

for $m \gg 0$

<u>Proposition</u> (Oort + C.)

(1) $(G_n)_{n\in\mathbb{N}}$ has a natural structure as a p-divisible group over κ

(2) If X_0, Y_0 are both isoclinic and $\text{slope}(X_0) \leq \text{slope}(Y_0)$,
then $G = (G_n)_{n\in\mathbb{N}}$ is isoclinic, $\text{slope} = \text{slope}(Y_0) - \text{slope}(X_0)$
$\text{ht} = \text{ht}(Y_0)\cdot \text{ht}(X_0)$

Moreover if κ is perfect, then
$D_*(G) =$ max. $W(\kappa)$ submodule of $\mathrm{Hom}_{W(\kappa)}(D_*(X_0), D_*(Y_0))$
stable under the natural action of F and V

covariant Dieudonné module

Rmk. (a) $G = (G_n)_{n \in \mathbb{N}}$ represents the maximal p-divisible sub of the functor $R \longmapsto \mathrm{Ext}_R(X_0, Y_0)$

comm. Artinian augmented κ-algebras

(b) This "internal Hom" construction is used in recent work of Caraiani – Scholze.

(c) $(\underline{\mathrm{Hom}}(X_0[p^n], Y_0[p^n]))_{n \in \mathbb{N}}$ is a commutative smooth formal group corresponding to the whole functor $R \longmapsto \mathrm{Ext}_R(X_0, Y_0)$; its Cartier module, if κ is perfect, is $\mathrm{Ext}^1_{\mathrm{Cart}_p(\kappa)}(D_*(X_0), BC_p(\kappa) \otimes_{\mathrm{Cart}_p(\kappa)} D_*(Y_0))$,

where $BC_p(\kappa)$ = Cartier module of the ∞-dim$^{\ell}$ smooth formal group

$$R \longmapsto \mathrm{Ker}(\mathrm{Cart}_p(R) \longrightarrow \mathrm{Cart}_p(\kappa))$$

with 3 compatible structure of modules over $\mathrm{Cart}_p(\kappa)$