LEAVES AND LOCAL HECKE SYMMETRY

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# Outline

1. Hecke symmetry on PEL moduli varieties
2. Rigidity problems
3. Results, obstacles and hope
4. A scheme-theoretic definition of leaves
5. Serre-Tate coordinates on leaves
6. A new approach to local Hecke symmetries
7. Details for the Lubin-Tate action
PEL type modular varieties

A PEL type modular variety $\mathcal{M}$ is the moduli space attached to a PEL input datum $\mathcal{D} = (D, *, \mathcal{O}_D, H, \langle \cdot, \cdot \rangle, h)$; points of $\mathcal{M}$ correspond to abelian varieties with imposed symmetry $(A, \rho : A \to A^t, \iota : \mathcal{O}_D \to \text{End}(A), \text{level structure})$ whose $H_1$ are modeled on the linear algebra structure $\mathcal{D}$.

Fix a prime number $p$, unramified for the PEL datum $\mathcal{D}$. We will focus on the equal characteristic $p$ situation unless otherwise specified: $\mathcal{M}$ is a moduli space over $\overline{\mathbb{F}}_p$.

Let $B = \text{End}_D(H)$, with involution $*_B$ induced by $*$. Let $G = \text{SU}(B, *_B)$ (or $\text{GU}(B, *_B)$).
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Hecke symmetry

Let $\widetilde{M}$ be the prime-to-$p$ tower for $M$; it is a profinite etale Galois cover of $M$ with group $G(\hat{\mathbb{Z}}(p))$.

The group $G(\mathbb{A}_f^{(p)})$ operates on $\widetilde{M}$, inducing Hecke correspondences on $M$.

($M = \widetilde{M}/G(\hat{\mathbb{A}}^{(p)});$ the Hecke correspondences on $M$ is the remnant of the $G(\mathbb{A}_f^{(p)})$-action on $\widetilde{M}$.)

Example: $M = \mathcal{A}_g$ = the moduli space classifying $g$-dimensional principally polarized abelian varieties, $G = \text{Sp}_{2g}$ (or $\text{GSp}_{2g}$).
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Given a point $x \in \mathcal{M}(\overline{F}_p)$, corresponding to a quadruple $(A_x, \rho_x : A_x \to A_x^t, \iota_x : \mathcal{O}_D \to \text{End}(A_x), \text{level structure})$.

Let $\mathcal{M}/x$ be the formal completion of $\mathcal{M}$ at $x$.

Let $H_x := U(\text{End}_D^0(A_x), *_{\text{Ros}})(\mathbb{Z}(p))$, and let $G_x := U(\text{End}_D^0(A_x[p^{\infty}]), *_{\text{Ros}})(\mathbb{Z}_p)$.

The Serre-Tate deformation theorem implies that there is a natural action of the compact $p$-adic group $G_x$ on $\mathcal{M}/x$, by “changing the marking”.

This action can be regarded as a local version of the global Hecke symmetries.
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Local stabilizer subgroups

We call $G_x$ the local stabilizer subgroup at $x$. The group $H_x$ can be thought of as the “intersection” of $G_x$ with the global Hecke symmetries on $\mathcal{M}$.

**Lemma** (Local stabilizer principle) If a closed subvariety $Z \subset \mathcal{M}$ is stable under all Hecke symmetries, then $Z/\!\!/x \subset \mathcal{M}/\!\!/x$ is stable under the action of the $p$-adic closure of $H_x$ in $G_x$.

**Examples.** For a “general” $x \in \mathbb{A}_g(\overline{\mathbb{F}_p})$ (in particular $x$ is ordinary), the Zariski closure of $H_x$ is a $g$-dimensional torus, while the Zariski closure of $G_x$ is $\text{GL}_g$.

For a supersingular point $x \in \mathbb{A}_g(\overline{\mathbb{F}_p})$, $H_x$ is $p$-adically dense in $G_x$, and the Zariski closure of $G_x$ is a twist of $\text{Sp}_{2g}$. 
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The global rigidity problem

(Oort’s Hecke orbit conjecture)

**Prediction.** Let $Z \subset \mathcal{M}/\overline{\mathbb{F}}_p$ be a reduced closed subset of $\mathcal{M}$ stable under all prime-to-$p$ Hecke correspondences. Then $Z$ contains the leaf $C(x)$ passing through $x$ for every point $x \in Z(\overline{\mathbb{F}}_p)$.

(Every Hecke-invariant closed subset of $\mathcal{M}/\overline{\mathbb{F}}_p$ is a union of leaves; the latter can be regarded as “generalized Shimura subvarieties in char. $p$”.)
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Definition and examples of leaves

- A leaf $C(x)$ in $\mathcal{M}/\overline{\mathbb{F}}_p$ is the locus in $\mathcal{M}/\overline{\mathbb{F}}_p$ where all $p$-adic invariants have the same “value” as those of $x$.\(^1\)
- The ordinary locus $\mathcal{A}_g^{\text{ord}} \subset \mathcal{A}_g/\overline{\mathbb{F}}_p$ is a leaf in $\mathcal{A}_g/\overline{\mathbb{F}}_p$.
- The leaf passing through a supersingular point in $\mathcal{A}_g$ is finite.
- The leaf passing through a point in $\mathcal{A}_3$ corresponding to a 3-dimensional abelian variety with slopes $\{1/3, 2/3\}$ is two-dimensional. Such leaves form a one-dimensional family in the slopes $\{1/3, 2/3\}$ locus of $\mathcal{A}_3$. (The latter locus has dimension three.)

\(^1\)The universal $p$-divisible group $A[p^\infty]$ (with imposed symmetries) over $C(x)$ is a “twist” of the constant $p$-divisible group $A_x[p^\infty]$. 
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Remark. In application(s) to Iwasawa theory pioneered by Hida, certain strong versions of the global rigidity problem appear naturally:

- The assumption on $Z$ is weakened to: $Z$ is stable under the action of a “not-to-small” subset of Hecke correspondences.
- The desired conclusion is that $Z$ is a union of leaves in the reduction of certain Shimura subvarieties.
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Local rigidity problems

**Set-up.** $Z \subset \mathcal{M}/x$ is a reduced closed formal subscheme of $\mathcal{M}/x$, stable under the action of a “not-too-small” subgroup of $G_x$.

**Restricted** local rigidity problem (to make it easier):
Assume in addition that $Z \subset C(x)/x$.

**Desired conclusion.** $Z$ has a (very) special form (e.g. defined by a finite collection of Tate cycles.)
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Irreducibility criterion through Hecke symmetry

(a biproduct)

Hecke transitive $\implies$ irreducibility

for Hecke-invariant subvarieties which are not generically supersingular/basic.

Sample application.\(^2\)

**Proposition.** (i) Every non-supersingular Newton polygon stratum in $\mathcal{A}_g$ is (geometrically) irreducible.

(ii) Every non-supersingular leaf in $\mathcal{A}_g$ is irreducible.

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Results on the restricted local rigidity problem

**Proposition.** Restricted local rigidity holds for $\mathcal{A}_g$, in the case when $A_x$ has only two slopes.

(Aside/Fact) $C(x)^/x$ has a natural structure as a torsor for an isoclinic $p$-divisible formal group $X_x$.

If $Z \subset C(x)^/x$ is stable under a not-too-small subgroup of $G_x$, then $Z_x$ is a torsor for a $p$-divisible subgroup of $X_x$. 
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Restricted local rigidity: an example and consequences

An example. Let $Z$ be an irreducible formal subscheme of a formal torus $\hat{G}_m^r$ over $\overline{\mathbb{F}}_p$. Suppose that $Z$ is closed under the action of $[1 + p^m]$ for some $m \geq 2$. Then $Z$ is a formal subtorus of $\hat{G}_m^r$. (exercise)

Consequence of restricted local rigidity: linearization of the global rigidity problem, helped by considerations of local and global $p$-adic monodromy.
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Consequence of restricted local rigidity:
linearization of the global rigidity problem, helped by considerations of local and global $p$-adic monodromy.
Theorem. Global rigidity holds for $\mathcal{A}_g$.

Remarks. (1) Besides the restricted local rigidity and monodromy arguments, the proof uses a trick: Every point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is contained in a Hilbert modular subvariety of $\mathcal{A}_g$.³

(2) This “Hilbert trick” fails for PEL modular varieties of type A or D.

(3) A strong global rigidity statement holds for Hilbert modular varieties (and many other modular varieties associated to semisimple groups of $\mathbb{Q}$-rank one).⁴

³Global rigidity is substantially easier for these “small” modular varieties.

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(2) This “Hilbert trick” fails for PEL modular varieties of type A or D.

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\(^3\)Global rigidity is substantially easier for these “small” modular varieties.

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Results on global rigidity using the Hilbert trick

**Theorem.** Global rigidity holds for $\mathcal{A}_g$.

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Every point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is contained in a Hilbert modular subvariety of $\mathcal{A}_g$.

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Leaves of $p$-divisible groups

The notion of leaves was announced by F. Oort in 2001 (Texel).

(old) **Definition.** A $p$-divisible group $X$ over a scheme $S/\mathbb{F}_p$ is *geometrically fiberwise constant* if any two fibers $X_{s_1}, X_{s_2}$ are isomorphic when based-changed to a common algebraically closed field $K$ which contains both $\kappa(s_1)$ and $\kappa(s_2)$.

**Remark:** (i) This definition, though awkward, is reasonable when the base scheme $S$ is normal.
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An update 12 years after

**Definition.** (with F. Oort) Let $X$ be a $p$-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$.

1. A $\text{BT}_n$-group $\mathcal{X}_n$ over a $k$-scheme $U$ is X-sustained if there exists an fppf morphism $f : V \to U$ and an isomorphism $f^* \mathcal{X}_n \cong X[p^n] \times_{\text{Spec} k} V$.

2. A $p$-divisible group $\mathcal{X}$ over a $k$-scheme is X-sustained if the $\text{BT}_n$-group $\mathcal{X}[p^n]$ is sustained for every $n \geq 1$.

**Remark.** The category of all X-sustained $p$-divisible groups is not an Artin stack. For instance an X-sustained $p$-divisible group $\mathcal{X}$ may not be locally isomorphic to $X$ for the fppf topology.
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A differential criterion

Let $S$ be a scheme of finite type over $k$.
Let $S_1$ be the formal completion of $S \times S$ along the diagonal $\Delta_S \subset S_1 \times S_1$.

Let $\mathcal{X}$ be a $p$-divisible group over $S$.
Assume (for simplicity) that each fiber of $\mathcal{X}$ has two slopes.

**Proposition.** $\mathcal{X}$ is sustained iff the following conditions hold.

1. $\mathcal{X}$ admits a slope filtration $0 \to \mathcal{Z} \to \mathcal{X} \to \mathcal{Y} \to 0$ over $S$, where $\mathcal{Y}, \mathcal{Z}$ are isoclinic sustained $p$-divisible groups. (Therefore both $\mathcal{Y}$ and $\mathcal{Z}$ are Galois twists of constant $p$-divisible groups.) Moreover slope($\mathcal{Y}$) < slope($\mathcal{Z}$).

2. The extension class $[\text{pr}_1^* \mathcal{X}] - [\text{pr}_1^* \mathcal{X}] \in \text{Ext}(\mathcal{Y}, \mathcal{Z})(S_1)$ is $p$-divisible in the sense that it belongs to $\text{Ext}(\mathcal{Y}, \mathcal{Z})_{p\text{div}}(S_1)$.

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Hom schemes for $p$-divisible groups

Let $Y$, $Z$ be $p$-divisible groups over $k$.

$G_n := \text{Hom}(Y[p^n], Z[p^n])$, a group scheme of finite type over $k$,

$i_{n+1,n} : G_n \hookrightarrow G_{n+1}$ is the natural “inclusion homomorphism”,

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Remark. $\lim_n G_n = \text{Hom}(Y, Z)$ as sheaves on the flat big site of $\text{Spec} \ k$. It is representable by an affine scheme over $k$ but often not very useful.

For instance in the case when $Y$ is a one-dimensional $p$-divisible group of height 3 and $Z$ is the Serre dual of $Y$, one has $\text{Hom}(Y, Z) = \text{Spec} R/I$, where $R = k[x_0, x_1, x_2, x_3, \ldots]$ and $I$ is the ideal of $R$ generated by $x_0, x_1, x_2$ and $x_{i+3}^p - x_i$, $i \geq 0$.

The maximal ideal of this local ring $R$ is the nil radical of $R$, however $R$ seems to “come from a 3-dimensional thing”.
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**Leaves and Local Hecke Symmetry**

Ching-Li Chai

Hecke symmetry on PEL moduli varieties

Rigidity problems

Results, obstacles and hope

A scheme-theoretic definition of leaves

Serre-Tate coordinates on leaves

A new approach to local Hecke symmetries

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Stabilization for the $G_n$’s

Proposition. (i) There exists a constant $c$ such that

\[
\begin{align*}
G_{n+j}/i_{n+j,j}(G_j) &\sim \text{im}(r_{n,n+j} : G_{n+j} \to G_n) \\
G_{n+i}/i_{n+i,i}(G_i) &\sim \text{im}(r_{n,n+i} : G_{n+i} \to G_n)
\end{align*}
\]

for all $j \geq i \geq c$. Denote this finite group scheme over $k$ by $H_n$, which is naturally a subgroup scheme of $G_n$.

(ii) The homomorphisms $H_n \hookrightarrow H_{n+1}$ and $H_{n+1} \to H_n$ induced by $i_{n+1,n}$ and $r_{n,n+1}$ give $(H_n)_{n \geq 1}$ a structure of a $p$-divisible group $\mathcal{H}(Y,Z)$ over $k$.

(iii) $\mathcal{H}(Y,Z) = 0$ if $\lambda_Y > \lambda_Z$ for all slopes $\lambda_Y$ of $Y$ and all slopes $\lambda_Y$ of $Y$. 
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Let $T_p(Y)$ be the formal $p$-adic Tate module, i.e. the projective system of the $Y[p^n]$'s. Let $V_p(Y) = T_p(Y) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The push-out construction, applied to the short exact sequence
\[ 0 \to T_p(Y) \to V_p \to Y \to 0, \]
gives a map
\[ \delta_{Y,Z} : \mathcal{G}(Y, Z) \to \text{Ext}(Y, Z) \]
between sheaves for the flat topology.
Serre-Tate coordinates on leaves

Assume that $Y, Z$ are isoclinic with slopes $\lambda_Y, \lambda_Z$, and $\lambda_Y < \lambda_Z$.

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Serre-Tate coordinates on leaves, statement

**Theorem.** (1) The restriction of the sheaf $\text{Ext}(Y, Z)$ (respectively $\tilde{\mathcal{G}}(Y, Z)$) to the category of artinian local $k$-algebras is a smooth formal group $\mathcal{E}(Y, Z)$ (respectively $\mathcal{G}(Y, Z)$) over $k$.

(2) $\dim(\mathcal{E}(Y, Z)) = \dim(\mathcal{G}(Y, Z)) = \dim(Z) \cdot \dim(Y')$.

(3) $\delta_{Y,Z} : \mathcal{G}(Y, Z) \sim \mathcal{E}(Y, Z)$, and induces an isomorphism $\mathcal{H}(Y, Z) \sim \mathcal{E}(Y, Z)_{\text{pdiv}}$, where $\mathcal{E}(Y, Z)_{\text{pdiv}} = \text{the maximal } p\text{-divisible subgroup of } \mathcal{E}(Y, Z)$.

(4) $\mathcal{H}(Y, Z) \sim \mathcal{E}(Y, Z)_{\text{pdiv}}$ is naturally identified with the $X$-sustained locus in the universal $p$-divisible group over the deformation space $\text{Def}(Y \times Z)$.

(5) The $p$-divisible group $\mathcal{H}(Y, Z)$ is isoclinic of slope $\lambda_Z - \lambda_Y$, height $\text{ht}(Y) \cdot \text{ht}(Z)$ and dimension

$$(\lambda_Z - \lambda_Y) \text{ht}(Y) \text{ht}(Z) = \dim(Z) \cdot \dim(Y') - \dim(Y) \cdot \dim(Z').$$
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(2) $\text{dim}(\mathcal{E}(Y, Z)) = \text{dim}(\mathcal{G}(Y, Z)) = \text{dim}(Z) \cdot \text{dim}(Y^t)$.

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The Cartier module of $\mathcal{H}(Y, Z)$

Let $Y, Z$ be isoclinic $p$-divisible groups over $k$ with $\lambda_Y < \lambda_Z$.

Let $M, N$ be the Cartier modules of $Y$ and $Z$; both are finite free $W(k)$-modules, with semi-linear operators $F$ and $V$.

Let $H := \operatorname{Hom}_{W(k)}(M, N)$. We have natural semi-linear actions on $H \otimes_{W(k)} W(k)[1/p]$,

$$F_H : h \mapsto F_N \circ h \circ V_M, \quad V_H : h \mapsto p^{-1} V_N \circ h \circ F_M.$$ 

Let $H_1 :=$ the largest submodule of $H$ stable under $F_H$ and $V_H$.

**Proposition.** The Cartier module of the $p$-divisible group $\mathcal{H}(Y, Z)$ is naturally isomorphic to $H_1$.

**Remark.** The smallest submodule of $H \otimes_{W(k)} W(k)[1/p]$ which contains $H$ and stable under both $F_H$ and $V_H$ is the Cartier module of the maximal $p$-divisible quotient of $\mathcal{H}(Y, Z)$. 
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The Cartier module of $\mathcal{G}(Y,Z)$

Let $\text{Cart}_{p,k}$ be the Cartier ring functor on commutative $k$-algebras; it is an infinite dimensional commutative smooth formal group over $k$.

Let $\text{BC}_p(k)$ be the set of all $p$-typical formal curves in $\text{Cart}_{p,k}$.

- The ring structure of $\text{Cart}_{p,k}$ gives $\text{BC}_p(k)$ a $\text{Cart}_p(k)$-$\text{Cart}_p(k)$-bimodule structure.
- $\text{BC}_p(k)$, being the set of all $p$-typical curves in a commutative smooth formal group, has a natural $\text{Cart}_p(k)$-module structure which commutes with the above bimodule structure.

**Proposition.** The Cartier module of the smooth formal group $\mathcal{G}(Y,Z)$ is naturally isomorphic to

$$\text{Ext}_{\text{Cart}_p(k)}(M, \text{BC}_p(k) \otimes_{\text{Cart}_p(k)} N);$$

the third $\text{Cart}_p(k)$-module structure on $\text{BC}_p(k)$ gives the above Ext group a natural $\text{Cart}_p(k)$-module structure.
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The Cartier module of $G(Y,Z)$

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**Proposition.** The Cartier module of the smooth formal group $G(Y,Z)$ is naturally isomorphic to

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Global Serre-Tate coordinates

We illustrate the idea in the case of a leaf $C = C(x) \subset \mathcal{A}_g$, where $x$ corresponds to a $g$-dimensional principally polarized abelian variety $A_x$ over $k$ with two slopes $\lambda, 1 - \lambda$, $\lambda < 1/2$.

Let $C_1 = (C \times C)/\Delta C$, the formal completion along the diagonal. $pr_1 : C_1 \rightarrow C$ is formally smooth of relative dimension $(1 - 2\lambda)g(g + 1)/2$.

**Proposition.** $pr_1 : C_1 \rightarrow C$ has a natural structure as an isoclinic sustained $p$-divisible group over $C$, with slope $1 - 2\lambda$ and height $g(g + 1)/2$. 
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We illustrate the idea in the case of a leaf $\mathcal{C} = \mathcal{C}(x) \subset \mathcal{A}_g$, where $x$ corresponds to a $g$-dimensional principally polarized abelian variety $A_x$ over $k$ with two slopes $\lambda, 1 - \lambda, \lambda < 1/2$.

Let $\mathcal{C}_1 = (\mathcal{C} \times \mathcal{C})/\Delta \mathcal{C}$, the formal completion along the diagonal. $\text{pr}_1 : \mathcal{C}_1 \rightarrow \mathcal{C}$ is formally smooth of relative dimension $(1 - 2\lambda)g(g + 1)/2$.

**Proposition.** $\text{pr}_1 : \mathcal{C}_1 \rightarrow \mathcal{C}$ has a natural structure as an isoclinic sustained $p$-divisible group over $\mathcal{C}$, with slope $1 - 2\lambda$ and height $g(g + 1)/2$. 
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We illustrate the idea in the case of a leaf $\mathcal{C} = \mathcal{C}(x) \subset A_g$, where $x$ corresponds to a $g$-dimensional principally polarized abelian variety $A_x$ over $k$ with two slopes $\lambda, 1 - \lambda, \lambda < 1/2$.

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The **holy grail** for the rigidity problems:⁵

To pry actionable intelligence out of the action of the local stabilizing subgroup.

**Main obstacle:** Our poor understanding of this action (so cannot deploy enhanced interrogation techniques).

- Don’t have helpful (exact or approximate) formulas (have tried Norman’s algorithm).
- Linearization via crystalline techniques leads to formulas with high powers of $p$ in denominators.

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A tantalizing dream

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The Lubin-Tate case

We will explain a method to obtain an approximate (or even asymptotic) formula for the action of the local stabilizer subgroup, in the first non-trivial case,

where $\mathcal{M}/x = \text{Def}(G_0)$ is the Lubin-Tate moduli deformation space for a one-dimensional formal group $G_0$ of finite height $h$ over $\overline{\mathbb{F}}_p$. 
The Lubin-Tate case

We will explain a method to obtain an approximate (or even asymptotic) formula for the action of the local stabilizer subgroup, in the first non-trivial case, where $\mathcal{M}/x = \text{Def}(G_0)$ is the Lubin-Tate moduli deformation space for a one-dimensional formal group $G_0$ of finite height $h$ over $\overline{\mathbb{F}}_p$. 
How to compute the Lubin-Tate action

1. Go from the category of 1-dimensional formal groups to the category of \( p \)-typical formal group laws. (This is a “rigidification process”.)

- The latter category is a groupoid with infinitely many affine coordinates \( t_1, t_2, t_3, \ldots \)
- Honda’s formalism\(^6\) provides many objects and arrows in the groupoid.

One such object, a one-dimensional \( p \)-typical groupoid over the \( h-1 \)-dimensional base \( W(\overline{\mathbb{F}_p})[[t_1, \ldots, t_{h-1}]] \) with

\[ t_h = 1, 0 = t_{h+1} = t_{h+2} = \cdots \]

represents universal one-dimensional formal group \( \tilde{G} \) over \( \text{Def}(G_0) \).

**Goal:** Given an automorphism \( \gamma \) of \( G_0 \), compute an automorphism \( \tilde{\gamma} \) of \( \tilde{G} \) over an automorphism \( \rho_{\gamma} \) of the base (formal) scheme of \( \tilde{G} \) which induces \( \gamma \) over the closed fiber \( G_0 \).

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\(^6\) or Hazewinkel’s “functional equation lemma”
How to compute the Lubin-Tate action

1. Go from the category of 1-dimensional formal groups to the category of $p$-typical formal group laws. (This is a “rigidification process”.)

- The latter category is a groupoid with infinitely many affine coordinates $t_1, t_2, t_3, \ldots$

- Honda’s formalism$^6$ provides many objects and arrows in the groupoid.

One such object, a one-dimensional $p$-typical groupoid over the $h - 1$-dimensional base $W(\overline{\mathbb{F}_p})[[t_1, \ldots, t_{h-1}]]$ with $t_h = 1, 0 = t_{h+1} = t_{h+2} = \cdots$ represents universal one-dimensional formal group $\tilde{G}$ over $\text{Def}(G_0)$.

**Goal:** Given an automorphism $\gamma$ of $G_0$, compute an automorphism $\tilde{\gamma}$ of $\tilde{G}$ over an automorphism $\rho_\gamma$ of the base (formal) scheme of $\tilde{G}$ which induces $\gamma$ over the closed fiber $G_0$.

$^6$or Hazewinkel’s “functional equation lemma”
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How to compute the Lubin-Tate action, continued

PROCEDURE: Perform the computation of $(\tilde{\gamma}, \rho_\gamma)$ in several steps (corresponding to a factorization of $(\tilde{\gamma}, \rho_\gamma)$), in the groupoid of $p$-typical formal group laws.

2. (key observation) There is an integral recursive formula (no $p$ in the denominators) involving the naive Frobenius lifting, for the universal strict isomorphism between $p$-typical formal group laws.\(^7\)

3. The above integral formula, coupled with the (infinite dimensional version of) inverse function theorem, provides a way to compute local Hecke symmetries. The patterns are encoded in recursive formulas for the coordinates of the $p$-typical formal group laws involved.

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Let $h$ be a positive integer.

Let $G_1$ be the one-dimensional formal group over $\mathbb{Z}_p$ with logarithm

$$\sum_{j \in \mathbb{N}} p^{-j} x^{p^j h} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots$$

(so it is a Lubin-Tate formal group for $W(\mathbb{F}_{p^h})$.)

Let $G_0$ be the base extension to $\overline{\mathbb{F}}_p$ of the closed fiber of $G_1$; it is a one-dimensional formal group over $\mathbb{F}_p$ of height $h$.

It is well-known that $\text{End}(G_0)$ is the maximal order of $\text{End}^0(G_0) = a$ central division algebra over $\mathbb{Q}_p$ of dimension $h^2$. 

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**Notation**

- Universal p-typical group laws
- Formula for coeff
- Universal strict isom
- Integral recursive formula for universal strict isom
- Param for Fc

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**Leaves and Local Hecke Symmetry**

Ching-Li Chai

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Let $\mathcal{M}_h := \text{Def}(G_0)$; it is a smooth formal scheme over $W(F_p)$ of relative dimension $h - 1$.

Let $G_{\text{univ}} \to \mathcal{M}_h$ be the universal formal group over $\mathcal{M}_h$.

The compact $p$-adic group $\text{Aut}(G_0) = \text{End}(G_0)^\times$ operates on $\mathcal{M}_h$ by functoriality, as follows.

$\forall \gamma \in \text{Aut}(G_0), \exists!$ formal scheme automorphism $\rho(\gamma)$ of $\mathcal{M}_h$ and a formal group isomorphism

$$\tilde{\rho}(\gamma) : G_{\text{univ}} \to \rho(\gamma)^* G_{\text{univ}}$$

such that $\tilde{\rho}(\gamma)|_{G_0} = \gamma$

Remark. This action $\gamma \mapsto \rho(\gamma)$ of $\text{Aut}(G_0)$ on the Lubin-Tate moduli space $\mathcal{M}_h$ was first studied by Lubin and Tate in 1966. It is also known as (the essential part of) the Morava stabilizer subgroup action in chromatic homotopy theory.
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The universal $p$-typical formal group law

Let $\tilde{R} = \mathbb{Z}_p[[v]] = \mathbb{Z}_p[[v_1, v_2, v_3, \ldots]]$, and let $\sigma : \tilde{R} \to \tilde{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$.

Let $G_v(x) \in \tilde{R}[[x, y]]$ be the one-dimensional $p$-typical formal group law over $\tilde{R}$ whose logarithm

$$g_v(x) \in \tilde{R}[1/p][[x]] = \sum_{n \geq 1} a_n(v) \cdot x^{p^n}$$

satisfies

$$g_v(x) = x + \sum_{i=1}^{\infty} \frac{v_i}{p} \cdot g_v^{(\sigma^i)}(x^{p^i})$$

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$$g_v^{(\sigma)}(x) = \frac{x}{1 + \sigma(x)}$$
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Remarks on the formal group law $G_v$

Remarks. (1) The above “functional equation” is a recursive formula for the coefficients $a_n(v) \in p^{-n} \cdot \mathbb{Z}(p)[v_1, v_2, \ldots, v_n]$ of $g_v(x)$.

(2) Explicitly:

$$a_n(v) = \sum_{i_1, i_2, \ldots, i_r \geq 1 \atop i_1 + \cdots + i_r = n} p^{-r} \cdot \prod_{s=1}^{r} v_{i_s}^{p^{i_1+i_2+\cdots+i_{s-1}}}
= \sum_{i_1, i_2, \ldots, i_r \geq 1 \atop i_1 + \cdots + i_r = n} p^{-r} \cdot v_{i_1}^{p^{i_1}} \cdot v_{i_2}^{p^{i_1+i_2}} \cdots v_{i_r}^{p^{i_1+\cdots+i_{r-1}}}
$$

Note that $a_n(v)$ is a homogeneous polynomial in $v_1, \ldots, v_n$ of weight $p^n - 1$ when $v_j$ is given the weight $p^j - 1 \; \forall j \geq 1$.

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The universal formal group over $\mathcal{M}_h$ made explicit

Let $R = R_h = W(\overline{\mathbb{F}}_p)[[w_1, w_2, \ldots, w_{h-1}]]$.

Let $\pi = \pi_h : \tilde{R} \to R$ be the ring homomorphism such that

$$\pi(v_i) = \begin{cases} w_i & \text{if } 1 \leq i \leq h-1 \\ 1 & \text{if } i = h \\ 0 & \text{if } i \geq h+1 \end{cases}$$

The classifying morphism $\text{Spf}(R) \to \mathcal{M}_h$ for the deformation $\pi_* G^\vee$ of $G_0$ is an isomorphism.

We will identify $\mathcal{M}_h$ with $\text{Spf}(R)$ and the universal deformation $G_{\text{univ}}$ of $G_0$ with the formal group underlying the formal group law $G_R := \pi_* G^\vee$. 
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\[ \]
The universal strict isomorphism

Let $\mathbb{Z}_p[v, t] = \mathbb{Z}_p[v_1, v_2, v_3, \ldots; t_1, t_2, t_3, \ldots]$, and let $\sigma : \mathbb{Z}_p[v, t] \to \mathbb{Z}_p[v, t]$ be the obvious Frobenius lifting as before, with $\sigma(v_i) = v_i^p$ and $\sigma(t_i) = t_i^p \ \forall \ i \geq 1$.

Let $G_{v,t}(x, y)$ be the one-dimensional formal group law over $\mathbb{Z}_p[v, t]$ whose logarithm $g_{v,t}(x)$ satisfies

$$g_{v,t}(x) = x + \sum_{i=1}^{\infty} t_i \cdot x^{p^i} + \sum_{j=1}^{\infty} \frac{v_j}{p} \cdot g_{v,t}^{(\sigma^j)}(x^{p^j})$$
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integral recursive formula for universal strict isomorphism
The universal strict isomorphism, continued

It is known that $\alpha_{v,t} := g_{v,t}^{-1} \circ g_v \in \mathbb{Z}(p)[v,t][[x]]$, and defines a strict isomorphism

$$\alpha_{v,t} : G_v \to G_{v,t}$$

between $p$-typical formal group laws over $\mathbb{Z}(p)[v,t]$.

(A strict isomorphism is an isomorphism between formal group laws which is $\equiv x$ modulo higher degree terms in $x$.)

Moreover $\alpha_{v,t}$ is “the” universal strict isomorphism between one-dimensional $p$-typical formal group laws.
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The universal strict isomorphism, continued

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Moreover $\alpha_{v,t}$ is “the” universal strict isomorphism between one-dimensional $p$-typical formal group laws.
Parameters of $G_{\nu,t}$

By the universality $G_{\nu}$ for $p$-typical formal group laws, there exists a unique ring homomorphism

$$\eta : \mathbb{Z}(p)[\nu] \rightarrow \mathbb{Z}(p)[\nu, t]$$

such that

$$\eta_* G_{\nu} = G_{\nu, t}.$$ 

The elements

$$v_n = v_n(\nu, t) \in \mathbb{Z}(p)[\nu, t], \quad n \in \mathbb{N}_{\geq 1}$$

are the parameters of the $p$-typical formal group law $G_{\nu, t}$. 
Parameters of $G_{\overline{\nu}, t}$

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such that

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The elements

$$\overline{v}_n = \overline{v}_n(\nu, t) \in \mathbb{Z}(p)[\nu, t], \quad n \in \mathbb{N}_{\geq 1}$$

are the parameters of the $p$-typical formal group law $G_{\overline{\nu}, t}$. 
A known recursive formula for the parameters of $G_{v,t}$

\[
\bar{v}_n = v_n + pt_n + \sum_{i+j=n, i,j \geq 1} (v_j t_i^{p^j} - t_i \bar{v}_j^{p^j}) \\
+ \sum_{j=1}^{n-1} a_{n-j}(v) \cdot \left( v_j^{p^{n-j}} - \bar{v}_j^{p^{n-j}} \right) \\
+ \sum_{k=2}^{n-1} a_{n-k}(v) \cdot \sum_{i+j=k, i,j \geq 1} \left( v_j^{p^{n-k}} t_i^{p^{n-i}} - t_i^{p^{n-k}} \bar{v}_j^{p^{n-j}} \right)
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(This formula contains high power of $p$ in the denominators. Consequently it is not very useful for our purpose.)
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An integral recursion formula for $\bar{v}_n(v, t)$

(useful for computing the Lubin-Tate action)

$$
\bar{v}_n = v_n + pt_n - \sum_{j=1}^{n-1} t_j \cdot \bar{v}^{p^j}_{n-j} +
$$

$$
+ \sum_{l=1}^{n-1} v_l \sum_{k=1}^{n-l-1} \frac{1}{p} \cdot a_{n-k-l}(v)^{(p^l)} \cdot \left\{ (\bar{v}^{(p^l)}_k)^{p^{n-l-k}} - (\bar{v}^{p^l}_k)^{p^{n-l-k}} \right\}
$$

$$
+ \sum_{i+j=k, i, j \geq 1} t_j^{p^{n-k}} \left[ (\bar{v}^{(p^l)}_i)^{p^{n-l-i}} - (\bar{v}^{p^l}_i)^{p^{n-l-i}} \right]
$$

$$
+ \sum_{l=1}^{n-1} v_l \cdot \left\{ \frac{1}{p} (\bar{v}^{(p^l)}_{n-l} - \bar{v}^{p^l}_{n-l}) + \sum_{i+j=n-l, i, j \geq 1} t_j^{p^l} \cdot \frac{1}{p} \left[ (\bar{v}^{(p^l)}_i)^{p^j} - (\bar{v}^{p^l}_i)^{p^j} \right] \right\}
$$

for every $n \geq 1$.  ▶ parameters for Fc  ▶ back to key observation
Step 1

Given an element $\gamma \in \text{Aut}(G_0)$, construct

- a $p$-typical one-dimensional formal group law $F = F_\gamma$ over $R$ whose closed fiber is equal to $G_0$, and
- an isomorphism

$$\overline{\psi} = \overline{\psi}_\gamma : F_{\overline{\mathbb{R}}} \to G_{\overline{\mathbb{R}}}$$

over $\overline{R} := R/pR = \overline{\mathbb{F}}_p[[w_1, \ldots, w_{h-1}]]$ whose restriction to the closed fibers is

$$(\psi|_{G_0} : G_0 \to G_0) = \gamma.$$ 

Here $F_{\overline{\mathbb{R}}} = F \otimes_R \overline{R}$, $G_{\overline{\mathbb{R}}} = G_R \otimes_R \overline{R}$.

Note that both the formal group law $F$ over $R$ and the isomorphism $\psi$ over $\overline{R}$ depends on the given element $\gamma \in \text{Aut}(G_0)$. 
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The formal group law $F_c, c \in W(\mathbb{F}_{p^h})^\times$

For $\gamma = [c] \in W(\mathbb{F}_{p^h})^\times = \text{Aut}(G_1)$, we can take $F_c$ to be the formal group over $R$ whose logarithm $g_c(x)$ satisfies

$$f_c(x) = x + \sum_{i=1}^{h} \frac{c^{-1+\sigma^i} \cdot w_i}{p} \cdot f_c(\sigma^i)(x^{p^i})$$

($w_h=1$ by convention).

Let

$$\psi_c(x) = \log_{R}^{-1} \circ (c \cdot f_c)$$

We have $\psi_c(x) \in R[[x]]$ and $\psi_c$ defines an isomorphism from $F_c$ to $G_R$ over $R$ (not just over $\overline{R}$!) with $\psi_c|_{G_0} = [c]$. 
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Compute the parameters

\[(u_i = u_i(w_1, \ldots, w_{h-1}))_{i \in \mathbb{N}_{\geq 1}}\]

for the $p$-typical group law $F = F_\gamma$ over $R$.

The above condition means that

\[\xi^* G_{\tilde{v}} = F,\]

where

\[\xi = \xi_\gamma : \tilde{R} \rightarrow R\]

is the ring homomorphism such that

\[\xi(v_i) = u_i \quad \forall i \geq 1.\]
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The above condition means that

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is the ring homomorphism such that

\[\xi(v_i) = u_i \quad \forall i \geq 1.\]
Parameters for $F_c$, $c \in W(\mathbb{F}_{p^h})^\times$

In the case when $\gamma \in \text{Aut}(G_0)$ lifts to an element $[c]$ with $c \in W(\mathbb{F}_{p^h})^\times \simeq \text{Aut}(G_1)$, we have the following integral recursive formula for the parameters $u_n = u_n(c; w)$.

\[
\begin{align*}
  u_n(c; w) &= c^{-1+\sigma^n}w_n \\
  &\quad + \sum_{j=1}^{n-1} c^{-1+\sigma^j} \cdot \frac{1}{p} \left[ u_{n-j}(c; w)^{(p^j)} - u_{n-j}(c; w)p^j \right] \cdot w_j \\
  &\quad + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j-1} \frac{1}{p} a_{n-i-j}(w)^{(p^j)} \cdot c^{-1+\sigma^{n-i}} \cdot \\
  &\quad \left[ (u_i(c; w)^{(p^j)})^{p^{n-i-j}} - (u_i(c; w)^{p^j})^{p^{n-i-j}} \right] \cdot w_j
\end{align*}
\]

where $w_h = 1$, $w_m = 0 \ \forall m \geq h + 1$ by convention.
Remark. The above recursive formula for the parameters $u_n(c; w)$ can be turned into an explicit “path sum” formula for $u_n(c, w)$, with terms indexed by “paths”.
Step 3

Find/compute the uniquely determined element

$$\tau_n \in m_R, \quad n \in \mathbb{N}_{\geq 1}$$

and

$$\hat{u}_1 \in m_R, \ldots, \hat{u}_{h-1} \in m_R, \hat{u}_h \in 1 + m_R$$

such that

$$\bar{v}_n(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h, 0, 0, \ldots; \tau) = u_n \quad \forall n \geq 1.$$
**Remark.** (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over \( \tilde{\mathbb{R}} \), applied to the “vector-valued” function with components \( \bar{v}_n \) in the integral recursion formula discussed before.

(2) This step is a substitute for the operation *taking the quotient of the group “changes of coordinates”* in a space of formal group laws.

(3) The approximate solution coming from the linear term in the \( \tau_j \) variables is often good enough for our application.
Remark. (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over $\tilde{\mathbb{R}}$, applied to the “vector-valued” function with components $\bar{v}_n$ in the integral recursion formula discussed before.

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(3) The approximate solution coming from the linear term in the $\tau_j$ variables is often good enough for our application.
A congruence formula for $\overline{v}_n$

The follow formula helps to explain the last remark.

\[
\overline{v}_n \equiv v_n - \sum_{j=1}^{n} t_j \cdot v_{n-j}^p \\
+ \sum_{i,j,t,s_1,s_2,\ldots,s_t \geq 1 \atop s_1 + \ldots + s_t + i + j = n} (-1)^{t-1} t_i \cdot v_{j}^p \cdot v_1^{(p^{s_1} + p^{s_2} + \cdots + p^{s_t} - t)/(p-1)} \cdot v_{n-s_1}^{p^{s_1} - 1} \cdot v_{n-s_1-s_2}^{p^{s_2} - 1} \cdots v_{n-s_1-\cdots-s_t}^{p^{s_t} - 1} \\
\mod (pt_a, t_b)_{a,b \geq 1} \mathbb{Z}[v,t]
\]
Step 4

**Rescale** \( \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h \) as follows:

\[
\exists! \, \tau_0 \in m_R \text{ such that }
(1 + \tau_0)^{p^h-1} \cdot \hat{u}_h = 1.
\]

Let

\[
\hat{v}_i := (1 + \tau_0)^{p^i-1} \cdot \hat{u}_i \quad \text{for } i = 1, \ldots, h - 1.
\]

Let \( \omega : \tilde{R} \to R \) be the ring homomorphism such that

\[
\omega(v_i) = \hat{u}_i \quad \forall i \geq 1.
\]

Let \( \rho : R \to R \) be the \( W(\overline{\mathbb{F}_p}) \)-linear ring homomorphism such that

\[
\rho(w_i) = \hat{v}_i \quad \forall i \geq 1.
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The meaning of Steps 3 and 4

The universal strict isomorphism $\alpha_{\mathfrak{v},t}$ specializes to a strict isomorphism

$$\alpha = \alpha_{\mathfrak{u},\tau} : F \to \omega_\ast G_{\mathfrak{v}}$$

with $\alpha|_{G_0} = \text{Id}_{G_0}$.

The rescaling in step 4 gives an isomorphism (not necessarily a strict isomorphism)

$$\beta : \omega_\ast G_{\mathfrak{v}} \to \rho_\ast G_R$$

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The universal strict isomorphism $\alpha_{v,t}$ specializes to a strict isomorphism

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Conclusion

Combined with $\bar{\psi}$, we obtain an isomorphism

$$\bar{\psi} \circ \bar{\alpha}^{-1} \circ \bar{\beta}^{-1} : \bar{\rho}^* G_{\bar{R}} \to G_{\bar{R}}$$

whose restriction to the closed fiber $G_0$ is equal to the given element $\gamma \in \text{Aut}(G_0)$.
(Here $\bar{\alpha} = \alpha \otimes_R \bar{R}$ and $\bar{\beta} = \beta \otimes_R \bar{R}$.)

Conclusion. The given element $\gamma \in \text{Aut}(G_0)$ operates on the equi-characteristic deformation space $\text{Spf}(\bar{R})$ of $G_0$ via the ring automorphism $\rho$.
(Notice that $\bar{\psi}$, $\alpha$ and $\beta$ all depend on $\gamma$.)
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Local rigidity for the Lubin-Tate moduli space: the first non-trivial case

**Proposition.** Let $Z \subset \mathcal{M}_3_{\overline{\mathbb{F}}_p} = \text{Spf}(\overline{\mathbb{F}}_p[[w_1, w_2]])$ be an irreducible closed formal subscheme of $\mathcal{M}_3$ over $\overline{\mathbb{F}}_p$ corresponding to a height one prime ideal of $\overline{\mathbb{F}}_p[[w_1, w_2]]$. If $Z$ is stable under the action of an open subgroup of $W(\mathbb{F}_p^3)^\times$, then $Z = \text{Spf}(\overline{\mathbb{F}}_p[[w_1, w_2]]/(w_1))$. 
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