Siegel Modular Varieties and Hecke symmetry

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Goal: survey Hecke symmetry on the moduli space of (principally polarized) abelian varieties of characteristic $p>0$ and related rigidity phenomenon

- history:

  elliptic curves ↗ curves of higher genera

  ↘ abelian varieties ⇝ moduli spaces of abelian varieties ↻ Hecke symmetry

- phenomena and structures in characteristic $p>0$; predictions (= conjectures)

- new tools / methods applicable to other problems

§1 From elliptic curves to abelian varieties and their moduli

1.1. What is an elliptic curve? several approaches

(a) algebra $E : \{ y^2 = 4x^3 - g_2 x - g_3 \}$, $\Delta := g_2^3 - 27 g_3^2$, $j = 1728\, g_2^3/\Delta$

(b) geometry $E(\mathbb{C}) \longleftarrow \mathrm{Lie}(E)/H_1(E(\mathbb{C}),\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ $\tau \in \mathfrak{h} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$

$$P \longmapsto \int_\infty^P \frac{dx}{y}$$

(c) analysis
$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda_\tau}{}' \left[\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} \right]$$

$$\rightsquigarrow \left(\frac{d}{dz} \wp(z,\tau) \right)^2 = 4 \wp(z;\tau)^3 - g_2(\Lambda_\tau)\wp(z,\tau) - g_3(\Lambda_\tau).$$

$$g_2(\Lambda_\tau) = 60 \cdot \sum_{\gamma \in \Lambda_\tau}{}' \frac{1}{\gamma^4}, \quad g_3(\Lambda_\tau) = 140 \sum_{\gamma \in \Lambda_\tau}{}' \frac{1}{\gamma^6}$$

1.2. The origin of elliptic curves

A. (Diophantine equation)

- Fermat : $x^4 - y^4 = z^2$ has no nontrivial rational solution
  (infinite descent - actually a 2-descent : w.r.t. $2 = (1+\sqrt{-1})(1-\sqrt{-1})$
  - often treated in college-level number theory)

- Gauss (last entry of Gauss's mathematical diary, 1814)

  $E_{aff} := \{ 1 = x^2 + y^2 + x^2y^2 \}$ $\qquad (a+\sqrt{-1}\,b)\cdot\mathbb{Z}[\sqrt{-1}]$ prime ideal
  s.t. $a+\sqrt{-1}\,b \equiv 1 \mod (1+\sqrt{-1})^3$

  The congruence equation $1 \equiv x^2 + y^2 + x^2y^2 \pmod{a+\sqrt{-1}\,b}$

  has $(a-1)^2 + b^2$ solutions, including the 4 solutions at ∞:

  $(x=\infty,\ y=\pm\sqrt{-1}),\ (x=\pm\sqrt{-1},\ y=\infty)$

B. (elliptic integral,

December 1751. paper by Fagnano reached Euler in Berlin)

- Fagnano : $\dfrac{dx}{\sqrt{1-x^4}} = \dfrac{dy}{\sqrt{1-y^4}}$ has rational solutions.

i.e $\displaystyle\int_0^x \frac{d\rho}{\sqrt{1-\rho^4}} = \int_0^y \frac{d\psi}{\sqrt{1-\psi^4}}$ admits solution where $y =$ a rational function of x

- Euler $\displaystyle\int_0^r \frac{d\rho}{\sqrt{1-\rho^4}} \underset{\uparrow}{=} \sqrt{2}\int_0^t \frac{d\xi}{\sqrt{1+\xi^4}}$, $\displaystyle\int_0^t \frac{d\xi}{\sqrt{1+\xi^4}} \underset{\uparrow}{=} \sqrt{2}\int_0^u \frac{d\eta}{\sqrt{1-\eta^4}}$

$r^2 = \dfrac{2t^2}{1+t^4}$ $\qquad$ $t^2 = \dfrac{2u^2}{1-u^4}$

$$\int_0^r \frac{d\rho}{\sqrt{1-\rho^4}} \underset{\uparrow}{=} (1 \pm \sqrt{-1}) \int_0^v \frac{d\eta}{\sqrt{1-\eta^4}}$$

$$r = \pm \frac{2\sqrt{-1}\,v^2}{1-v^4}$$

inversion of abelian integrals

for $y^2 = f(x)$ i.e. for hyperelliptic curves

Abel 1827, Jacobi 1828

Jacobi 1829, Fundamenta Nova Theoriae Functionum Ellipticarum

(defined Jacobi theta functions)

1.3 (Riemann 1857. Theorie der Abel'sche Functionen)

Compact Riemann surfaces and their Jacobians

periods

$S = C(\mathbb{C})$ cpt Riemann surface; $\gamma_1, \dots, \gamma_{2g}$ $\mathbb{Z}$-basis of $H_1(S, \mathbb{Z})$

$\omega_1, \dots, \omega_g$: $\mathbb{C}$-basis of $\Gamma(S, \Omega^1_S)$. $\Delta = (\gamma_i \cdot \gamma_j) \in M_g(\mathbb{Z})$

$P = P(\omega_1, \dots, \omega_g; \gamma_1, \dots, \gamma_{2g}) = (P_{ri})_{\substack{1 \le r \le g \\ 1 \le i \le 2g}} \in M_{g \times 2g}(\mathbb{C})$.

$P_{ij} = \int_{\gamma_j} \omega_r$

Riemann bilinear relations

$$\boxed{\begin{array}{l} P \cdot \Delta^{-1} \cdot {}^t P = 0 \\ -\sqrt{-1} \cdot P \cdot \Delta^{-1} \cdot {}^t \overline{P} \gg 0_g \end{array}}$$

Torelli map: $C \longrightarrow \mathrm{Pic}^1(C)$

$\circlearrowleft$

$\mathrm{Pic}^0(C) = \mathrm{Jac}(C) = \Gamma(C, \Omega^1)^\vee / H_1(C(\mathbb{C}); \mathbb{Z})$

1.4. Abelian varieties

Def (i) (over $\mathbb{C}$). a compact complex torus $\mathbb{C}^g/Q\cdot\mathbb{Z}^{2g}$, $Q\in M_{g\times 2g}(\mathbb{C})$ is a complex abelian variety iff $\exists$ a skew-symmetric $E\in M_{2g}(\mathbb{Z})$ with $\det(E)\neq 0$ satisfying

"principal part of Q"

$$\begin{cases} Q\cdot E^{-1}\cdot {}^tQ = 0 \\ \sqrt{-1}\cdot Q\cdot E^{-1}\cdot {}^t\overline{Q} \gg 0 \end{cases}$$

(i)′ (equivalent to (ii)) a compact complex torus is an abelian variety iff it admits a holomorphic embedding into $\mathbb{P}^N(\mathbb{C})$ for some N

(ii) (algebraic definition) Weil 1948 An irreducible algebraic group variety over a field is an abelian variety if it is complete (i.e. proper over the base field)

Def^n (polarization of abelian varieties)

(i) A <u>polarization</u> of an abelian variety is an ample divisor on A up to algebraic equivalence

(ii) The polarization of an abelian variety A attached to an ample divisor D on A is <u>principal</u> if $D^g = g!$ (self-intersection g times).

(iii) The polarization on A attached to D is uniquely determined by the algebraic homomorphism

$$\varphi_{[D]} : A \longrightarrow A^t = \mathrm{Pic}^0(A) = \text{dual abelian variety, classifying line bundles on } D \text{ algebraically equivalent to } 0$$

$$x \longmapsto [\mathcal{O}_A(D-x)]$$

(iv) Over $\mathbb{C}$, the fundamental class $c([D]) \in H^2(A(\mathbb{C}), \mathbb{Z}(1))$ corresponds to a non-singular skew-symmetric pairing on $H_1(A(\mathbb{C}), \mathbb{Z})$ satisfying the Riemann bilinear relations (ie. a Riemann form).
A polarization [D] is principle iff the Riemann form is a perfect pairing $/\mathbb{Z}$

Over $\mathbb{C}$:

(i) Every principally polarized abelian variety of dimension g over $\mathbb{C}$ is of the form

$$A_\Omega := \mathbb{C}^g / \Omega\cdot\mathbb{Z}^g + \mathbb{Z}^g \quad \text{with principal part} \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$$

for some $\Omega \in \mathfrak{h}_g := \{\Omega \in M_g(\mathbb{C}) \mid {}^t\Omega = \Omega,\ \mathrm{Im}(\Omega) \gg 0\}$ ← Siegel upper space

(ii) $(A_{\Omega_1}, \lambda_{\Omega_1}) \cong (A_{\Omega_2}, \lambda_{\Omega_2})$ iff $\exists \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$

preserves polarizations

i.e. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix} = \begin{pmatrix} I_g & 0 \\ 0 & I_g \end{pmatrix}$

such that $(A\Omega_1 + B)\cdot(C\Omega_1 + D)^{-1} = \Omega_2$

1.5. Moduli space

Idea / phenomenon:

- The set of all isomorphism classes of all algebraic varieties of a given type (esp. with fixed discrete invariants) often has a natural structure as an algebraic variety.
- A subvariety of such a moduli space corresponds to an algebraic family of algebraic varieties of the given type

Ex. The set of all isomorphism classes of elliptic curves is parametrized by $\mathbb{A}^1$, $E \mapsto j(E)$

Ex. $g \geq 2$. $\mathcal{M}_g$ = the moduli space classifying all proper smooth algebraic curves of genus $g \geq 2$

* Ex. $\mathcal{A}_g$ = the moduli space of g-dimensional principally polarized abelian varieties

(Existence of $\mathcal{M}_g$ and $\mathcal{A}_g$ as schemes over $\mathbb{Z}$: Mumford 1965)

Over $\mathbb{C}$:

$$\mathcal{M}_g(\mathbb{C}) = {}_{\Gamma_g}\backslash^{\mathcal{T}_g}$$

$\mathcal{T}_g$ = Teichmüller space of genus g

Γ_g = mapping class group for an oriented connected smooth closed surface of genus g

$$\mathcal{A}_g(\mathbb{C}) = {}_{Sp_{2g}(\mathbb{Z})}\backslash^{\mathfrak{h}_g}$$

Remark

$$\begin{array}{cccc} T_g : & \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ & \Psi & & \Psi \\ & [C] & \longmapsto & [Jac(C)] \end{array}$$

Torelli map

$T_g(k): \mathcal{M}_g(k) \hookrightarrow \mathcal{A}_g(k)$

for every algebraically closed field k

$/\mathbb{C}$: Torelli 1914

general k: Weil 1957

Over an arbitrary algebraically closed field k:

- $\mathcal{M}_g/k$ is irreducible
  - char$(k)=0$ follows from the case $k=\mathbb{C}$
    case $k=\mathbb{C}$: immediate from uniformization
  - char$(k)=p>0$ Deligne-Mumford 1969

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§2 Hecke symmetry on $\mathcal{A}_g$

2.1 Definitions

Complex version: (transcendental) $\forall \gamma \in Sp_{2g}(\mathbb{Q})$, $\quad Sp_{2g}(\mathbb{Z}) \cdot \overset{\text{double coset}}{\gamma} \cdot Sp_{2g}(\mathbb{Z})$ induces an algebraic correspondence on $\underbrace{Sp_{2g}(\mathbb{Z})}_{\Gamma(1)} \backslash \mathfrak{h}_g$:

$$\begin{tikzcd}
\Gamma(1)\backslash \mathfrak{h}_g & \mathfrak{h}_g \arrow[l, two heads, "pr"'] \arrow[r, "\gamma \cdot"] \arrow[dl, "pr"] & \mathfrak{h}_g \arrow[d] \\
(\Gamma(1) \cap \gamma^{-1} \cdot \Gamma(1) \cdot \gamma)\backslash \mathfrak{h}_g \arrow[u] \arrow[rr, "\gamma \cdot"'] & & \Gamma(1)\backslash \mathfrak{h}_g
\end{tikzcd}$$

<u>Rmk</u>: These algebraic correspondences are "remnants" of the transitive action of $Sp_{2g}(\mathbb{R})$ on $\mathfrak{h}_g$, after quotient by $\Gamma(1) = Sp_{2g}(\mathbb{Z})$

algebraic version:

<u>Def</u> $[(A_1,\lambda_1)]$, $[(A_2,\lambda_2)] \in \mathcal{A}_g(k)$ are in the same (prime-to-p) <u>Hecke orbit</u> if $\exists$ an isogeny $\alpha: A_1 \longrightarrow A_2$ and $n \in \mathbb{Z}_{>0}$ (with $\gcd(n,p)=1$ if $\mathrm{char}(k)=p>0$) such that $\alpha^*(\lambda_2) = n\lambda_1$

$\alpha_2^t \circ \lambda_2 \circ \alpha$

adelic picture: $\mathbb{A}_f^{(p)} := \prod'_{\substack{\ell \text{ prime} \\ \ell \neq p}} \mathbb{Q}_\ell$ (restricted product)

$k \supseteq \mathbb{F}_p$ alg. closed

Let $\mathcal{A}_{g,n/k}$ = moduli space of triples $(A, \lambda, A[n] \underset{\text{symplectic}}{\xleftarrow{\sim}} (\mathbb{Z}/n\mathbb{Z})^{2g})$

$$Sp_{2g}(\mathbb{A}_f^{(p)}) \circlearrowright \tilde{\mathcal{A}}_{g/k}^{(p)} = \varprojlim_{\gcd(n,p)=1} \mathcal{A}_{g,n/k}$$

much bigger than $Sp_{2g}(\hat{\mathbb{Z}}^{(p)})$

$$\tilde{\mathcal{A}}_{g/k}^{(p)} \xrightarrow{\quad} \mathcal{A}_g$$

Galois with group $Sp_{2g}(\hat{\mathbb{Z}}^{(p)})$ $\qquad \hat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$

prime-to-p Hecke orbits on $\mathcal{A}_g$ $\leftrightsquigarrow$ $Sp_{2g}(\mathbb{A}_f^{(p)})$-orbits on $\tilde{\mathcal{A}}_{g/k}^{(p)}$

2.2. p-adic invariants of abelian varieties

$k = \bar{k} \supseteq \mathbb{F}_p$

(*) Every prime-to-p symplectic isogeny between principally polarized abelian varieties over k preserve all p-adic invariants

Examples of p-adic invariants

(a) slopes / Newton polygon of an abelian variety A/k

- compare $Fr_A^{(p)}: A \to A^{(p)}$ and its iterates $Fr_A^{(p^n)}: A \to A^{(p^n)}$

slopes = p-adic valuation of "eigenvalues" of $Fr_A^{(p)}$

("eigenvalues" do not make sense)

$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2g} \leq 1 \qquad \lambda_i \in \mathbb{Q} \ \forall i \qquad \lambda_i + \lambda_{2g+1-i} = 1 \ \forall i$

$\text{denominator}(\lambda_i) \mid \text{multiplicity}(\lambda_i) \qquad \forall i$

Properties of slopes and Newton polygons

- They measure asymptotic divisibility properties of the action of $Fr^{(p^n)}$ on $H^1_{cris}(A)$ as $n \to \infty$
- A is ordinary $\underset{def}{\Longleftrightarrow}$ slopes are 0 and 1

  ordinary abelian varieties of genus g form an open dense subset of $\mathcal{A}_{g/k}$
- A is supersingular $\Longleftrightarrow$ all slopes are $1/2$

  most special Newton polygon

  i.e. asymptotically $Fr^{(p^n)}$ is comparable to $p^{\frac{n}{2}}$ on $H^1_{cris}(A)$

Ex. (b) isomorphism class of $(A[p], \lambda[p])$

Ker$([p]: A \to A)$

(c) isomorphism class of $\left(A[p^\infty],\ \lambda[p^\infty]\right) \dashleftarrow\dashrightarrow$ "$H_1(A)$"

$\varinjlim_n A[p^n]$ $\qquad$ $A[p^n] := \mathrm{Ker}\left([p^n]: A \to A\right)$

This is the "ultimate p-adic invariant" of an abelian variety

$\rightsquigarrow$ leaves in $\mathcal{A}_g$

Thm (CLC, 1995) $\forall x = [(A,\lambda)] \in \mathcal{A}_g(k)$ with A ordinary, the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{A}_g$.

Q. What is the general phenomenon when we zoom in to the non-ordinary locus of $\mathcal{A}_g$?

Will see: new structure emerges

2.3 p-divisible groups

Tate 1967, Grothendieck 1970

Def^n: A <u>p-divisible group</u> $X \to S$ is an inductive system of commutative finite locally free group schemes

$$\left((X_n \to S)_{n \in \mathbb{N}} \, . \; i_{n+1,n} : X_n \hookrightarrow X_{n+1} \, . \quad \pi_{n,n+1} : X_{n+1} \xrightarrow{\text{faithfully flat}} X_n \right)$$

such that $\quad i_{n+1,n} \circ \pi_{n+1} = [p]_{X_{n+1}} \quad \forall n$

<u>Fact</u> $\exists\, h : S \to \mathbb{N}$, locally constant, such that $\mathrm{rk}(X_n) = p^{nh} \;\; \forall n$

Main example: $A \to S$ abelian scheme $\rightsquigarrow$ $A[p^\infty] = \varinjlim A[p^n]$ is a p-divisible group

- May think of $A[p^\infty]$ as a substitute for Lie algebra in characteristic $p>0$

2.4 Leaves in $\mathcal{A}_g$

Def^n (Oort, 1999) $\underset{[(A,\lambda)]}{\overset{}{x}} \in \mathcal{A}_g(k)$, $k = \bar{k} \supseteq \mathbb{F}_p$

The leaf $\mathcal{C}(x)$ through x is the locally closed subvariety such that

whose existence requires a finiteness property

$$\mathcal{C}(x)(k) = \left\{ (B,\mu) \in \mathcal{A}_g(k) \;\middle|\; (B[p^\infty], \mu[p^\infty]) \cong (A[p^\infty], \lambda[p^\infty]) \right\}$$

<u>Fact</u> Every leaf in $\mathcal{A}_g$ is smooth, and stable under all prime-to-p Hecke correspondences

<u>Conj</u>. (Oort) Let $\mathcal{C}$ be a leaf in $\mathcal{A}_g$. For every $x \in \mathcal{C}(k)$, the prime-to-p Hecke orbit of x is Zariski dense in $\mathcal{C}$.

Remark (i) This conjecture can be formulated also for moduli spaces of PEL type in characteristic $p > 0$, which classify abelian varieties with a fixed type of polarization, endomorphisms and level structure

(ii) The Hecke orbit conjecture holds for $\mathcal{A}_g$ (Oort + CLC) and more generally for moduli spaces of PEL-type C such that p is unramified for the PEL-structure.

Hower the proof uses a special property ("Hilbert trick"), and the conjecture is completely open for PEL-types A and D.

3. New tools, structures and conjectures/predictions/phenomena related to Hecke symmetry

$k = \bar{k} \supseteq \mathbb{F}_p$

3.1. Monodromy and irreducibility results

Proposition A. Let $Z \subset \mathcal{A}_g$ be a positive dimensional locally closed subvariety stable under all prime-to-p Hecke correspondences. If Hecke operates transitively on $\pi_0(Z)$, then Z is irreducible

(reducing irreducibility to Hecke transitivity)

Proposition B. Let $C \subseteq \mathcal{A}_g$ be a positive dimensional leaf on $\mathcal{A}_g$. then the naive p-adic monodromy for C is maximal

Prop. C. Every non-supersingular Newton stratum in $\mathcal{A}_g$ is irreducible

Prop. D. Every non-supersingular leaf in $\mathcal{A}_g$ is irreducible

[A: CLC ; B, C.D : C + Oort]

Note: a leaf in $\mathcal{A}_g$ is supersingular iff it is finite

3.2. Local structure of leaves

- simplest case (2 slopes)

$\mathcal{A}_g \supseteq \mathcal{C} \ni x_0 = [(A_0, \lambda_0)]$ $\quad k = \bar{k} \supseteq \mathbb{F}_p$. slopes of $A_0 = \{\lambda, 1-\lambda\}$
$\lambda < \frac{1}{2}$

Proposition: $\mathcal{C}^{/x_0}$ = the formal completion of $\mathcal{C}$ at x_0 has a natural structure as an isoclinic p-divisible group with slope $\underset{(1-\lambda)-\lambda}{1-2\lambda}$ and height $\underset{\dim(\mathcal{A}_g)}{g(g+1)/2}$.

Remark: general case: $\mathcal{C}^{/x_0}$ is "built up" from p-divisible formal groups by a sequence of fibrations (with p-divisible groups as fibers"

not assuming that A_0 has only two slopes

3.3. Local stabilizer principle $\qquad k = \bar{k} \supseteq \mathbb{F}_p$

Proposition. Let $Z \subseteq \mathcal{A}_g$ be a locally closed subvariety, stable under all prime-to-p Hecke correspondences. $x_0 = [(A_0, \lambda_0)] \in Z(k)$. Then $Z^{/x_0} \subseteq \mathcal{A}_g^{/x_0}$ is stable under the natural action of an open subgroup of $U(\mathrm{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p)$ on $\mathcal{A}_g^{/x_0}$.

unitary group $\qquad$ semi-simple algebra with involution

Explanation:

Serre - Tate theorem

$$\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty]) \text{ acts on } \mathcal{A}_g^{/x_0} = \mathrm{Def}(A_0, \lambda_0) \xrightarrow{\sim} \mathrm{Def}(A_0[p^\infty], \lambda_0[p^\infty])$$

$$\cup$$

$$U(\mathrm{End}(A_0), *_{\lambda_0})$$

via functoriality of deformation theory

Rosati involution on $\mathrm{End}(A_0) \otimes \mathbb{Q}$

3.4. Rigidity

$k = \bar{k} \supseteq \mathbb{F}_p$

Theorem (CLC, local rigidity) X: p-divisible formal group over k

$Z \subseteq X$ irreducible formal subvariety.

Suppose $\exists$ a subgroup $G \subset \mathrm{Aut}(X)$ such that $X^G =$ trivial and

subgroup of X fixed by G

Z is stable under G. Then Z is a p-divisible formal subgroup of X

$\hat{\mathbb{G}}_m^h = \mathrm{Spf}(k[[T_1, \dots, T_h]])$

"Exer" Case $X = \hat{\mathbb{G}}_m^h$ group law of $\hat{\mathbb{G}}_m$ given by $k[[T_1, \dots, T_h]] \xrightarrow{\psi} k[[u_1, \dots, u_h, v_1, \dots, v_h]]$

$T_i \longmapsto u_i + v_i + u_i v_i$.

$G = \mathbb{Z}_p^\times$

$Z \longleftrightarrow$ a prime ideal $P \subseteq k[[T_1, \dots, T_h]]$

Statement is: If P is stable under $f(T_1, \dots, T_h) \longmapsto f((1+T_1)^{1+p^2} - 1, \dots, (1+T_h)^{1+p^2} - 1)$

then $\psi(P) \subseteq (pr_1^*(P), pr_2^*(P))$

$pr_1^* f(\underline{t}) = f(\underline{u})$, $pr_2^* f(\underline{t}) = f(\underline{v})$

Simple application: E_0: an ordinary elliptic curve $/k = \bar{k} \supseteq \mathbb{F}_p$

$A_0 = \underbrace{E_0 \times \cdots \times E_0}_{g\text{-times}}$ λ_0 = product polarization on A_0

Then the prime-to-p Hecke orbit of x_0 is dense in $\mathcal{A}_g$

Pf: $\mathcal{A}_g^{/x_0} \cong \widehat{\mathbb{G}}_m^{\,g(g+1)/2}$, $U(\mathrm{End}(A_0), *_{\lambda_0})(\mathbb{Z}_p) \cong GL_g(\mathbb{Z}_p)$

character group

action of $GL_g(\mathbb{Z}_p)$ on $X^*(\widehat{\mathbb{G}}_m^{\,g(g+1)/2}) \cong \mathbb{Z}_p^{\,g(g+1)/2}$

$\cong S^2$(standard representation of $GL_g(\mathbb{Z}_p)$ on $\mathbb{Z}_p^g$)

Global rigidity Conjecture

Suppose $Z \subset \mathcal{A}_g^{ord}$ ← ordinary locus in $\mathcal{A}_g$, $x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g^{ord}(k)$, $k = \bar{k} \supseteq \mathbb{F}_p$

assume: $Z^{/x_0} \subset \mathcal{A}_g^{/x_0}$ = Serre-Tate formal torus is a formal subtorus.

Then Z is the reduction of a Shimura subvariety of $\mathcal{A}_g$

Remark: Known if $Z \subset$ a Hilbert modular subvariety (CLC)

This case has application in Iwasawa theory (Hida, Ann. Math 2012) a special case of the

Local rigidity conjecture

G_0: 1-dimensional smooth formal group over $\overline{\mathbb{F}}_p$, $ht(G_0)=h$

i.e. $\mathrm{slope}(G_0)=\frac{1}{h}$

$\mathcal{M}$ = equi-characteristic deformation space of G_0

$\cong \mathrm{Spf}(\overline{\mathbb{F}}_p[[x_1,\dots,x_{h-1}]])$ $\qquad x_1$ = Hasse invariant

Lubin-Tate 1966

$Z \subseteq \mathcal{M}$ irreducible formal subscheme, $x_1|_Z \neq 0$ (i.e. Z is generically ordinary)

If Z is stable under the natural action of an open subgroup of $\mathrm{Aut}(G_0)$, $\cong$ group of units in a central division algebra over $\mathbb{Q}_p$ with Brauer invariant $1/h$

then $Z=\mathcal{M}$

3.5. New notion : sustained p-divisible group

Motivation : Find a good (scheme-theoretic) definition of leaves

Remark : The original definition of leaves is "pointwise"; shortcomings include difficulty with deformation theory

<u>Definition</u> (Oort + C.) κ = a field of characteristic $p>0$.

$X_0/_\kappa$: p-divisible group over κ $\quad$ $S/_\kappa$: scheme over κ

A p-divisible group $X \longrightarrow S$ is <u>strongly κ-sustained</u> modeled on $X_0/_\kappa$ if

$$\underline{\mathrm{Isom}}_S(X_0[p^n]\times S,\ X[p^n]) \longrightarrow S \text{ is faithfully flat } \forall_{n\in\mathbb{N}}.$$

scheme of isomorphisms