Sustained p-divisible groups and a foliation of moduli spaces of abelian varieties

with Frans Oort

§ 1 Introduction

1.1. central leaves in $\mathbb{A}^d, d, n/\mathbb{F}_p$ (Oort, Texel 1999)

(a) gfc polarized p-divisible group
(b) $C(x_0) \subseteq \mathbb{A}^d, d, n/\mathbb{F}_p$

$x_0 \in \mathbb{A}^d, d, n(\mathbb{F}_p)$

properties of $C(x_0)$: (i) smooth, locally closed

(ii) stable under all prime-to-\(p\) Hecke correspondences

(iii) characteristic-\(p\) analogue of Shimura varieties

example

1) $\mathbb{A}^d, d, n$ is a central leaf

2) $\mathbb{A}^d, d, n(\mathbb{F}_p) \ni x_0 = [(A_0, \mu_n)] \ni A_0[p^n] \cong Y \times Y^t$

$C(x_0) \text{ dim } = 2$

$C(x_0)_{x_0}$ has a natural structure as a p-divisible

formal group, isoclinic of slope $1/3$

$ht = 6 \text{ (dim } A_3, 4), \text{ dim } = 2$

Question:

functional/scheme-theoretic definition of $C(x_0)$

1.2. New insight:

(a) Let $(x, \mu)$ = restriction to $C(x_0)$ of the polarized p-divisible group

(b) $x$ admits a slope filtration attached to the universal

$C(x_0)$
(b) \( \forall n \in \mathbb{N}, \exists T_n \to C(x_0) \) faithfully flat + of finite presentation
and a \( T_n \)-isomorphism
\[
(A_n, \mu_n) [p^n] \times T_n \stackrel{\sim}{\rightarrow} (A, \mu) [p^n] \times T_n / C(x_0)
\]

\[8.2. \text{Definition and first properties of (strongly) sustained (polarized) }\]
\( p \)-divisible groups

**Def. 2.1.** Strongly \( \kappa \)-sustained \( p \)-divisible group \((X, \mu) \to S\) modeled on \((X_0, \mu_0) / \kappa\)
\( S = \kappa \)-scheme
\( \kappa = \mathbb{F}_p \) : a field

**Prop. 2.2.** \( X \to S \) strongly \( \kappa \)-sustained modeled on \( X_0 / \kappa \) \( \Rightarrow X \to S \) admits a canonical slope filtration \( \text{F}^S(X) \) is strongly \( \kappa \)-sustained over \( S \) \( \forall S \)

**Thm. 2.3.** (Backward compatibility)
\( S / \kappa \text{ reduced } \kappa = \mathbb{F}_p \neq \emptyset \)
\((X, \mu) \to S\) polarized \( p \)-divisible group
\((X, \mu)\) is strongly \( \kappa \)-sustained \( \iff \) it is \( gfc \)

\[8.3. \text{Stabilized } \text{Hom}-, \text{End}-, \text{Aut}-\text{schemes} \]
\( Y, Z : p \)-divisible groups over \( \kappa = \mathbb{F}_p \) \( \kappa = \text{a field} \)

**Def. 3.1(a).** \( \text{Hom}^{st}(Y, Z) : = \text{Im} \left( \text{Hom} \left( Y[p^{n+1}], Z[p^n] \right) \to \text{Hom} \left( Y[p^n], Z[p^n] \right) \right) \)

\( N >> 0 \)

**Def. 3.1(b).** \( \text{Hom}' (Y, Z) = \text{inductive system} \left( \text{Hom} \left( Y[p^n], Z[p^n] \right)_{n \in \mathbb{N}} \right) \)

\( Y[p^n] \xrightarrow{h} Z[p^n] \)

\( \sigma_p \uparrow \quad \downarrow \)

\( Y[p^{n+1}] \xrightarrow{\nu_{n+1, h}(h)} Z[p^{n+1}] \)

**Def. 3.1(c).** \( \text{Hom}' (Y, Z)_{p\text{-div}} = \text{inductive system of} \left( \text{Hom}^{st} \left( Y[p^n], Z[p^n] \right)_{h \in \mathbb{N}} \right) \)
Prop 3.2 (a) \( \text{Hom}'(Y, Z) \) is a smooth formal group over \( \kappa \) (not necessarily connected)

(b) \( \text{Hom}'(Y, Z)_{p\text{-div}} \) is a \( p \)-divisible formal group over \( \kappa \)

= the maximal \( p \)-divisible subgroup of \( \text{Hom}'(Y, Z) \)

Def 3.3 \( \text{End}^\ast(Y) = \) the projective system \( (\text{End}^\ast(Y)_n)_{n \in \mathbb{N}} \)

\( \text{Aut}^\ast(Y) = \) the projective system \( (\text{Aut}^\ast(Y)_n)_{n \in \mathbb{N}} \)

3.4. Key property:
\( \text{Aut}^\ast(Y) \) has a natural (finite decreasing) slope filtration

s.t.
\[ \text{Fil}^s \text{Aut}^\ast(Y) / \text{Fil}^{s+1} \text{Aut}^\ast(Y) = \text{the projective system given by the } p \text{-divisible group} \]
\[ \text{Fil}^s \text{End}^\ast(Y) / \text{Fil}^{s+1} \text{End}^\ast(Y) \]

3.5. Relation with strongly sustained \( p \)-divisible groups

\[ \left\{ \begin{array}{l}
\text{strongly } n \text{-sustained } p \text{-div group } X \to S \text{ modeled on } Y/\kappa \\
\text{compatible projective families of right } \text{Aut}^\ast(Y)_n \text{-torsors over } S
\end{array} \right\} \]

\[ X/S \nrightarrow \left( T_n = \text{Isom}^\ast(Y[p^n]_S, X[p^n]) \right)_{n \in \mathbb{N}} \]

\[ \left( T_n \times \text{Aut}^\ast(Y)_{nS} \right) \nrightarrow \left( T_n \right)_{n \in \mathbb{N}} \]
§4 Deformation of strongly sustained $p$-divisible groups

**Theorem 4.1** $\text{Def}(Y)_{\text{sus}} : (R, m) \to X \to \text{Spec}(R)$, strongly $\kappa$-sustained

$$R/m \cong \kappa + X \times \text{Spec}(R/m) \leftarrow Y \times \text{Spec} R$$

The sustained deformation functor $\text{Def}(Y)$ is smooth.

**Sketch of proof:**

$$(R', m'') / \kappa : \text{small ext}^n \text{ of } (R, m) \quad S' = \text{Spec}(R'), \quad S = \text{Spec}(R),$$

i.e. $R = R'/J, \quad J \cdot m'' = 0 \quad R/m = (R'/m'') = \kappa$

$$X \to \text{Spec}(R) = S \quad \text{strongly } \kappa \text{-sustained} \quad \text{right compatible family of } \text{Aut}^\text{s+}(Y)_n \text{-torsors over } S$$

$$\longleftrightarrow (T_n)_{n \in \mathbb{N}}$$

**Illusie 1)** Have $\ell^\text{perfect complex, amplitude } \in [-1, 0]$ functorial

2) **Obstruction of lifting $T_n$ to $S'$**

$$\in H^2(S_0, \ell^\text{LL}_{T_n \times S_0} / R_0, J) = 0$$

3) **All liftings of $T_n$ to $S'$**

$a$ a torsor for $H^4(S, \ell^\text{LL}_{T_n \times S_0} / R_0, J) =: \nu_{T_n \times S_0} \otimes R_0 J$

4) $\nu_{T_n \times S_0 / R_0} \otimes J \cong \nu_{T_n \times S_0 / R_0} \otimes J$

By dévissage, using the slope filtration on $(T_n)_{n \in \mathbb{N}}$ and similar maps for a $p$-divisible group are isomorphisms. "Q.E.D."
Thm 4.3. \( \text{Def}_\text{sus} (Y, \nu) \) is smooth over \( \kappa \)

Def 4.4. (Tate-linear formal subvarieties of \( \text{Def}(Y)_{\text{sus}} \))

A smooth closed formal subsheaf \( Z \subseteq \text{Def}(Y)_{\text{sus}} \)

is strongly Tate-linear if there exists a projective family \( (H_n)_{n \in \mathbb{N}} \) of subgroup schemes of \( (\mathbb{P}_n) = (\text{Aut}^\text{st}(Y))_n \) such that

\[
Z = \text{Im} \left( \text{Def} \left( \text{the trivial right } \text{Aut}^\text{st}(Y)\text{-torsor} \right) \to \text{Def} \left( \text{the trivial right } \text{Aut}^\text{st}(Y)\text{-torsor} \right) \right)
\]

\[
\text{Def}(Y)_{\text{sus}}
\]

4.5. Remark: \( M = \mathbb{A}^g \), PEL modular subvariety

\( C_M(z) = C_{\mathbb{A}^g}(x_0) \) and \( C_M(z)/\mathbb{Z}_p \) is a Tate-linear formal subsheaf

4.6. Examples

(a) \((A, \mu)\) \(g\)-dimensional polarized abelian variety

\( \text{slope}_0 = s, 0 < s < \frac{1}{2} \)

\( \text{Al}_p \) is isoclinic of slope \( s \)

\( \text{Def}(\text{Al}_p)_{\text{sus}} \) is a \( p \)-divisible formal group, isoclinic of slope \( 1 - 2s \)

\( \text{height} = g^2 \)

(\( \text{Def}(\text{Al}_p, \mu)_{\text{sus}} \) is a \( p \)-divisible formal group, isoclinic, slope \( 1 - 2s \), height \( g(g+1)/2 \)

(b) \((A, \mu)\), \( A[p^\infty] = Y_1 \times Y_2 \times Y_3 \)

\( Y_i \) is isoclinic of slope \( S_i \), height \( h_i \)

\( S_1 < S_2 = \frac{1}{2} < S_3 = 1 - S_4 \)

\( h_1 = h_3 \)

\( \text{Def}(\text{Al}_p)_{\text{sus}} \) is a biextension of \( p \)-divisible formal groups

\( (X_1, X_2) \) by \( X_3 \), \( X_1, X_2 \) isoclinic, slope \( \frac{1}{2} - s_1 \)

\( \text{height} = h_1, h_2 \)

\( X_3 \) is isoclinic, slope \( 1 - 2s_1 \)

\( \text{height} = h_3 \)
§5. Rigidity and linearity

Thm 5.1 (local rigidity)

(a) for $p$-divisible formal groups

(b) for bi-extensions of $p$-divisible formal groups

5.2 Expectation (local rigidity for $\text{Def}(Y)_{\text{sus}}$)

5.3 Conjecture global rigidity.