Moduli of Abelian Varieties

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Goal: survey the geometry of the moduli space of (principally polarized) abelian varieties, with emphasis in the case of positive characteristic $p > 0$, and the related rigidity phenomena.

- **History:** elliptic curves $\rightarrow$ curves of higher genera $\rightarrow$ moduli space of curves $\mathcal{M}_g$.
- Abelian varieties $\rightarrow$ moduli space of abelian varieties $\mathcal{A}_g$.
- Hecke symmetry.
- Definitions, $p$-divisible groups.
- Phenomena and structures in characteristic $p > 0$.
  - Predictions (= conjectures).
- New tools/methods applicable to other problems.
§1 From elliptic curves to abelian varieties and their moduli

1.1 What is an elliptic curve? several approaches

(a) algebra \[ E = \left\{ y^2 = 4x^3 - g_2 x - g_3 \right\}, \quad \Delta = g_2^3 - 27g_3^2 \neq 0, \quad j = \frac{g_2^3}{\Delta} \]

(b) geometry \( E(\mathbb{C}) \sim \text{Lie}(E)/H_1(E(\mathbb{C}), \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} + \mathbb{Z} \quad \tau \in \mathfrak{g} = \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \} \)

(c) analysis \[ \vartheta(z; \tau) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda_{\tau} \setminus \{0\}} \left[ \frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2} \right] \]

\[ \left( \frac{d}{dz} \vartheta(z; \tau) \right)^2 = 4 \vartheta(z; \tau)^3 - g_2(\Lambda_{\tau}) \vartheta(z; \tau) - g_3(\Lambda_{\tau}) \]

where \[ g_2(\Lambda_{\tau}) = 60 \sum_{\gamma \in \Lambda_{\tau} \setminus \{0\}} \frac{1}{\gamma^4}, \quad g_3(\Lambda_{\tau}) = 140 \sum_{\gamma \in \Lambda_{\tau} \setminus \{0\}} \frac{1}{\gamma^6} \]
1.2. The origin of elliptic curves: Diophantine equations, elliptic integrals

- Fermat \( x^4 - y^4 = z^2 \) has no non-trivial rational solution (infinite descent)

- Gauss \( E_{a,b} := \{1=x^2+y^2+x^2y^2\} \quad (a+b) \cdot \mathbb{Z}[i] \): prime ideal at \( a+b \equiv 1 \mod (a+i)^2 \) (1814)
  \# \( E(Z[i]/(a+b) \cdot Z[i]) = (a-1)^2 + b^2 \)

- December 1751, paper by Fagnano reached Euler in Berlin

\[
\frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}} \quad \text{has rational solutions, i.e.} \quad \int_{0}^{x} \frac{dp}{\sqrt{1-p^4}} = \int_{0}^{y} \frac{dp}{\sqrt{1-p^4}}
\]

admits solutions where

\[ y = \text{a rational function of } x \]

Note: The elliptic curves involved in the above are all twists of

"the same" curve with CM by \( \mathbb{Z}[i] \), \( j = 1728 \)
1.3. Periods of compact Riemann surfaces/smooth projective algebraic curves

\[ S = C(C) \text{ compact Riemann surface}; \quad \gamma_1, \ldots, \gamma_g : \mathbb{Z}-\text{basis of } H_1(S; \mathbb{Z}) \]
\[ \omega_1, \ldots, \omega_g : C-\text{basis of } \Gamma(S, \Omega^1_S) \]
\[ \Delta = (\gamma_i \cdot \gamma_j) \in M_{2g}(\mathbb{Z}) \]
\[ P = (\sum_{1 \leq r \leq s \leq g} \omega_r \omega_s) = (P_{r,j}) \in M_{g \times 2g}(\mathbb{C}) \]

Riemann bilinear relations

\[ P \cdot \Delta^{-1} \cdot P = 0 \]
\[ -\sqrt{-1} \cdot P \cdot \Delta^{-1} \cdot P \gg 0_g \]

\[ C \rightarrow \text{Pic}^1(C) \]

\[ \text{Pic}^0(C) = \text{Jac}(C) = \Gamma(C, \Omega^1)^{\perp}/H_1(C(C), \mathbb{Z}) \]

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\[ \text{Pic}^0(C) = \text{Jac}(C) = \Gamma(C, \Omega^1)^{\perp}/H_1(C(C), \mathbb{Z}) \]
1.4. Abelian varieties

Defn (i) (over C) a compact complex torus $\mathbb{C}^g/Q \cdot \mathbb{Z}^g$, $Q \in \text{M}_{g \times g}(\mathbb{C})$ is a complex abelian variety iff $\exists$ a skew-symmetric $E \in \text{M}_{g}(\mathbb{Z})$ with $\text{det}(E) \neq 0$ s.t.

$$\begin{cases} Q \cdot E^{-1} \cdot ^t Q = 0 \\ \sqrt{-1} \cdot Q \cdot E^{-1} \cdot ^t \overline{Q} \gg 0 \end{cases}$$

(i)', (equivalent to (i)) a compact complex torus is a complex abelian variety iff it admits a holomorphic embedding into $\mathbb{P}^N(\mathbb{C})$ for some $N$.

(ii) (Weil 1948) An irreducible algebraic group over a field $k$ which is a complete (i.e. proper) over $k$ is an abelian variety.
Defn (polarization of abelian varieties)

(i) A polarization of an abelian variety $A$ is an algebraic equivalence class of an ample divisor on $A$.

(ii) A polarization of an abelian variety $A$ represented by an ample divisor $D$ on $A$ is a principal polarization if $D^g = g!$.

Fact: (i) The polarization attached to an ample divisor $D$ on $A$ is uniquely determined by the algebraic homomorphism

$$
\varphi_{[D]} : A \longrightarrow \text{Pic}^0(A) = \text{dual abelian variety}, \text{classifying line bundles on } A \text{ which are algebraically equivalent to } 0.
$$

(2) Over $\mathbb{C}$, $c(D) \in H^2(A(\mathbb{C}); \mathbb{Z}(1))$ corresponds to a non-singular skew-symmetric pairing

$$
H_1(A(\mathbb{C}); \mathbb{Z}) \times H_1(A(\mathbb{C}); \mathbb{Z}) \longrightarrow \mathbb{Z}(1)
$$

$[D]$ is a principal polarization $\iff$ the above is a perfect pairing over $\mathbb{Z}$.
Every principally polarized abelian variety of dimension $g$ over $\mathbb{C}$ is isomorphic to $A_\Omega := \mathbb{C}^g / \Omega \cdot \mathbb{Z}^g + \mathbb{Z}^g$

for some $\Omega \in H_g := \{ \Omega \in \mathbb{M}_g(\mathbb{C}) \mid ^t \Omega = \Omega, \text{Im}(\Omega) \gg g \}$

2) $(A_{\Omega_1}, \lambda_{\Omega_1}) \cong (A_{\Omega_2}, \lambda_{\Omega_2})$ iff $\exists (AB \atop CD) \in Sp_{2g}(\mathbb{Z})$ such that $(A_{\Omega_1} + B) \cdot (CC\Omega_1 + D^{-1}) = \Omega_2$

Note: $Sp_{2g}(\mathbb{R})$ acts on $H_g$ by

$$(AB \atop CD) : \Omega \mapsto (A\Omega + B) \cdot (CC\Omega + D)^{-1}$$

This is a transitive action
1.5. The moduli space of \( g \)-dimensional principally polarized abelian varieties \( \mathcal{A}_g \)

Idea/phenomenon

- The set of all isomorphism classes of \( g \)-dimensional abelian varieties (with level-\( n \) structure) has a natural structure as an algebraic variety.
- A subvariety of \( \mathcal{A}_g \) corresponds to a family of abelian varieties.

\[ \uparrow \]

or more generally, a morphism \( S \to \mathcal{A}_g \)

**Ex.** \( g = 1 \). The set of all isomorphism classes of elliptic curves is parametrized by \( \mathbb{A}^1 : E \to j(E) \)

**Ex.** \( \mathcal{A}_g(\mathbb{C}) \cong \frac{\mathfrak{Sp}_g(\mathbb{Z})}{\mathfrak{Sp}_g(\mathbb{Q})} \)
- Existence of $A_g$ over $\mathbb{Z}$: Mumford 1965

- Fact: $A_g / k$ is irreducible $k$-field $k$
  - Case $k = \mathbb{C}$: immediate from complex uniformization $A_g(\mathbb{C}) \cong \frac{\mathbb{H}}{\mathbb{Z}_g}$
  - char $(k) = 0$: follows from the case $k = \mathbb{C}$
  - char $(k) = p > 0$: Faltings - C. 1984
§2 Hecke symmetry on $\mathbb{A}^g$

2.1 Definition

*Complex version:*

For $g \in Sp_{2g}(\mathbb{Q})$, $Sp_{2g}(\mathbb{Z}) \cdot g \colon Sp_{2g}(\mathbb{Z})$ induces an algebraic correspondence on $Sp_{2g}(\mathbb{Z})$.

\[
\begin{array}{ccc}
\Gamma(1) \backslash \mathfrak{H}_g & \leftarrow & \mathfrak{H}_g \\
\downarrow \text{pr} & & \downarrow \gamma \\
\mathfrak{H}_g & \rightarrow & \mathfrak{H}_g \\
\uparrow \text{pr} & & \downarrow \gamma \\
(\Gamma(1) \cap \Lambda \Gamma(1) \Lambda) \mathfrak{H}_g & \rightarrow & \Gamma(1) \mathfrak{H}_g
\end{array}
\]

(Think of \( \Gamma(1) \mathfrak{H}_g \) as a "multi-valued map" from \( \mathfrak{H}_g \) to \( \Gamma(1) \mathfrak{H}_g \).)
algebraic version:

* \([[(\lambda_1), \lambda_1]], \quad [(\lambda_2), \lambda_2] \in \hat{A_g}(\mathbb{Q})\) are in the same \((\text{prime-to-p})\) Hecke orbit if there exists an \(\text{isogeny} \ \alpha : A_1 \to A_2\) and \(n \in \mathbb{Z}_{\geq 0}\) (with \(\gcd(n, p) = 1\)) such that \(\alpha^*(\lambda_2) = n \cdot \lambda_1\)

Adelic picture:

\[
\text{Sp}_g \left( \mathbb{A}^{(p)}_f \right) \subset \hat{A}_g(p) = \varprojlim_{\gcd(n, p) = 1} \hat{A}_g, \hat{A}_g \subset \mathbb{Q}_l
\]

\[
\Gamma(n) \backslash \hat{A}_g
\]

\[
\text{prime-to-p Hecke orbits on } \hat{A}_g \leftrightarrow \text{Sp}_g(\mathbb{A}^{(p)}_f)\text{-orbits on } \hat{A}_g(p)
\]

phenomenon: Every Hecke orbit is large ("as large as possible")
2.2. $p$-divisible groups

Tate 1967

Definition: A $p$-divisible group $X \to S$ is an inductive system of commutative finite locally free group schemes

$$((X_n \to S)_{n \in \mathbb{N}}, \ i_{n+1,n} : X_n \hookrightarrow X_{n+1})$$

together with faithfully flat homomorphisms

$$\pi_{n,n+1} : X_{n+1} \to X_n$$

such that $i_{n+1,n} \circ \pi_{n,n+1} = [p]_{X_{n+1}}, \pi_{n,n+1} \circ i_{n+1,n} = [p]_{X_n}$ \forall n

Fact: \exists! locally constant function $h : S \to \mathbb{N}$ such that $\text{rk}(X_n/S) = p^n \cdot h$ \forall n

Main Example: $A \to S$ abelian scheme $\Rightarrow A[p^\infty] = \left( \lim_{n \to \infty} A[p^n] \right)_{n \in \mathbb{N}}$

is a $p$-divisible group of rank $2 \cdot \dim(A/S)$

$\uparrow$ a substitute for Lie algebra in char. $p > 0$
2.3 p-adic invariants of abelian varieties

- All p-adic invariants of an abelian variety $A/k$, $k = \mathbb{F}_p$, come from the p-divisible group $A[p^\infty]$
- Every prime-to-$p$ Hecke symmetry/ symplectic isogeny between principally polarized abelian varieties over $k = \mathbb{F}_p$ preserves all p-adic invariants.

Examples of p-adic invariants

(a) slopes / Newton polygon of a p-divisible group $X/k$

- idea: compare $F_r^{(p)} : X \rightarrow X(p^n)$ with $X[p^n]$
- slopes = p-adic valuation of "eigenvalues" of $F_r^{(p)}$

Every g-dim abelian variety has 2g slopes $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2g} \leq 1$, $\lambda_i \in \mathbb{Q} \times \mathbb{Z}$, $\lambda_i + \lambda_{g+1-i} = 1 \forall i$. 

denominator $(\lambda_i) \mid$ multiplicity $(\lambda_i) \forall i.$
A is ordinary $\iff$ slopes are 0 or 1
A is supersingular $\iff$ all slopes are $\frac{1}{2}$
ordinary abelian varieties in $A_g$ form an open dense subset of $A_g/k$

Example (b) $(A,\lambda) \mapsto$ isom. class of $(A[p], \lambda[p])$
(c) $(A,\lambda) \mapsto$ isom class of $(A[p^\infty], \lambda[p^\infty])$

Thm (C. 1995) $\forall x = [(A,\lambda)] \in A_g(k)$ with $A$ ordinary, the prime-to-$p$ Hecke orbit of $x$ is Zariski dense in $A_g$
§3 Leaves in $\mathcal{A}_g/k = k_\ell$,

**Def** (Oort, 1999) Given $x = [(A, \lambda)] \in A(k)$, the leaf $C(x)$ through $x$

is the locally closed subvariety of $\mathcal{A}_g$ s.t.

$C(x)(k) = \{ [(B, \mu)] \in A_g(k) \mid (B[p^\infty], \mu[p^\infty]) \cong (A[p^\infty], \lambda[p^\infty]) \}$

**Fact:** Every leaf in $\mathcal{A}_g$ is smooth, and stable under all prime-to-$p$ Hecke correspondences

**Conjecture** (Oort) Given a leaf $C \subset \mathcal{A}_g$ and $x \in C(k)$,

the prime-to-$p$ Hecke orbit of $x$ is Zariski dense in $C$

**Ans.** True (Oort+C., 2006)

proof uses a special property of $\mathcal{A}_g$ ("Hilbert trick")

The conjecture for general PEL-type moduli spaces remains open
§3 New tools, structures and conjectures/predictions/phenomena related to Hecke symmetry

3.1 Monodromy and irreducibility results

Prop A. (reducing irreducibility to Hecke transitivity)
Let $Z \subseteq A_g$ be a positive dimensional locally closed subvariety stable under all prime-to-$p$ Hecke correspondences. If Hecke symmetries are transitive on $\pi_0(Z)$, then $Z$ is irreducible.

Prop B. Let $C \subseteq A_g$ be a positive-dimensional leaf on $A_g$ (equivalently, $C$ is not supersingular), then the naive $p$-adic monodromy of $C$ is maximal, so is the $l$-adic monodromy of $C \forall l \neq p$. 
Prop. C. Every non-supersingular Newton stratum in $A_g$ is irreducible.

Prop. D. Every non-supersingular leaf in $A_g$ is irreducible.

Note: Prop. A is used in the proof of B-D. i.e., the maximality/irreducibility results in B-D are proved using Hecke symmetry.
3.2. Local structure of leaves (the 2-slope case) \( k = \overline{k} = \Omega_F \)

\[ A_g \supseteq C \ni x_0 = [(A_0, \lambda_0)] \quad \text{slope of } A_0 = \{ \lambda, 1-\lambda \} \quad \lambda < \frac{1}{2} \]

\[ \text{Prop. } C^{x_0} = \text{the formal completion of } C \text{ at } x_0 \text{ has a natural structure as a (trivial torsor for) an isoclinic } p\text{-divisible formal group with slope } 1-2\lambda \text{ and height } g \cdot (g+n)/2 \]
3.3. Local stabilizer principle \( k = \bar{k} = F_p \)

Prop. Let \( Z \subseteq \mathcal{A}_q \) be a locally closed subvariety stable under all prime-to-\( p \) Hecke correspondences, \( x_0 = [\mathcal{A}_0, \lambda_0] \in Z(k) \).

Then \( \mathcal{Z}^{x_0} \subseteq \mathcal{A}_q^{x_0} \) is stable under the natural action of an open subgroup of \( \begin{array}{c}
U(\text{End}(A_0), \ast_{x_0}) (\mathbb{Z}_p) \end{array} \) on \( \mathcal{A}_q^{x_0} \).

Explanation:

\[ \text{Aut}(A_0[p^\infty], \lambda_0[p^\infty]) \] acts on \( \mathcal{A}_q^{x_0} = \text{Def}(A_0, \lambda_0) \xrightarrow{\text{Serre-Tate}} \text{Def}(A_0[p^\infty], \lambda_0[p^\infty]) \)

U (End(A_0), \ast_{x_0})

"locally stabilizer subgroup at \( x_0 \)," corresponding to Hecke symmetries fixing \( x_0 \).
3.4. Rigidity phenomena

\[ \mathcal{H} = \overline{\mathcal{H}} \cong \mathcal{H}_p, \quad X/\mathcal{H} : p\text{-divisible formal group} \]

Theorem (C., local rigidity for p-divisible groups): Let \( Z \subseteq X \) be an irreducible formal subvariety.

- \( G \subseteq \text{Aut}(X) \), a p-adic Lie group acting on \( X \)
- Assume:
  - No open subgroup of \( G \) operates trivially on any non-zero subquotient of \( X \) (\( G \) operates "strongly non-trivially" on \( X \))
  - \( Z \) is stable under \( G \)

Then \( Z \) is a p-divisible formal subgroup of \( X \)

Remark: (Recent progress: Local rigidity holds for bi-extensions of p-divisible groups)
Exercise" Case  \( \hat{G}_m^h = \text{Spf}(\overline{\mathbb{F}_p}[T_1, \ldots, T_h]) \)

Group law:  \( \psi: \overline{\mathbb{F}_p}[T_1, \ldots, T_h] \rightarrow \overline{\mathbb{F}_p}[U_1, \ldots, U_h, V_1, \ldots, V_h] \)

\[ T_i \rightarrow U_i + V_i + U_i V_i \]

\[ (1+U_i) (1+V_i) = 1 + U_i + V_i + U_i V_i \]

\( G = 1 + p^2 \mathbb{Z}_p \subset \mathbb{Z}_p^\times \) operates on  \( \overline{\mathbb{F}_p}[T_1, \ldots, T_h] \)

\[ [1+p^2]^* \]:  \( f(T_1, \ldots, T_h) \rightarrow f((1+T_1)^{hp^2} - 1, \ldots, (1+T_h)^{hp^2} - 1) \)

The statement is: If  \( P \subseteq \overline{\mathbb{F}_p}[T_1, \ldots, T_h] \) is a prime ideal s.t.

\[ 1+p^2 \mathbb{Z}_p^* (P) \subseteq P \), then  \( \psi(P) \subseteq (P_1^*(P), P_2^*(P)) \)

where  \( P_1^*(f(T_1, \ldots, T_h)) = f(U_1, \ldots, U_h) \)

\( P_2^*(f(T_1, \ldots, T_h)) = f(V_1, \ldots, V_h) \)
Application (Exercise + local stabilizer principle)

\[ E_0 : \text{an ordinary elliptic curve over } \overline{\mathbb{F}_p} \]

\[ A_0 = E_0 \times \ldots \times E_0, \quad \lambda_0 = \text{product polarization on } A_0 \]

\[ X_0 = [(A_0, \lambda_0)] \in A_g(\overline{\mathbb{F}_p}) \]

Then the prime-to-\( p \) Hecke orbit of \( X_0 \) is Zariski dense in \( A_g \)

\[ A_g^{/X_0} \cong \hat{\mathbb{G}}_m^{g(\mathfrak{g} + n)/2}, \quad U(\text{End}(A_0), \ast_{\lambda_0})(\mathbb{Z}_p) \cong \text{GL}_g(\mathbb{Z}_p) \]

The action of \( \text{GL}_g(\mathbb{Z}_p) \) on \( \chi(\hat{\mathbb{G}}_m, g(\mathfrak{g} + n)/2) \cong \mathbb{Z}_p \times \hat{\mathbb{G}}_m^{g(\mathfrak{g} + n)/2} \)

\[ \cong S^2 \text{ (standard representation of } \text{GL}_g(\mathbb{Z}_p) \text{ on } \mathbb{Z}_p^{2g}) \]
3.5 Global rigidity conjecture

**Conj.** Suppose $Z \subseteq A_g^{\text{ord}}$, $x_0 = [(A_0, \lambda_0)] \in A_g^{\text{ord}}(\mathbb{F}_p)$. Assume that $Z^{x_0} \subseteq A_g^{x_0} (= \text{Serre-Tate formal torus})$ is a formal subtorsor. Then $Z$ is the reduction of a Shimura subvariety of $A_g$.

**Remark.** True if $Z \subset$ a Hilbert modular subvariety (C.).

This is the main geometric ingredient of Hida's proof (together with the local rigidity for $p$-divisible groups) of the non-vanishing of the $\mu$-invariant for Katz $p$-adic $L$-functions (Ann. Math. 2012).
3.6 A (special case of a) local rigidity conjecture

\[ G_0 = \text{a 1-dim smooth formal group over } \overline{\mathbb{F}_p}, \quad \text{ht}(G_0) = h \]

\[ M = \text{equi-characteristic } p \text{ deformation space of } G_0 \]

\[ \cong \text{Spf}(\mathbb{F}_p[[x_1, \ldots, x_{n-1}]]) \quad \text{with } x_i = \text{Hasse invariant} \]

\[ \text{Lubin-Tate 1966} \]

Conj. Suppose \( Z \subseteq M \) is an irreducible formal subvariety such that \( \frac{x_1}{Z} \neq 0 \) (i.e. \( Z \) is generically ordinary), and \( Z \) is stable under the action of an open subgroup of \( \text{Aut}(G_0) \).

then \( Z = M \)

\[ \text{group of units } \mu \text{ in a central division algebra over } \mathbb{Q}_p, \quad \dim_{\mathbb{Q}_p} = h^2 \]

with Brauer invariant \( \frac{1}{h} \)
3.7. New/better definition of leaves

Definition. Let $\kappa$ be a field of characteristic $p > 0$.
$X_0/\kappa$: a $p$-divisible group over $\kappa$
$S/\kappa$: a $\kappa$-scheme

A $p$-divisible group $X \to S$ is strongly $\kappa$-sustained modeled on $X_0$ if

$$\text{Isom}_S (X_0[p^n]_{p \neq \kappa}, X[p^n]) \to S$$

is faithfully flat for all $s \in S$. 