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Harvard-MIT
Alg. Geom. Seminar

Sustained p-divisibe groups and a foliation of moduli spaces of abelian varieties

with Frans Oort

§1 Introduction:

1.1. central leaves in $\mathcal{A}_{g,d,n/\overline{\mathbb{F}_p}}$ (Oort, Texel 1999)

(a) gfc polarized p-divisible group

(b) $C(x_0) \subseteq \mathcal{A}_{g,d,n/\overline{\mathbb{F}_p}}$ $\qquad x_0 \in \mathcal{A}_{g,d,n}(\overline{\mathbb{F}_p})$

properties of $C(x_0)$: (i) smooth, locally closed

(ii) stable under all prime-to-p Hecke correspondences

(iii) characteristic-p analogue of Shimura varieties

example

1) $\mathcal{A}_{g,1,n}^{ord}$ is a central leaf

2) $\mathcal{A}_{3,1}(\overline{\mathbb{F}_p}) \ni x_0 = [(A_0, \mu_0)]$ s.t. $A_0[p^\infty] \cong Y \times Y^t$

$C(x_0)$: $\dim = 2$ $\qquad \dim(Y)=1$, slope$(Y)=\frac{1}{3}$, $ht(Y)=3$

$C(x_0)^{/x_0}$: has a natural structure as a p-divisible formal group, isoclinic of slope $1/3$, $ht = 6$ ($= \dim \mathcal{A}_{3,1}$), $\dim = 2$

Question:

functorial / scheme-theoretic definition of $C(x_0)$

1.2. new insight:

Let $(\mathcal{X},\mu)$ = Restriction to $C(x_0)$ of the polarized p-divisible group attached to the universal abelian scheme

(a) $\mathcal{X}_{/C(x_0)}$ admits a slope filtration

(b) $\forall n \in \mathbb{N}$, $\exists\ T_n \to C(x_0)$ faithfully flat + of finite presentation and a T_n-isom

$$(A_0, \mu_0)[p^n] \times T_n \xrightarrow{\sim} (\mathcal{X}, \mu)[p^n] \times_{C(x_0)} T_n$$

§2. Definition and first properties of (strongly) sustaind (polarized) p-divisible groups

<u>Def^n 2.1</u> strongly κ-sustained polarized p-divisible group $(X, \mu) \to S$ modeled on $(X_0, \mu_0)/\kappa$

S = κ-scheme

$\kappa \supseteq \mathbb{F}_p$: a field

<u>Prop. 2.2</u> $X \to S$ strongly κ-sustained modeled on X_0/κ $\Rightarrow$ $X \to S$ admits a canonical slope filtration $\mathrm{Fil}^{\cdot} X$

$gr^s(X)$ is strongly κ-sustained over S $\forall s$

<u>Thm 2.3</u> (backward compatibility)

S/κ reduced $\kappa = \bar{\kappa} \supseteq \mathbb{F}_p$, $S(\kappa) \neq \phi$

$(X, \mu) \to S$ polarized p-divisible group

(X, μ) is strongly κ-sustained iff it is gfc

§3. Stabilized Hom-, End-, Aut-schemes

Y, Z : p-divisible groups over $\kappa \supseteq \mathbb{F}_p$ κ = a field

<u>Def^n 3.1</u> (a) $\underline{\mathrm{Hom}}^{st}(Y, Z)_n := \mathrm{Im}\left(\underline{\mathrm{Hom}}(Y[p^{n+N}], Z[p^{n+N}]) \to \underline{\mathrm{Hom}}(Y[p^n], Z[p^n])\right)$ $N >> 0$

(b) $\underline{\mathrm{Hom}}'(Y, Z)$ = inductive system $\left(\underline{\mathrm{Hom}}(Y[p^n], Z[p^n])\right)_{n \in \mathbb{N}}$

$$\begin{tikzcd} Y[p^n] \arrow[r, "h"] & Z[p^n] \arrow[d, hook] \\ Y[p^{n+1}] \arrow[u, "\times p"] \arrow[r, "i_{n+1,n}(h)"'] & Z[p^{n+1}] \end{tikzcd}$$

(c) $\underline{\mathrm{Hom}}'(Y, Z)_{p\text{-div}}$ = inductive system of $\left(\underline{\mathrm{Hom}}^{st}(Y[p^n], Z[p^n])\right)_{n \in \mathbb{N}}$

Prop 3.2 (a) $\underline{\mathrm{Hom}}'(Y,Z)$ = a smooth formal group over κ (not necessarily connected)

(b) $\underline{\mathrm{Hom}}'(Y,Z)_{p\text{-div}}$ = a p-divisible formal group over κ

= the maximal p-divisible subgroup of $\underline{\mathrm{Hom}}'(Y,Z)$

Def^n 3.3 $\underline{\mathrm{End}}^{st}(Y)$ = the projective system $\left(\underline{\mathrm{End}}^{st}(Y)_n\right)_{n\in\mathbb{N}}$

$\underline{\mathrm{Aut}}^{st}(Y)$ = the projective system $\left(\underline{\mathrm{Aut}}^{st}(Y)_n\right)_{n\in\mathbb{N}}$

3.4. Key property:

$\underline{\mathrm{Aut}}^{st}(Y)$ has a natural (finite decreasing) slope filtration s.t.

$\mathrm{Fil}^{\geq s}\,\underline{\mathrm{Aut}}^{st}(Y)/\mathrm{Fil}^{>s}\,\underline{\mathrm{Aut}}^{st}(Y)$

= the projective system given by the p-divisible group

$\mathrm{Fil}^{\geq s}\,\underline{\mathrm{End}}^{st}(Y)/\mathrm{Fil}^{>s}\,\underline{\mathrm{End}}^{st}(Y)$

3.5. Relation with strongly sustained p-divisible groups

$$\left\{\begin{array}{c}\text{strongly } \kappa\text{-sustained p-div}\\ \text{group } X\to S \text{ modeled on } Y/\kappa\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{compatible projective families}\\ \text{of right } \underline{\mathrm{Aut}}^{st}(Y)_n\text{-torsors over } S\end{array}\right\}$$

$$X/S \rightsquigarrow \left(T_n = \underline{\mathrm{Isom}}^{st}(Y[p^n]_S, X[p^n])\right)_{n\in\mathbb{N}}$$

$$\left(T_n \times^{\underline{\mathrm{Aut}}^{st}(Y)_{n/S}} Y[p^n]_S\right)_{n\in\mathbb{N}} \leftarrow\!\!\!\sim\!\!\sim (T_n)_{n\in\mathbb{N}}$$

§4 Deformation of strongly sustained p-divisible groups

<u>Theorem</u> 4.1 $\mathrm{Def}(Y)_{sus}$: $(R,\mathfrak{m}) \rightsquigarrow X \to \mathrm{Spec}(R)$, strongly κ-sustained

$R/\mathfrak{m} \cong \kappa$ $\quad + \; X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{m}) \xleftarrow{\sim} Y \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\, R$

The sustained deformation functor $\mathrm{Def}(Y)$ is <u>smooth</u>

Sketch of proof:

$(R', \mathfrak{m}')_{/\kappa}$: small ext^n of $(R, \mathfrak{m})$. $\quad S' = \mathrm{Spec}(R')$, $S := \mathrm{Spec}(R)$,

i.e. $R = R'/J$, $\quad J \cdot \mathfrak{m}' = (0)$ $\quad R/\mathfrak{m} = (R'/\mathfrak{m}') = \kappa$ $\quad S_0 = \mathrm{Spec}(\kappa)$

$X \to \mathrm{Spec}(R) = S$ strongly κ-sustained

$\longleftrightarrow$ $(T_n)_{n \in \mathbb{N}}$ compatible family of right $\underbrace{\underline{\mathrm{Aut}}^{st}(Y)_n}_{\textstyle \Gamma_n}$-torsors over S

Illusie 1) Have $\ell_{T_n/S}$: perfect complex, amplitude $\subseteq [-1, 0]$ functorial

2) Obstruction of lifting T_n to S'

$\in H^2(\underset{\substack{\| \\ \mathrm{Spec}(R/\mathfrak{m})}}{S_0}\; \ell^{\vee}_{T_n \times_S S_0} \overset{\mathbb{L}}{\otimes}_{R_0} J) = (0)$

3) All liftings of T_n to S'

$=$ a torsor for $H^1(S, \ell^{\vee}_{T_n \times_S S_0} \overset{\mathbb{L}}{\otimes}_{R_0} J) =: \nu_{T_n \times_S S_0/S_0} \otimes_{R_0} J$

4) $\nu_{T_{n+1} \times_S S_0/S_0} \otimes_{R_0} J \xrightarrow{\sim} \nu_{T_n \times_S S_0/S} \otimes_{R_0} J$

by dévissage, using the slope filtration on $(\Gamma_n)_{n \in \mathbb{N}}$

<u>and</u>: Similar maps for a p-divisible group are isomorphisms

"QED"

(Y, ν) polarized p-divisible group over κ

Thm 4.3. $\mathrm{Def}(Y,\nu)_{sus}$ is smooth over κ

Def^n 4.4. (Tate-linear formal subvarieties of $\mathrm{Def}(Y)_{sus}$)

A smooth closed formal subscheme $Z \subseteq \mathrm{Def}(Y)_{sus}$ is strongly Tate-linear if $\exists$

a projective family $(H_n)_{n \in \mathbb{N}}$ of subgroup schemes of $(T_n) = (\underline{\mathrm{Aut}}^{st}(Y)_n)$ s.t. $H_{n+1} \twoheadrightarrow H_n \ \forall n$

such that

$$Z = \mathrm{Im}\left(\mathrm{Def}\left(\begin{matrix}\text{the trivial}\\ (H_n)\text{-torsor}\end{matrix} \right) \to \underset{\displaystyle \mathrm{Def}(Y)_{sus}}{\overset{\|}{\mathrm{Def}\left(\begin{matrix}\text{the trivial}\\ \text{right } \underline{\mathrm{Aut}}^{st}(Y)\text{-torsor}\end{matrix} \right)}} \right)$$

4.5 Remark: $\mathcal{M} \subset \mathcal{A}_g$ PEL modular subvariety

$$\begin{matrix} \mathcal{M} & \subset & \mathcal{A}_g \\ \cup & & \cup \\ C_{\mathcal{M}}(z_0) & \subset & C_{\mathcal{A}_g}(x_0) \end{matrix} \rightsquigarrow C_{\mathcal{M}}(z_0)^{/z_0} \text{ is a Tate-linear formal subscheme}$$

4.6. Examples

(a) (A, μ) g-dim^l polarized abelian variety $/ k = \bar{k} \supseteq \mathbb{F}_p$

slopes $= s, 1-s$ $\quad 0 \leq s < \frac{1}{2}$ $\quad A[p^\infty] = Y \times Z$ $\quad$ Y: isoclinic of slopes s, Z: isoclinic of slope $1-s$

$\rightsquigarrow$ $\mathrm{Def}(A[p^\infty])_{sus}$ = a p-divisible formal group, isoclinic of slope $1-2s$
height $= g^2$

$\mathrm{Def}(A[p^\infty], \mu[p^\infty])_{sus}$ = a p-divisible formal group, isoclinic,
slope $= 1-2s$, height $= g(g+1)/2$

(b) (A, μ), $A[p^\infty] = Y_1 \times Y_2 \times Y_3$ $\quad$ Y_i = isoclinic of slope s_i, height h_i
$s_1 < s_2 = \frac{1}{2} < s_3 = 1 - s_1$
$h_1 = h_3$

$\mathrm{Def}(A[p^\infty])_{sus}$ = a biextension of p-divisible formal groups
(X_1, X_2) by X_3, $\quad$ X_1, X_2 = isoclinic, slope $= \frac{1}{2} - s_1$
height $= h_1 \cdot h_2$
X_3 = isoclinic, slope $= 1 - 2s_1$
height $= h_1^2$

§5. Rigidity and linearity

<u>Thm</u> 5.1 (local rigidity)

(a) for p-divisible formal groups

(b) for bi-extensions of p-divisible formal groups

5.2 <u>Expectation</u> (local rigidity for $\mathrm{Def}(Y)_{sus}$)

5.3 <u>Conjecture</u> global rigidity.