Moduli of abelian varieties: symmetry and rigidity

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Goal: survey Hecke symmetry on the moduli spaces of abelian varieties in positive characteristic $p > 0$

- history: elliptic curves $\mapsto$ curves of higher genera
  $\mapsto$ abelian varieties
  $\mapsto$ moduli spaces $\mapsto$ Hecke symmetry

- phenomena
- structures in characteristic $p > 0$, conjectures
- new tools / results
§1 From elliptic curves to abelian varieties and moduli

1.1. Several approaches to elliptic curves

(algebra) \[ E : \quad y^2 = 4x^3 - g_2 x - g_3, \quad \Delta = g_2^3 - 27g_3^2, \quad j = 1728 \frac{g_2^3}{\Delta} \]

(geometry) \[ E(\mathbb{C}) \sim \text{Lie}(E) / H_1(E(\mathbb{C}), \mathbb{Z}), \quad \mathfrak{P} \mapsto \int_0^\mathfrak{P} \frac{dx}{y} \]

(analysis) \[ \varphi(z; \tau) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda_z} \left[ \frac{1}{z - \gamma} - \frac{1}{\gamma^2} \right] \]

\[ \left( \frac{d}{dz} \varphi \right)^2 = 4 \varphi^3 - \varphi \varphi' - \varphi' \]

\[ g_2 = 60 \sum_{\gamma \in \Lambda_\tau} \frac{1}{\gamma^4}, \quad g_3 = 140 \sum_{\gamma \in \Lambda_\tau} \frac{1}{\gamma^6} \]
1.2 Origin of elliptic curves

A. (Diophantine) Fermat: \( E \) \( x^4 - y^4 = z^2 \) has no non-trivial rational solution

infinite descent:

\[
E' \quad s^4 + 4t^4 = u^2
\]

\[
\begin{align*}
(x_1, y_1, z_1) \quad & \quad x_1 + y_1 = 4t^2 \\
& \quad z_1 = 4ust
\end{align*}
\]

primitive solution to equation \( E \)

\[
\begin{align*}
(s, t, u) \quad & \quad x_1 = z^2 \\
& \quad y_1 = 2s^2
\end{align*}
\]

primitive sol'n to eq. \( E' \), \( u > 0 \)

\[
\begin{align*}
(x_2, y_2, z_2) \quad & \quad s^2 = x_2^4 - y_2^4 \\
& \quad t^2 = x_2^2 y_2^2 \\
& \quad u = x_2^4 + y_2^4 \\
& \quad s = z_2
\end{align*}
\]

prim. sol'n to \( E \)

Note: This is \( z \)-descent via \( z = (1 + \sqrt{-1}) \cdot (1 - \sqrt{-1}) \):

\[ z^2 = x^4 - 1 \] defines an elliptic curve over \( \mathbb{Q} \) with CM by \( \mathbb{Z}[i] \).
B. (elliptic integral)

Fagnano (December, 1751, paper by Fagnano reached Euler in Berlin)

Euler

Fagnano: \( \frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}} \) has rational solutions

i.e. \( \int_{0}^{x} \frac{d\rho}{\sqrt{1-x^4}} = \int_{0}^{y} \frac{d\psi}{\sqrt{1-y^4}} \) has soln

\( y = \) a rational function of \( x \)

Euler: \( \frac{m \, dx}{\sqrt{1-x^4}} = \frac{n \, dy}{\sqrt{1-y^4}} \)

\[ \int_{0}^{r} \frac{d\rho}{\sqrt{1-\rho^4}} = \sqrt{2} \int_{0}^{t} \frac{d\xi}{\sqrt{1+\xi^4}} \]

\[ t^2 = \frac{2u^2}{1-u^4} \]

\[ r^2 = \frac{2t^2}{1+t^4} \]

\[ \int_{0}^{r} \frac{d\rho}{\sqrt{1-\rho^4}} = (1 \pm \sqrt{-1}) \int_{0}^{t} \frac{d\eta}{\sqrt{1-\eta^4}} \]

\( r = \pm \frac{2\sqrt{1-v^2}}{1-v^4} \)
inversion of elliptic integrals
(for $y^2 = f(x)$, i.e. hyperelliptic curves)

Abel 1827, Jacobi 1828.

Jacobi 1829. Fundamenta Nova Theoriae Functionum Ellipticarum

defined Jacobi theta functions
1.3. Curves and their Jacobians

Riemann 1857, Theorie der Abelsche Functionen

$S = C(C)$ compact Riemann surface i.e. $C =$ smooth projective connected algebraic curve over $C$

$y_1, \ldots, y_2g$ : $\mathbb{Z}$-basis of $H_1(S, \mathbb{Z})$

$\omega_1, \ldots, \omega_g$ : $C$-basis of $\Gamma(S, \Omega^1_S)$

$P = P(\omega_1, \ldots, \omega_g; y_1, \ldots, y_2g) = (P_{ri})_{1 \leq r \leq g \atop 1 \leq i \leq 2g}$

$P \cdot \Delta^{-1} \cdot tP = 0$

$-\sqrt{-1}P \cdot \Delta^{-1} \cdot tP \gg 0_g$

Riemann bilinear relations

$\text{Pic}^1(C) \leftarrow \text{Pic}^0(C) = \text{Jac}(C)$

principal homog space

$\text{Jac}(C)(C) \cong \Gamma(C, \Omega^1_C) / H_1(C(C), \mathbb{Z})$
1.4. Abelian varieties

Def. A $g \times g$ matrix $Q \in M_{g \times g}(\mathbb{C})$ is a Riemann matrix if $\exists$ a skew symmetric $2g \times 2g$ matrix $E \in M_{2g}(\mathbb{Z})$ with $\det(E) \neq 0$ satisfying

$$\begin{cases} Q \cdot E^{-1} \cdot tQ = q_g \\ \sqrt{-1} Q \cdot E^{-1} \cdot tQ \gg q_g \end{cases}$$

($E$ is called the principle part)

Def. (Abelian varieties)

(i) (over $\mathbb{C}$): a compact complex torus $\mathbb{C}^g/Q \cdot \mathbb{Z}^g$ is an abelian variety if $Q$ is a Riemann matrix.

(ii) (over $\mathbb{C}$): a compact complex torus is an abelian variety if it admits a holomorphic embedding to $\mathbb{P}^N(\mathbb{C})$ for some $N$.

(ii) An irreducible algebraic group variety $A$ over a field $k$ is an abelian variety if $A$ is complete (i.e., proper over $k$).
Rmk (a) The classical proof of (i) $\iff$ (ii) uses Riemann's theta function.

(b) The algebraic theory of abelian varieties was due to André Weil (1948).

Def. (polarization of abelian varieties)

(i) A polarization of an abelian variety $A$ is an ample divisor on $A$ up to algebraic equivalence (equivalently, up to translation).

(ii) The polarization of $A$ given by an ample divisor $D$ on $A$ is principal if $D^g = g!$

Rmk. (a) The polarization induced by $D$ is uniquely determined by

\[ \Psi_D : A \to A^b = \text{Pic}^0(A) = \text{dual abelian variety} \]

\[ x \mapsto \Omega_A(D-x) \]

(b) The fundamental class $c([D]) \in H^2(A(C), \mathbb{Z}(1))$ corresponds over $C$ to a Riemann form.
Recap: Over $\mathbb{C}$

(1) Every principally polarized abelian variety of dimension $g$ is of the form $A_g = \mathbb{C}^g/\Omega \cdot \mathbb{Z}^g + \mathbb{Z}^g$ with principal part $\begin{bmatrix} \Omega_g & \Omega_g \\ -\Omega_g & \Omega_g \end{bmatrix}$ for some $\Omega \in \mathcal{H}_g = \{ \Omega \in \mathcal{M}_g(\mathbb{C}) : ^t\Omega = \Omega, \text{ Im}(\Omega) \gg \mathbb{Q} \} =$ Siegel's upper space

(ii) $(A_{\Omega_1}, \lambda_{\Omega_1}) \cong (A_{\Omega_2}, \lambda_{\Omega_2}) \iff \exists (A \ B) \in \text{Sp}_{2g}(\mathbb{Z})$ such that $\lambda_{\Omega_1}$ preserves polarization $\Rightarrow (A_{\Omega_1} + B) \cdot (C_{\Omega_2} + D)^{-1} = \Omega_2$
1.5 Moduli spaces (or, classifying spaces)

Idea / phenomenon:
The set of all isomorphism classes of all algebraic varieties with fixed discrete invariants often has a natural structure as an algebraic variety (or an algebraic structure close to an algebraic variety).

Ex. \( M_g \) = the moduli stack classifying all smooth proper curves of genus \( g \geq 2 \).

\( A_g \) = the moduli stack classifying \( g \)-dimensional principally polarized abelian varieties

(First existence proof by Mumford, 1965)
Over $\mathbb{C}$:

$$M_g(\mathbb{C}) = \Gamma_g \backslash \mathcal{J}_g,$$
where $\mathcal{J}_g = \text{Teichmüller space of genus } g$

$$\Gamma_g = \text{mapping class group for an oriented connected smooth closed surface of genus } g$$

$$A_g(\mathbb{C}) = Sp_{2g}(\mathbb{Z}) \backslash \mathcal{H}_g$$

Rmk. $T_g : M_g \xrightarrow{\text{Torelli map}} A_g$
$$[C] \xrightarrow{\text{Torelli map}} [\text{Jac}(C)]$$

Torelli theorem:

$$T_g : M_g(\mathbb{C}) \xrightarrow{\text{Torelli}} A_g(\mathbb{C})$$
$$\mathbb{C} : \text{Torelli 1914}$$
$$\text{general } \mathbb{F} : \text{Weil 1957}$$
Over an arbitrary algebraically closed field $k$

- $Mg/k$ is irreducible
  
  $\text{char}(k) = 0$: from uniformization + Lefschetz principle
  $\text{char}(k) = p > 0$: Deligne-Mumford, 1969

- $Ag/k$ is irreducible
  
  $\text{char}(k) = 0$: from uniformization
  $\text{char}(k) = p > 0$: Faltings-C. 1984
§2 Hecke symmetry on \( \mathcal{A} \quad \)

2.1 Definitions

Over \( \mathbb{C} \): \( \forall \gamma \in \text{Sp}_2g(\mathbb{Q}) \), the double coset \( \text{Sp}_2g(\mathbb{Z}) \cdot \gamma \cdot \text{Sp}_2g(\mathbb{Z}) \) induces an algebraic correspondence on \( \text{Sp}_2g(\mathbb{Z})/\mathcal{H}_g = \Gamma_g(1)/\mathcal{H}_g \)

\[
\begin{array}{ccc}
\Gamma_g(1)/\mathcal{H}_g & \xrightarrow{\text{pr}} & \mathcal{H}_g \\
\downarrow & & \downarrow \text{pr} \\
(\Gamma_g(1) \cdot \gamma \cap \Gamma_g(1)) & \xrightarrow{\text{pr}} & \Gamma_g(1)/\mathcal{H}_g \\
\gamma & \xrightarrow{\gamma} & \Gamma_g(1)/\mathcal{H}_g
\end{array}
\]

These are remnants of the transitive action of \( \text{Sp}_2g(\mathbb{R}) \) on \( \mathcal{H}_g \), after quotient by \( \Gamma_g(1) = \text{Sp}_2g(\mathbb{Z}) \).
algebraic version:

Def. \([A_1, \lambda_1] , \ [(A_2, \lambda_2)] \in A_g(\mathbb{F})\) \(\lambda_i : A_i \overset{\sim}{\rightarrow} A_i^t\) principal pencil are in the same Hecke orbit if \(\exists\) isogeny \(\alpha : A_1 \rightarrow A_2\)

\(\alpha^*(\lambda_2) = n \cdot \lambda_1\) for some \(n \in \mathbb{N}\)

\(\alpha^t \circ \lambda_2 \circ \alpha\)

Note: Suppose \(\text{char}(\mathbb{F}) = p > 0\), say \([A_1, \lambda_2]\) and \([A_2, \lambda_2]\) are in the same prime-to-\(p\) Hecke orbit if one can choose \(\alpha\) to be an isogeny s.t. \(\text{rank}(\text{Ker}(\alpha))\) is prime to \(p\);

\(\equiv\) \(\gcd(n, p) = 1\) in Def above.
Adelic picture: \( \hat{A}^{(p)}_f = \prod_{l: \text{prime}} \hat{Q}_l \) (restrict product)

\( k \equiv \mathbb{F}_p, \text{ alg. closed} \)

\[ \mathcal{G}_{P_{2g}(\hat{A}^{(p)}_f)} \mathcal{C}_f \mathcal{A}_{g}/k \xrightarrow{\sim} (\text{moduli stack of triples} (A, \lambda, A[n] \subseteq (\mathbb{Z}/n\mathbb{Z})^g) \text{ symplectic}) \]

prime-to-\( p \) Hecke orbits on \( \mathcal{A}_g/k \) \( \leftrightarrow \) orbits of \( \mathcal{G}_{P_{2g}(\hat{A}^{(p)}_f)} \) on \( \mathcal{A}^{(p)}_g \)
Remark. The Hecke correspondences also act on automorphic vector bundles (and automorphic local systems) on $A_g$, such as $e^*\Omega^1_{A/Ag}$, $\det(e^*\Omega^1_{A/Ag})$ (and $R^1\pi_*Q_\epsilon$) therefore induce linear action (via trace) of Hecke algebra(s) on automorphic forms on $A_g$. (Hecke - case $g=1$)
2.2 p-adic invariants \( k = \overline{k} = \overline{F_p} \)  
base field

(*) Every prime-to-\( p \) symplectic isogeny between principally polarized abelian varieties over \( k \) preserves p-adic invariants

Example of p-adic invariants

(a) slopes / Newton polygon of an abelian variety \( A/k \) 
\( \Phi_A^{(p)} : A \to A^{(p)} \) and its iterates \( \Phi_A^{(p^n)} : A \to A^{(p^n)} \)

slopes (with multiplicity) \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_g \) 
\( \lambda_i \in \mathbb{Q} \ \forall i, \ \lambda_i + \lambda_{g+1-i} = 1 \ \forall i \)  
\( \text{denom} (\lambda_i) | \text{multiplicity} (\lambda_i) \ \forall i \)
properties of slopes / Newton polygon

- They measure asymptotic divisibility properties (by powers of \( p \)) of the action of \( \text{Fr}(p^n) \) on \( H^m_{\text{cris}}(A) \) as \( n \to \infty \)

- \( A \) is ordinary \( \iff \) slopes are 0 and 1 \( \iff \) Hasse invariant of \( A \) does not vanish

  When \( A_{/\overline{\mathbb{F}_q}} \) is ordinary \( \iff \) half of the eigenvalues of \( \text{Fr}_{A_{/\overline{\mathbb{F}_q}}}: A \to A \) are \( p \)-adic units

- \( A \) is supersingular \( \iff \) all slopes = \( \frac{1}{2} \)
(b) Isomorphism class of \((A[p], \lambda[p]) \leftrightarrow (H_{1, \text{dR}}(A) + \text{symplectic pairing})\)

In both (a) and (b), the discrete invariants have natural partial ordering, compatible with specialization

\[ \rightarrow \text{stratification of } \bar{A}/\mathbb{G}_m \text{ by locally closed subschemes} \]

In each of the above two stratifications of \(\bar{A}/\mathbb{G}_m\), there is a dense open stratum: the open subscheme \(\bar{A}^{\text{ord}}\) of \(\bar{A}\) corresponding to ordinary principally polarized abelian varieties.
Thm (CLC, 1995) \( \forall x = [A, \lambda] \in \mathcal{A}_g/k \) with \( A \) ordinary, the prime-to-\( p \) Hecke orbit of \( x \) is Zariski dense in \( \mathcal{A}_g/k \)

\[ \Delta \] Trivial fact: every prime-to-\( p \) Hecke orbit of \( \mathcal{A}_g/\overline{\mathbb{Q}} \) is Zariski dense in \( \mathcal{A}_g/\overline{\mathbb{Q}} \). However the above theorem does not follow from this fact and the Serre-Tate canonical lifting of ordinary abelian varieties (to characteristic 0)

Q. How to generalize this result on Hecke orbits to non-ordinary abelian varieties?
2.3. p-divisible groups (or, Barsotti-Tate groups)

Defn. A p-divisible group \( X \to S \) is an inductive system of commutative finite locally free group schemes

\[
\left( (X_n \to S)_{n \in \mathbb{N}}, \ i_{n,n+1} : X_n \hookrightarrow X_{n+1}, \ \pi_{n,n+1} : X_{n+1} \xrightarrow{\text{faithfully flat}} X_n \right)
\]

such that \( i_{n,n+1} \circ \pi_{n,n+1} = [p]_{X_{n+1}} \forall n \)

Fact: \( \exists h : S \to \mathbb{N}, \text{ locally constant, such that } \text{height}_{\text{height}}(X_n) = p^{nh} \forall n \)

Primary example: \( A \to S \) abelian scheme \( \to (A[p^n])_{n \in \mathbb{N}} \) is a p-divisible group

* \( A[p^n] \) : substitute for Lie algebra in characteristic \( p > 0 \) (or mixed characteristics)
* \( A[p^{\infty}] \) "gives all \( p \)-adic invariants" of \( A \).
2.4 Leaves (or central leaves) in moduli spaces

Definition (Oort, 1999) \( \overline{k} = \overline{\mathbb{F}_p}, \ x \in \mathcal{A}_g(\overline{k}), \ x = [(A, \lambda)] \)

The leaf \( E(x) \) through \( x \) is the locally closed subvariety of \( \mathcal{A}_g \) such that

\[ E(x)(k) = \{ (B, \mu) \in \mathcal{A}_g(k) \mid (B[p^\infty], \mu[p^\infty]) \cong (A[p^\infty], \lambda[p^\infty]) \} \]

Fact: Every leaf in \( \mathcal{A}_g \) is smooth, and stable under all prime-to-\( p \) Hecke correspondences.

Conjecture (Oort) Let \( C \) be a leaf in \( \mathcal{A}_g \). For any \( x \in C(k) \), the prime-to-\( p \) Hecke orbit of \( x \) is Zariski dense in \( C \).

Note: \( \mathcal{A}_g^{ord} \) is a leaf; have seen that the conjecture holds for \( \mathcal{A}_g^{ord} \).
Remark (i) This Hecke orbit conjecture can be formulated for moduli spaces of PEL type (in characteristic p, classify moduli spaces with fixed type of polarization, endomorphism and level structure).

(ii) This conjecture holds for \( \mathbb{A}^g \) \((\text{Oort} + \text{CLC})\); proof uses a special property of \( \mathbb{A}^g \). Open for PEL type A and D.
§3 New tools, structures and conjectures related to Hecke symmetry

3.1. Irreducibility

\[ \overline{k} = \overline{\mathbb{F}_p} \]

**Proposition A** Let \( Z \subseteq \mathbb{A}_g \) be a positive dimensional locally closed subvariety which is stable under all prime-to-\( p \) Hecke correspondences. If Hecke operates transitively on \( \mathcal{P}_0(Z) \), then \( Z \) is irreducible.

**Proposition B** Let \( C \subseteq \mathbb{A}_g \) be a positive dimensional leaf on \( \mathbb{A}_g \), then the naive \( p \)-adic monodromy for \( C \) is maximal

\[ \Rightarrow \quad \text{Aut} \left( A_x[p^{\infty}], \lambda_x[p^{\infty}] \right) \quad \forall x \in C \]

Note: Prop. B is useful in Iwasawa theory: “irreducibility of Igusa towers”
**Prop C** \((\text{Oort} + C)\) Every non-supersingular Newton stratum in 
\(\text{Ag}\) is irreducible

Note: (i) supersingular \(\Rightarrow\) all slopes \(= \frac{1}{2}\)

(ii) The supersingular stratum has dimension \(\left[\frac{g^2}{4}\right]\)

**Prop D** \((\text{Oort} + C)\) Every non-supersingular leaf in \(\text{Ag}\) is irreducible

Note: a leaf \(C\) in \(\text{Ag}\) is supersingular \(\iff\) \(\dim(C) = 0\)
3.2 Local structure of leaves

2-slope case: $C \ni x_0 = [(A_0, \lambda_0)] \in A_g(k), \quad k = \overline{k} = \kappa$ slopes of $A_0$

$\lambda \leq \frac{1}{2}$

Prop. $C^\times_{x_0}$ (formal completion of $C$ at $x_0$)

has a natural structure as a (neutral torsor for $a$)

isoclinc $p$-divisible group with slope $1 - 2\lambda$ and height $8(g+1)/2$
3.3 Local stabilizer principle \( k = \overline{k} \supseteq \mathbf{F}_p \)

Prop. Let \( Z \subseteq \mathcal{A}_g \) be a locally closed subvariety, stable under all prime-to-\( p \) Hecke correspondences, \( x_0 = [(\lambda_0, \lambda_0)] \in Z(k) \).

Then \( Z//x_0 \subseteq \mathcal{A}_g//x_0 \) is stable under the natural action of an open subgroup of \( U(\text{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p) \) on \( \mathcal{A}_g//x_0 \).

\[
\text{Aut}(A_0[\mathbf{p}^\infty], \lambda_0[\mathbf{p}^\infty]) \supseteq \mathcal{A}_g//x_0 \cong \text{Def}(A_0, \lambda_0) \xrightarrow{\text{Def}} \text{Def}(A_0[\mathbf{p}^\infty], \lambda_0[\mathbf{p}^\infty])
\]

Serre-Tate Theorem

(Prime-to-\( p \)) Hecke correspondences with \( x_0 \) as a fixed point
3.4. Rigidity

**Theorem (Local rigidity)** $X$: $p$-divisible group over $\mathbb{F}_p = \bar{\mathbb{F}}_p$, connected $p$-adic Lie group $\mathbb{G}$ an irreducible formal subvariety. Suppose $\mathcal{E}$ subgroup $G \subseteq \text{Aut}(X_\mathbb{G})$ such that $X^G$ trivial and $\mathbb{E}$ is stable under $G$.

Then $\mathbb{E}$ is a $p$-divisible formal subgroup of $X$.

"Exer." Case $X = G_m^h$, $G = (1 + p^2 \mathbb{Z}_p \cdot \text{Id}_X)$.
Example/Cor. \( E_0 \): an ordinary elliptic curve \( k = \overline{k} \cong \mathbb{F}_p \)

\[ x_0 := [(A_0, \lambda_0)] \]

Then the prime-to-\( p \) Hecke orbit of \( x_0 \) is dense in \( \mathcal{A}_g \)

Proof: \( \mathcal{A}_g \xrightarrow{\sim} \hat{G}_m \xrightarrow{\text{Serre-Tate}} \mathcal{G}_m \)

\[ \mathcal{U}(\text{End}(A_0), \star \lambda_0)(\mathbb{Z}_p) \cong \text{GL}_g(\mathbb{Z}_p) \]

\[ \text{Aut}(A_0[p\infty], \lambda_0[p^\infty]) \]

Action of \( \text{GL}_g(\mathbb{Z}_p) \) on \( X^*(\hat{G}_m^{g(g+1)/2}) \)

\[ \cong S^2 \left( \text{standard representation of } \text{GL}_g(\mathbb{Z}_p) \text{ on } \mathbb{Z}_p^g \right) \]

irreducible! QED
Conjecture: Suppose \( Z \subset A_g^{\text{ord}}, \ x_0 = [(A_0, \lambda_0)] \in A_g^{\text{ord}}(k), \ k = \bar{k} \cong \mathbb{F}_p \)

and \( Z/\!\!/x_0 \subset A_g/\!\!/x_0 \) is a formal subtorus of \( A_g/\!\!/x_0 \) Serre-Tate formal torus.

Then \( Z = (\text{reduction of}) \) a Shimura subvariety of \( A_g \).

Remark: Known if \( Z \) is a Hilbert modular (sub)variety. This case has application in Iwasawa theory.

(geometric input in Hida, non-vanishing of the \( \mu \)-invariant, Ann. of Math. 2012)

"density of CM points"
Conjecture (Local rigidity, special case)

\[ G_0 = \text{1-dimensional smooth formal group over } \overline{\mathbb{F}}_p, \ \text{ht} (G_0) = h \]
\[ M = \text{equi-char. deformation space of } G_0 \quad \text{Lubin-Tate, 1966} \]
\[ \cong \text{Spf } (\overline{\mathbb{F}}_p [x_1, \ldots, x_{h-1}]) \]
\[ Z \subseteq M \text{ irreducible formal subscheme} \]

Assume (i) \( Z \) is stable under the natural action of an open subgroup of \( \text{Aut}(G_0) \) \leftarrow \text{units in a central division algebra over } \mathbb{Q}_p \)
with Brauer invariant \( \frac{1}{h} \)

(ii) \( Z \) is generically ordinary (i.e. \( Z \) \neq \text{zero locus of Hasse invariant})

Then \( Z = M \)

Rmk: A proof will have application in chromatic homotopy theory
3.6 Sustained p-divisible groups

Motivation: Find a good (scheme-theoretic) definition of leaves

One benefit: can study local structure of leaves using deformation theory

Definition (Oort+C.) \( \kappa = \text{a field of char. } p > 0 \) (not nec. alg. closed)

\( X_0/\kappa: p\text{-divisible group. } S/\kappa: \text{scheme over } \kappa \)

A p-divisible group \( X \to S \) is strongly \( \kappa \)-sustained modeled on \( X_0 \) if

\[
\text{Isom}_{S}(X[p^n], X_0[p^n] \times_{\text{Spec } \kappa} S) \to S \text{ is faithfully flat } \forall n \in \mathbb{N}
\]
3.7 "Internal Hom" of $p$-divisible groups

(Actually these are "internal Ext")

$\kappa$: field of char $p > 0$, $X_0, Y_0$: $p$-divisible groups over $\kappa$

$G_n := \text{Image } (\text{Hom}(X_0[p^n], Y_0[p^n]) \rightarrow \text{Hom}(X_0[p^n], Y_0[p^n]))$

for $m \gg 0$

Proposition (Oort + C.)

1. $(G_n)_{n \in \mathbb{N}}$ has a natural structure as a $p$-divisible group over $\kappa$

2. If $X_0, Y_0$ are both isoclinic and $\text{slope}(X_0) \leq \text{slope}(Y_0)$, then $G = (G_n)_{n \in \mathbb{N}}$ is isoclinic, $\text{slope} = \text{slope}(Y) - \text{slope}(X_0)$

Moreover if $\kappa$ is perfect, then

$D_{\kappa}(G) = \max. W(\kappa)$ submodule of $\text{Hom}_{W(\kappa)} (D_{\kappa}(X_0), D_{\kappa}(Y_0))$

stable under the natural action of $F$ and $V$

\text{covariant}

\text{Dieudonné module}
Rmk. (a) $G = (G_n)_{n \in \mathbb{N}}$ represents the maximal $p$-divisible sub of the functor $R \mapsto \text{Ext}_R(X_0, Y_0)$.

(b) This "internal Hom" construction is used in recent work of Caraiani - Scholze.

(c) $\left( \text{Hom}(X_0[p^n], Y_0[p^n]) \right)_{n \in \mathbb{N}}$ is a commutative smooth formal group corresponding to the whole functor $R \mapsto \text{Ext}_R(X_0, Y_0)$; its Cartier module is

$$\text{if } k \text{ is perfect } \text{Ext}^1_{\text{Cont}_p(k)}(D_*(X_0), BC_p(k) \otimes_{\text{Cont}_p(k)} D_*(Y_0)),$$

where $BC_p(k)$ is Cartier module of the $\infty$-dim smooth formal group

$$R \mapsto \ker \left( \text{Cont}_p(R) \rightarrow \text{Cont}_p(k) \right)$$

with 3 compatible structure of modules over $\text{Cont}_p(k)$.