Siegel Modular Varieties and Hecke symmetry

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Goal: survey Hecke symmetry on the moduli space of (principally polarized) abelian varieties of characteristic $p > 0$ and related rigidity phenomenon.
* history:
  - elliptic curves → curves of higher genera
  - abelian varieties → moduli spaces of abelian varieties

* phenomena and structures in characteristic \( p > 0 \);
  predictions (= conjectures)

* new tools/methods applicable to other problems
§ 1. From elliptic curves to abelian varieties and their moduli

1.1. What is an elliptic curve? Several approaches

(a) algebra \[ E : \{ y^2 = 4x^3 - g_2x - g_3 \} , \quad \Delta := g_2^3 - 27g_3^2 , \quad j = 1728 \frac{g_2^3}{\Delta} \]

(b) geometry \[ E(\mathbb{C}) \leftarrow \text{Lie}(E)/H_1(E(\mathbb{C}),\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} \tau + \mathbb{Z} \quad \tau \in \mathcal{H}_2 = \{ \tau \in \mathbb{C} \mid \Im(\tau) > 0 \} \]

\[ p \rightarrow \int_p^{\infty} \frac{dx}{y} \]

(c) analysis

\[ g(z; \tau) = \frac{1}{z^2} + \sum_{y \in \Lambda_{\tau}}' \left[ \frac{1}{(z-y)^2} - \frac{1}{y^2} \right] \]

\[ \left( \frac{d}{dz} g(z; \tau) \right)^2 = 4g(z; \tau)^3 - g_2(\Lambda_{\tau})g(z; \tau) - g_3(\Lambda_{\tau}) , \]

\[ g_2(\Lambda_{\tau}) = 60 \cdot \sum_{y \in \Lambda_{\tau}}' \frac{1}{y^4} , \quad g_3(\Lambda_{\tau}) = 140 \sum_{y \in \Lambda_{\tau}}' \frac{1}{y^6} \]
1.2. The origin of elliptic curves

A. (Diophantine equation)

- Fermat: \( x^4 - y^4 = z^2 \) has no nontrivial rational solution
  (infinite descent - actually a 2-descent: w.r.t. \( 2 = (1 + \sqrt{-1})(1 - \sqrt{-1}) \))
  - often treated in college-level number theory

- Gauss (last entry of Gauss's mathematical diary, 1814)

\[ \begin{align*}
E_{\text{aff}} := \{ & \, 1 = x^2 + y^2 + x^2 y^2 \} \\
 & (a + \sqrt{-b}) \cdot \mathbb{Z}[\sqrt{-1}] \text{ prime ideal} \\
\text{The congruence} & \quad 1 \equiv x^2 + y^2 + x^2 y^2 \pmod{(a + \sqrt{-b})^3} \\
\text{has} & \quad (a - 1)^2 + b^2 \text{ solutions, including the 4 solutions at } \infty:
\end{align*} \]

\((x = \infty, y = \pm \sqrt{a}), (x = \pm \sqrt{a}, y = \infty)\)
B. (elliptic integral,

December 1751, paper by Fagnano reached Euler in Berlin)

- Fagnano: \( \frac{dx}{\sqrt{1-x^4}} = \frac{dy}{\sqrt{1-y^4}} \) has rational solutions

  i.e. \( \int_0^x \frac{dp}{\sqrt{1-p^4}} = \int_0^y \frac{dy}{\sqrt{1-y^4}} \) admits solution where \( y = a \) rational function of \( x \)

- Euler

  \[ \int_0^r \frac{dp}{\sqrt{1-p^4}} = \sqrt{2} \int_0^t \frac{d\xi}{\sqrt{1+\xi^4}}, \quad \int_0^t \frac{d\xi}{\sqrt{1+\xi^4}} = \sqrt{2} \int_0^u \frac{d\eta}{\sqrt{1-\eta^4}} \]

  \[ r^2 = \frac{2t^2}{1+t^4}, \quad t^2 = \frac{2u^2}{1-u^4} \]

  \[ \int_0^r \frac{dp}{\sqrt{1-p^4}} \uparrow (1 \pm \sqrt{-1}) \int_0^v \frac{d\eta}{\sqrt{1-\eta^4}} \]

  \( r = \pm \frac{2\sqrt{1}v^2}{1-v^4} \)
inversion of abelian integrals

for \( y^2 = f(x) \) i.e. for hyperelliptic curves

Abel 1827, Jacobi 1828
Jacobi 1829, Fundamenta Nova Theoriae Functionum Ellipticarum
(defined Jacobi theta functions)
Compact Riemann surfaces and their Jacobians

\[ S = C(C) \text{ cpt Riemann surface; } \gamma_1, \ldots, \gamma_2g \text{ } \mathbb{Z}-\text{basis of } H_1(S, \mathbb{Z}) \]

\[ \omega_1, \ldots, \omega_g : \text{ } \mathbb{C}-\text{basis of } \Gamma(S, \Omega^1_S) \]

\[ \Delta = (\gamma_i : \gamma_j) \in \text{M}_g(\mathbb{Z}) \]

\[ P = P(\omega_1, \ldots, \omega_g; \gamma_1, \ldots, \gamma_2g) = (P_{ij})_{1 \leq i \neq j \leq 2g} \in \text{M}_{g \times 2g}(\mathbb{C}), \]

\[ P_{ij} = \int_{\gamma_j} \omega_i \]

Riemann bilinear relations

\[ P \cdot \Delta^{-1} \cdot tP = 0 \]

\[ -\sqrt{f} \cdot P \cdot \Delta^{-1} \cdot tP \gg \omega_g \]

Torelli map:

\[ C \longrightarrow \text{Pic}^1(C) \]

\[ \text{Pic}^0(C) = \text{Jac}(C) = \Gamma(C, \Omega^1)^\vee / H_1(C(\omega), \mathbb{Z}) \]
1.4. Abelian varieties

Def (i) (over $\mathbb{C}$), a compact complex torus \(\mathbb{C}^g/\mathbb{Q} \cdot \mathbb{Z}^g\), \(Q \in \text{M}_{g \times g} (\mathbb{C})\) is a complex abelian variety iff \(\exists\) a skew-symmetric \(E \in \text{M}_{2g} (\mathbb{Z})\) with \(\det (E) \neq 0\) satisfying
\[
\begin{align*}
Q \cdot E^{-1} \cdot tQ &= 0 \\
\sqrt{-1} \cdot Q \cdot E^{-1} \cdot tQ &> 0
\end{align*}
\]

"principal part of $Q$"

(i)' (equivalent to (i)) a compact complex torus is an abelian variety iff it admits a holomorphic embedding into \(\mathbb{P}^N(\mathbb{C})\) for some $N$

Weil 1948

(ii) (algebraic definition) An irreducible algebraic group variety over a field is an abelian variety if it is complete (i.e., proper over the base field)
Def \(^{n}\) (polarization of abelian varieties)

(i) A **polarization** of an abelian variety \( A \) is an ample divisor on \( A \) up to algebraic equivalence.

(ii) The polarization of an abelian variety \( A \) attached to an ample divisor \( D \) on \( A \) is **principal** if \( D^3 = g! \) (self-intersection \( g \) times).

(iii) The polarization on \( A \) attached to \( D \) is uniquely determined by the algebraic homomorphism:

\[
\phi_D: A \longrightarrow A^t = \text{Pic}^0(A) = \text{dual abelian variety}, \text{classifying line bundles on } D \text{ algebraically equivalent to } 0
\]

\[x \mapsto [O_A(D-x)]\]

(iv) Over \( \mathbb{C} \), the fundamental class \( c([D]) \in H^2(A(\mathbb{C}), \mathbb{Z}(1)) \) corresponds to a non-singular skew-symmetric pairing on \( H_1(A(\mathbb{C}), \mathbb{Z}) \) satisfying the Riemann bilinear relations (i.e. a **Riemann form**).

A polarization \([D]\) is principal iff the Riemann form is a perfect pairing \(\mathbb{Z}\).
Over $\mathbb{C}$:

(i) Every principally polarized abelian variety of dimension $g$ over $\mathbb{C}$ is of the form

$$A_\Omega := \mathbb{C}^g / \Omega \cdot \mathbb{Z}^g + \mathbb{Z}^g$$

with principal part $\begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$

for some $\Omega \in \mathfrak{H}_g := \{ \Omega \in \mathrm{M}_g(\mathbb{C}) \mid \Omega = \Omega^+, \text{ Im} \Omega > 0 \}$, $\text{Siegel upper space}$

(ii) $(A_{\Omega_1}, \lambda_{\Omega_1}) \sim (A_{\Omega_2}, \lambda_{\Omega_2})$ if and only if $\exists \begin{pmatrix} A & B \\ CD & D \end{pmatrix} \in \mathfrak{H}_g(\mathbb{Z})$

such that

$$(A_{\Omega_1} + B) \cdot (C \Omega_1 + D)^{-1} = \Omega_2$$

i.e.

$$\begin{pmatrix} A & B \\ CD & D \end{pmatrix} \begin{pmatrix} \Omega_1 & \Omega_1 \\ \frac{\Omega_1}{C} & \frac{\Omega_1}{A} \end{pmatrix} = \begin{pmatrix} \Omega_2 & \Omega_2 \\ \frac{\Omega_2}{C} & \frac{\Omega_2}{A} \end{pmatrix}$$
1.5. Moduli space

Idea / phenomenon:

- The set of all isomorphism classes of all algebraic varieties of a given type (with fixed discrete invariants) often has a natural structure as an algebraic variety.
- A subvariety of such a moduli space corresponds to an algebraic family of algebraic varieties of the given type

Ex. The set of all isomorphism classes of elliptic curves is parametrized by $\mathbb{A}^1$, $E \mapsto j(E)$
Ex. \( g > 2 \), \( M_g = \) the moduli space classifying all proper smooth algebraic curves of genus \( g > 2 \)

*Ex. \( A_g = \) the moduli space of \( g \)-dimensional principally polarized abelian varieties

(Existence of \( M_g \) and \( A_g \) as schemes over \( \mathbb{Z} \): Mumford 1965)
Over $\mathbb{C}$:

$$M_g(\mathbb{C}) = \Gamma_g \backslash J_g$$

$$A_g(\mathbb{C}) = \mathbb{H}/\mathbb{Z} \backslash \mathbb{H}_g$$

$J_g$ = Teichmüller space of genus $g$

$\Gamma_g$ = mapping class group for an oriented connected smooth closed surface of genus $g$

**Remark**

$\tilde{T}_g : M_g \rightarrow A_g$

$[C] \rightarrow [\text{Jac}(C)]$

$T_g(k) : M_g(k) \rightarrow A_g(k)$

For every algebraically closed field $k$

$\tilde{T}_g$ = Torelli 1914

General $k$: Weil 1957
Over an arbitrary algebraically closed field \( k \):

- \( M_g/k \) is irreducible
  - \( \text{char}(k) = 0 \): follows from the case \( k = \mathbb{C} \)
  - \( \text{char}(k) = p > 0 \): immediate from uniformization

- \( A_g/k \) is irreducible
  - \( \text{char}(k) = 0 \): follows from the case \( k = \mathbb{C} \)
  - \( \text{char}(k) = p > 0 \): Faltings - C. 1984

Deligne-Mumford 1969
§2 Hecke symmetry on $\mathbb{A}_g$

2.1 Definitions

Complex version: $\forall \gamma \in \text{Sp}_{2g}(\mathbb{Q}), \quad \text{Sp}_{2g}(\mathbb{Z}) \cdot \gamma \cdot \text{Sp}_{2g}(\mathbb{Z})$ induces an algebraic correspondence on $\mathbb{P}^g_{\mathbb{Q}}$.

(transcendental)

Double coset

Rmk: These algebraic correspondences are "remnants" of the transitive action of $\text{Sp}_{2g}(\mathbb{R})$ on $\mathbb{P}^g_{\mathbb{Q}}$, after quotient by $\Gamma(1) = \text{Sp}_{2g}(\mathbb{Z})$. 
Def: \([[(A_1, \lambda_1)], [(A_2, \lambda_2)]] \in \mathcal{A}_g(k)\) are in the same (prime-to-\(p\)) Hecke orbit if \(\exists\) an isogeny \(\alpha: A_1 \to A_2\) and \(n \in \mathbb{Z}_{>0}\) (with \(\gcd(n, p) = 1\) if \(\text{char}(k) = p > 0\)) such that \(\alpha^*(\lambda_2) = n \lambda_1\).

Adelic picture: \(\mathcal{A}_g^{(p)} := \prod_{l \neq p} \mathbb{Q}_l (\text{restricted product})\) Let \(\mathcal{A}_{g,n/k} = \text{moduli space of triples}\) \((A, \lambda, A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^g)\) symplectic

\[\text{Sp}_2g(\mathcal{A}_g^{(p)}) \supset \text{Sp}_2g(\mathcal{A}_g) \xrightarrow{\text{Gal of group}} \text{Sp}_2g(\mathcal{A}_g^{(p)}) \xrightarrow{\hat{\mathcal{Z}}^{(p)}} \prod_{l \neq p} \mathbb{Z}_l\]

prime-to-\(p\) Hecke orbits on \(\mathcal{A}_g \leftrightarrow \text{Sp}_2g(\mathcal{A}_g^{(p)})\)-orbits on \(\hat{\mathcal{A}}_g^{(p)}\)
2.2. $p$-adic invariants of abelian varieties

\[ \ell_c = \overline{\ell_c} \geq 1 \overline{F_p} \]

(*) Every prime-to-$p$ symplectic isogeny between principally polarized abelian varieties over $\mathbb{F}_p$ preserve all $p$-adic invariants

Examples of $p$-adic invariants

(a) slopes / Newton polygon of an abelian variety $A/k$

- compare $F_{A}^{(p)} : A \to A^{(p)}$ and its iterates $F_{A}^{(p^n)} : A \to A^{(p^n)}$

  slopes = $p$-adic valuation of "eigenvalues" of $F_{A}^{(p)}$

  ("eigenvalues" do not make sense)

  \[ 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2g} \leq 1 \]

  \[ \lambda_i \in \mathbb{Q} \quad \forall \ i \]

  \[ \lambda_i + \lambda_{g+1-i} = 1 \quad \forall \ i \]

  denominator $(\lambda_i)$ | multiplicity $(\lambda_i)$ \quad \forall \ i.
Properties of slopes and Newton polygons

- They measure asymptotic divisibility properties of the action of $F_{p^n}$ on $H^1_{	ext{crys}}(A)$ as $n \to \infty$

- $A$ is ordinary $\iff$ slopes are 0 and 1
  
  ordinary abelian varieties of genus $g$ form an open dense subset of $A_g/k$

- $A$ is supersingular $\iff$ all slopes are $\frac{1}{2}$
  
  i.e. asymptotically $F_{p^n}$ is comparable to $p^n$ on $H^1_{	ext{crys}}(A)$

Ex. (b) Isomorphism class of $(A, \chi_{[p]})$

$\uparrow$

$\text{Ker } ([p]: A \to A)$
(c) isomorphism class of \((A[p^\infty], \lambda[p^\infty]) \hookrightarrow \cdots \rightarrow H_1(A)\)

\[
\lim_{\rightarrow n} A[p^n] = A[p^n] = \ker([p^n]: A \to A)
\]

This is the "ultimate p-adic invariant" of an abelian variety

\[\sim \text{ leaves in } A_g\]
Thm (CLC, 1995) \( \forall x = [(A, \lambda)] \in A_g(k) \) with \( A \) ordinary, the prime-to-\( p \) Hecke orbit of \( x \) is Zariski dense in \( A_g \).

Q. What is the general phenomenon when we zoom in to the non-ordinary locus of \( A_g \)?

Will see; new structure emerges
2.3 \( p \)-divisible groups

Tate 1967, Grothendieck 1970

**Defn:** A \( p \)-divisible group \( X \to S \) is an inductive system of commutative finite locally free group schemes

\[
\left( (X_n \to S)_{n \in \mathbb{N}}, \ i_{n+1,n} : X_n \hookrightarrow X_{n+1}, \ \pi_{n+1,n} : X_{n+1} \longrightarrow X_n \right)_{\text{faithfully flat}}
\]

such that \( i_{n+1,n} \circ \pi_{n+1} = [p]_{X_{n+1}} \forall n \)

**Fact** \( \exists h : S \to \mathbb{N}, \text{ locally constant, such that } \text{rk} (X_n) = p^{nh} \forall n \)

**Main example:** \( A \to S \) abelian scheme \( \Rightarrow A[p^\infty] = \varinjlim A[p^n] \) is a \( p \)-divisible group

- May think of \( A[p^\infty] \) as a substitute for Lie algebra in characteristic \( p > 0 \)
2.4 Leaves in $\tilde{A}_g$

**Def** (Oort, 1999) $x \in \tilde{A}_g(k)$, $k = \overline{k} \cong H_p
$

The leaf $C(x)$ through $x$ is the locally closed subvariety such that

$$C(x)(k) = \left\{ (B, \mu) \in \tilde{A}_g(k) \mid (B[p^\infty], \mu[p^\infty]) \cong (A[p^\infty], \lambda[p^\infty]) \right\}$$

**Fact** Every leaf in $\tilde{A}_g$ is smooth, and stable under all prime-to-$p$ Hecke correspondences.

**Conj.** (Oort) Let $C$ be a leaf in $\tilde{A}_g$. For every $x \in C(k)$, the prime-to-$p$ Hecke orbit of $x$ is Zariski dense in $C$.
Remark (i) This conjecture can be formulated also for moduli spaces of PEL type in characteristic $p > 0$, which classify abelian varieties with a fixed type of polarization, endomorphisms and level structure.

(ii) The Hecke orbit conjecture holds for $Ag_p$ and more generally for moduli spaces of PEL-type $C$ such that $p$ is unramified for the PEL-structure.

However, the proof uses a special property ("Hilbert trick"), and the conjecture is completely open for PEL-types $A$ and $D$. 

\[ (\text{Oort} + \text{CLC}) \]
3. New tools, structures and conjectures/predictions/phenomena related to Hecke symmetry

\[ \ell_c = \ell_c \geq 1 \ell_p \]

3.1. Monodromy and irreducibility results

**Proposition A.** Let \( Z \subseteq \mathbb{A}^g \) be a positive dimensional locally closed subvariety stable under all prime-to-\( p \) Hecke correspondences. If Hecke operates transitively on \( \pi_0(Z) \), then \( Z \) is irreducible (reducing irreducibility to Hecke transitivity).

**Proposition B.** Let \( C \subseteq \mathbb{A}^g \) be a positive dimensional leaf on \( \mathbb{A}^g \). Then the naive \( p \)-adic monodromy for \( C \) is maximal.
Prop. C. Every non-supersingular Newton stratum in $\mathcal{A}_g$ is irreducible.

Prop. D. Every non-supersingular leaf in $\mathcal{A}_g$ is irreducible.

[$A: CLE \ ; \ B,C,D: C+O_{art}$]

Note: a leaf in $\mathcal{A}_g$ is supersingular iff it is finite.
3.2. Local structure of leaves

- Simplest case (2 slopes)

\[ A_g \cong C \cong x_0 = [(A_0, x_0)] \quad \text{for } k = \mathbb{F}_p. \]  
Slopes of \( A_0 = \{ \lambda, 1-\lambda \} \) \( \lambda < \frac{1}{2} \)

Proportion: \( C^{/x_0} = \text{the formal completion of } C \text{ at } x_0 \)

- Has a natural structure as an isoclinic \( p \)-divisible group with

  slope \( 1-2\lambda \) and height \( g(g+1)/2 \) \( \text{dim} (A_g) \)

Remark: General case: \( C^{/x_0} \) is "built up" from \( p \)-divisible formal groups

not assuming that \( A_0 \) has only two slopes

by a sequence of fibrations (with \( p \)-divisible groups as fibers)
3.3 Local stabilizer principle \( \mathfrak{g} = \mathfrak{g} \geq 1_{F_p} \)

Proposition. Let \( Z \subseteq A_g \) be a locally closed subvariety, stable under all prime-to-\( p \) Hecke correspondences, \( \mathfrak{X}_0 = \{(A_0, \lambda_0) \in Z(k)\} \).

Then \( Z^{/\mathfrak{X}_0} \subseteq A_g^{/\mathfrak{X}_0} \) is stable under the natural action of an open subgroup of \( U(\text{End}(A_0), \mathfrak{X}_0)(\mathbb{Z}_p) \) on \( A_g^{/\mathfrak{X}_0} \).

Explanation:

\[ \text{Aut}(A_0[p^\infty], \lambda_0[p^\infty]) \] acts on \( A_g^{/\mathfrak{X}_0} = \text{Def}(A_0, \lambda_0) \xrightarrow{\sim} \text{Def}(A_0[p^\infty], \lambda_0[p^\infty]) \) via functoriality of deformation theory.

\[ U(\text{End}(A_0), \mathfrak{X}_0) \] via Rosati involution on \( \text{End}(A_0) \otimes \mathbb{Q} \)

Senary-Tate theorem
3.4. Rigidity

\[ \overline{\kappa} = \overline{\kappa} \geq 1_p \]

Theorem (CIC, local rigidity) \( X \) : \( p \)-divisible group over \( \kappa \)

\( Z \leq X \) irreducible formal subvariety.

Suppose \( \exists \) a subgroup \( G \subset \text{Aut}(X) \) such that \( X^G = \) trivial and \( \hat{L} \), subgroup of \( X \) fixed by \( G \)

\( Z \) is stable under \( G \). Then \( Z \) is a \( p \)-divisible formal subgroup of \( X \)

"Exer" Case \( X = \hat{\Gamma}^h \)

\[ \hat{\Gamma}^h = \text{Spf}(\overline{\kappa}[T_1, \ldots, T_h]) \]

\[ \hat{\Gamma}^h = \mathbb{Z}_p^x \]

\( Z \) \( \nrightarrow \) a prime ideal \( \mathfrak{P} \subset \overline{\kappa}[T_1, \ldots, T_h] \)

Statement is: If \( \mathfrak{P} \) is stable under \( f(T_1, \ldots, T_h) \mapsto f((1+T_1)^{i+T_2}, \ldots, (1+T_h)^{i+T_h}) \)

then \( \psi(\mathfrak{P}) \leq (pr_1^*(\mathfrak{P}), pr_2^*(\mathfrak{P})) \)

\[ pr_1^*(f(t)) = f(u), \quad pr_2^*(f(t)) = f(v) \]
Simple application: \( E_0 \) : an ordinary elliptic curve /\( k = \bar{\mathbb{F}}_p \)
\( A_0 = E_0 \times \cdots \times E_0 \quad \chi_0 = \) product polarization on \( A_0 \) \( g \)-times

Then the prime-to-\( p \) Hecke orbit of \( \chi_0 \) is dense in \( A_g \)

**Pf:** \( A_g^{/\chi_0} \cong \hat{\mathbb{G}}_m^{g(g+1)/2} \). \( U(\text{End}(A_0), \chi_0)(\mathbb{Z}_p) \cong \text{GL}_g(\mathbb{Z}_p) \)

character group

action of \( \text{GL}_g(\mathbb{Z}_p) \) on \( X^*(\hat{\mathbb{G}}_m^{g(g+1)/2}) \cong \mathbb{Z}_p^{g(g+1)/2} \)

\( \cong S^g \text{ (standard representation of } \text{GL}_g(\mathbb{Z}_p) \text{ on } \mathbb{Z}_p^g \)
Global rigidity Conjecture

Suppose \( Z \subset \text{Ag}^{\text{ord}} \), \( x_0 = [(A_0, \lambda_0)] \in \text{Ag}^{\text{ord}}(k) \), \( k = \overline{k} \cong \mathbb{Q}_p \)

\[ Z^{x_0} \subset \text{Ag}^{x_0} = \text{Serre-Tate formal torus} \text{ is a formal subtorus}. \]

Then \( Z \) is the reduction of a Shimura subvariety of \( \text{Ag} \).

Remark: Known if \( Z \subset \) a Hilbert modular subvariety \( (\Sigma \mathcal{L} \Sigma) \)

This case has application in Iwasawa theory \( (\text{Hida, Ann. Math. 2012}) \)
a special case of the Local rigidity conjecture

\[ G_0 : \text{1-dimensional smooth formal group over } \overline{\mathbb{F}}_p, \; \text{ht}(G_0) = h \]

\[ M = \text{equi-characteristic deformation space of } G_0 \]

\[ \cong \text{Spf}(\overline{\mathbb{F}}_p \llbracket x_1, \ldots, x_{h-1} \rrbracket) \]

\[ x_1 = \text{Hasse invariant} \]

Lubin-Tate 1966

\[ Z \subseteq M \text{ irreducible formal subscheme, } x_1 |_Z \neq 0 \] (i.e. \( Z \) is generically ordinary)

If \( Z \) is stable under the natural action of an open subgroup of \( \text{Aut}(G_0) \),

then \( Z = M \)

\[ \cong \text{group of units in a central division algebra over } \mathbb{Q}_p \text{ with Brauer invariant } \frac{1}{h} \]
3.5. New notion: sustained $p$-divisible group

Motivation: Find a good (scheme-theoretic) definition of leaves

Remark: The original definition of leaves is "pointwise"; shortcomings include difficulty with deformation theory

Definition (Oort + C.) $\kappa = a$ field of characteristic $p > 0$.

$X_0/\kappa$: $p$-divisible group over $\kappa$

$S/\kappa$: scheme over $\kappa$

A $p$-divisible group $X \rightarrow S$ is strongly $\kappa$-sustained modeled on $X_0/\kappa$ if

$\text{Isom}_S(X_0[p^n] \times S, X[p^n]) \rightarrow S$ is faithfully flat for all $n \in \mathbb{N}$.

scheme of isomorphisms