10/31/2012

Algebraic interpretation of modular forms

§1. Notation

$$GL_+ = \{(\omega_1,\omega_2)\in\mathbb{C}^2 \mid \mathrm{Im}(\omega_2/\omega_1)>0\}$$

$$\mathcal{L} = \{\text{lattices } \Lambda\subseteq\mathbb{C}\} \longleftrightarrow \left\{(E,\omega) \,\middle|\, \begin{array}{l} E:\text{ elliptic curve}/\mathbb{C} \\ \omega = \text{generator of } H^0(E,\Omega^1_{E/\mathbb{C}})\end{array}\right\}\Big/\text{isom}$$

$$\begin{tikzcd}
 & GL_+ \arrow[dl, "\mathbb{C}^*"'] \arrow[dr, "SL_2(\mathbb{Z})"] & \\
\mathfrak{h} \arrow[dr] & & \mathcal{L} \arrow[dl] \\
 & SL_2(\mathbb{Z})\backslash\mathfrak{h} &
\end{tikzcd}$$

$(\omega_1,\omega_2) \mapsto \frac{\omega_2}{\omega_1} = \tau \in \mathfrak{h}$; $\quad \mathcal{L} \ni L$

$(\omega_1,\omega_2) \mapsto (\omega_1,\omega_2)\cdot\begin{pmatrix} d & b\\ c & a\end{pmatrix}$: right action

Note $\begin{pmatrix} d & b\\ c & a\end{pmatrix} = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\cdot{}^t\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$

$SL_2(\mathbb{Z})\backslash\mathfrak{h}$ ← "bare" modular curve

Recall Weierstraß theory: 12

$$x = \wp(z,L) = \frac{1}{z^2} + \sum_{0\neq\ell\in L}\left[\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}\right] \qquad L\in\mathcal{L}$$

$$y = \frac{d}{dz}\wp(z,L)$$

$$g_2 = 60\sum_{0\neq\ell\in L}\frac{1}{\ell^4}, \qquad g_3 = 140\sum_{0\neq\ell\in L}\frac{1}{\ell^6}$$

$$\rightsquigarrow \mathcal{L} \xrightarrow{\sim} \{(g_2,g_3)\in\mathbb{C}^2 \mid g_3^2 - 27g_2^3 \neq 0\} = \mathrm{Spec}\,\mathbb{C}[g_2,g_3,(g_3^2-27g_2^3)^{-1}](\mathbb{C})$$

Over $\mathcal{L}$, has an elliptic curve $\mathcal{E}/_{\mathcal{L}}$ with Weierstraß eq^n

$$\left(y^2 = 4x^3 - g_2\cdot x - g_3,\ \frac{dx}{y}\right)$$

Let $\eta := x\frac{dx}{y} = \wp(z,L)\,dz$

3

Legendre period relation: $\boxed{\det \begin{pmatrix} \omega_1 & \omega_2 \\ \eta(\omega_1;L) & \eta(\omega_2;L) \end{pmatrix} = 2\pi\sqrt{-1}}$

$\eta(\omega_i;L)$ = periods of η

Weistraß zeta

$$\zeta(z,L) = \frac{1}{z} + \sum_{0\neq \ell\in L} \left[\frac{1}{z+\ell} + \frac{z}{\ell^2} - \frac{1}{\ell}\right], \quad -d\zeta = \eta$$

$$\underset{\substack{\Cap \\ H^1_{dR}(E)}}{[\eta]} = \left[\ell \longmapsto \int_\ell \eta =: \eta(\ell,L) = \zeta(z,L) - \zeta(z+\ell,L)\right] \in \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{C})$$

reformulation: $\langle\omega,\eta\rangle_{top} = 2\pi\sqrt{-1}$ equiv. $\langle\omega,\eta\rangle_{DR} = 1$ $\left(\langle\,,\rangle_{DR} = \frac{1}{2\pi\sqrt{-1}}\langle\,,\rangle_{top}\right)$

Recall: Serre's def^n of $\langle\,,\rangle_{dR}$: On a curve $C/\mathbb{C}$ (or over a field)

α: dfk, β: dsk $\quad \langle\alpha,\beta\rangle_{dR} = -\sum_{P\in C} \mathrm{Res}_P(f_P\cdot\alpha) \quad f_P \in \hat{\mathcal{O}}_P,\ df_P = \beta$

4

Recall: On a Riemann surface C, a meromorphic differential is a differential of the 2nd kind if it is locally exact, i.e. all residues are 0.

$$H^1_{DR}(C) = H^1(C,\Omega^\bullet) \xrightarrow{\sim} \{\text{differential of the } 2^{nd} \text{ kind}\}/\{\text{exact}\}$$

$C = U_1 \cup U_2$ Čech cover

cocycles:

$(\omega_1,\omega_2,f_{12})$ $\quad \omega_i \in \Gamma(U_i,\Omega^1_{U_i})$

$\omega_1 - \omega_2 + df_{12} = 0 \quad f_{12}\in\Gamma(U_1\cap U_2, \mathcal{O}_{U_1\cap U_2})$ $\quad$ ($\rightsquigarrow$ ω_1,ω_2 are dsk)

$(\omega_1,\omega_2,f_{12}) \longmapsto \omega_1$

§2 Modular forms

modular forms of wt $k \in \mathbb{Z}$

(i) $F : GL_+ \longrightarrow \mathbb{C}$ s.t. $F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} F(\omega_1, \omega_2)$ $\forall \lambda \in \mathbb{C}^\times$ $\forall (\omega_1, \omega_2) \in GL_+$

↳ holomorphic, or C^∞

(More generally, of weights $(k, s) \in \mathbb{Z} \times \mathbb{C}$ if $F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} \cdot |\lambda|^{-2s} \cdot F(\omega_1, \omega_2)$)

(ii) $f : \mathfrak{h} \longrightarrow \mathbb{C}$ $\quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subsetneq \Gamma(1)$

congruence subgroup

($\rightsquigarrow F(\omega_1, \omega_2) := \omega_1^{-k} f(\omega_2/\omega_1)$ is of weight k and Γ-invariant)

$(i)_{alg}$ $(E, \omega) \longmapsto F(E, \omega)$ s.t. $F(E, \lambda\omega) = \lambda^{-k} F(E, \omega)$ $\forall \lambda \in R^\times$

E: elliptic curve over R

$R \cdot \omega = H^0(E, \Omega^1_{E/R})$

↑ depending only on the isomorphism class of (E, ω) and functorial w.r.t. base change $R \rightarrow R'$

(iii) $\mathcal{M}_\Gamma \overset{\pi}{\underset{\varepsilon}{\leftrightarrows}} \mathcal{E}_\Gamma$ universal elliptic curve $/\mathcal{M}_\Gamma$. $\underline{\omega} = \pi_* \Omega^1_{\mathcal{E}} = \varepsilon^* \Omega^1_{\mathcal{E}}$

a global section of $\underline{\omega}^{\otimes k}$

($F(E_R, \omega_{E/R}) \cdot \omega_{E/R}^{\otimes k}$ in the situation of $(i)_{alg}$)

§3 The q-expansion principle

q-expansion: $q = e^{2\pi\sqrt{-1}\tau}$

$$F(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = \sum_{n \in \frac{1}{N}\cdot\mathbb{Z}} a_n \cdot q^n$$

If F is invariant by $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$

i.e. $F(\omega_1, \omega_2 + N\omega_1) = F(\omega_1, \omega_2)$

algebraic interpretation of the q-expansion

via the Tate curve $(\mathrm{Tate}(q), \omega_{can})/_{\mathbb{Z}((q))}$ (or $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R$ R: a ground ring)

$\mathrm{Tate}(q) = $ " $\mathbb{G}_m / q^{\mathbb{Z}}$ " ω_{can} = descended from the invariant differential dt/t on $\mathbb{G}_m$

t: coord. on $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{Z}((q))[t, t^{-1}]$

Tate curve$/_{\mathbb{C}}$: $\mathbb{C}/(2\pi\sqrt{-1}\mathbb{Z} + 2\pi\sqrt{-1}\tau)$ over $\{ q \in \mathbb{C}^\times \mid 0 < |q| < 1 \}$

(holomorphic version) $\uparrow$ Tate(q) $q = e^{2\pi\sqrt{-1}\tau}$

$$\mathrm{Tate}(q): \quad X^2 = 4X^3 - g_2(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau)X - g_3(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau)$$

$$g_2(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = \frac{1}{12}\Big(1 + 240 \sum_{n\geq 1} \sigma_3(n)\, q^n\Big), \qquad \sigma_k(n) = \sum_{\substack{d|n \\ d\geq 1}} d^k$$

$$g_3(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = -\frac{1}{216}\Big(1 - 504 \sum_{n\geq 1} \sigma_5(n)\, q^n\Big)$$

Consider: Tate(q) over $\mathfrak{h} \ni \tau$

$X = \wp(w; 2\pi\sqrt{-1}\mathbb{Z} + 2\pi\sqrt{-1}\tau)$ $\omega_{can} = dw$

$Y = \frac{d}{dw}\wp(w; 2\pi\sqrt{-1}\mathbb{Z} + 2\pi\sqrt{-1}\tau)$ $\eta_{can} = \wp(w; 2\pi\sqrt{-1}\mathbb{Z} + 2\pi\sqrt{-1}\tau)\cdot dw$

classical convention $x = \wp(z; \tau\cdot\mathbb{Z} + \mathbb{Z})$

$y = \frac{d}{dz}(z; \tau\mathbb{Z} + \mathbb{Z})$ " $w = 2\pi\sqrt{-1}z$ "

q-expansion principle

q-expansion = evaluation at Tate(q)

("ignore" level structures)

$\mathcal{M}/_{\mathbb{Z}}$ modular curve (for some unspecified level structure)

$R \subseteq \mathbb{C}$ a subring

Thm:

A modular form $f(\tau)\cdot \omega_{can}^{\otimes k}$ of weight k has all Fourier coefficients in R iff it comes from an element of $\Gamma(\mathcal{M} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,R, \underline{\omega}^{\otimes k})$.

§4 $\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M})$ and the canonical splitting

Given any (E, ω) over $R \supseteq \mathbb{Z}[\frac{1}{6}]$,

$\exists$ unique $X, Y \in \Gamma(E \setminus \{\text{zero section}\}, \mathcal{O})$ $\quad X \in \Gamma(\mathcal{O}(2\infty))$, $Y \in \Gamma(\mathcal{O}(3\infty))$

write "∞" for this divisor $\quad \mathrm{ord}_\infty(X) = -2$, $\mathrm{ord}_\infty(Y) = 3$

s.t. $\begin{cases} Y^2 = 4X^3 - g_2 X - g_3 & \text{for some } g_2, g_3 \in R \\ \text{and } \omega = dX/Y & (\rightsquigarrow g_2, g_3 \text{ are then specified}) \end{cases}$

This construction also gives a splitting of $\quad \simeq \omega_{E/R}^{\otimes -1}$

$$0 \to \underline{\omega}_{E/R} \longrightarrow \underline{H}^1_{DR}(E/R) \longrightarrow \underline{H}^1(E, \mathcal{O}_E) \to 0$$

This splitting is canonical (functorial for isomorphisms of elliptic curves) but not functorial for homomorphism between elliptic curves

$$\mathbb{H}^1(\mathcal{O}_E \to \Omega^1_{E/R}) \xrightarrow{\sim} \mathbb{H}^1(\mathcal{O}_E(\infty) \longrightarrow \Omega^1_{E/R}(2\infty))$$
$$\xrightarrow{\sim} \operatorname{coker}\left(H^0(\mathcal{O}_E(\infty)) \to H^0(\Omega^1_{E/R}(2\infty))\right) \xrightarrow{\sim} H^0(\Omega^1_{E/R}(2\infty))$$
$$\because \ H^1(\mathcal{O}_E(\infty)) = 0 = H^1(\Omega^1_{E/R}(2\infty))$$

$\rightsquigarrow$ get, for (E,ω),

$$H^1_{DR}(E/R) \xleftarrow{\sim} \underline{\omega}_{E/R} \oplus \underline{\omega}_{E/R}^{\otimes -1} \qquad \text{canonical splitting}$$

$$a\omega + b\eta \longleftarrow a\omega + b\omega^{-1}$$

Note: The R-submodul $R\cdot\eta \subseteq H^1_{DR}(E/R)$ is independent of the choice of the R-basis ω of $H^0(\Omega^1_{E/R})$. i.e. the above splitting is functorial w.r.t isomorphisms between elliptic curves

§5 $\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M})$ and the Gauss-Manin connection

$\exists$ algebraic construction (omitted)

classical convention: $E_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ $\qquad \gamma_1 \longleftrightarrow \tau$ $\qquad z$: coord on $\mathbb{C}$

$\gamma_2 \longleftrightarrow 1$

$$\int_\gamma \nabla_\tau(\xi) = \frac{d}{d\tau}\int_\gamma \xi \qquad \forall \gamma \in \mathbb{Z}+\mathbb{Z}\tau$$
$$\forall \xi \in H^1_{DR}(E/\text{(holomorphic functions of } \tau)})$$

$$\nabla\left(\frac{d}{d\tau}\right)\begin{pmatrix}\omega\\ \eta\end{pmatrix} = \frac{-1}{2\pi\sqrt{-1}}\begin{pmatrix}\eta_2 & -1\\ (\eta_2)^2 - 2\pi\sqrt{-1}\cdot\frac{d}{d\tau}\eta_2 & \eta_2\end{pmatrix}\cdot\begin{pmatrix}\omega\\ \eta\end{pmatrix}$$

$$\omega = \frac{dx}{y}, \ \eta = \frac{x\,dx}{y}$$

Recall classical formulas:

$\omega_i = \int_{\gamma_i} \omega, \quad \eta_i = \int_{\gamma_i} \eta$

$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi\sqrt{-1}$

Note: Here we have a reversal of the ordering of ω_1, ω_2 compared with the convention for GL^+: Here $\mathrm{Im}(\omega_1/\omega_2) = \mathrm{Im}(\tau) > 0$.

$(\iff \langle \omega, \eta \rangle_{top} = 2\pi\sqrt{-1})$

i.e.

$$\begin{cases} \eta_1 - \tau\eta_2 = 2\pi\sqrt{-1} \\ \eta_2(\tau) = -\sum\limits_{m\in\mathbb{Z}} \sum\limits_{\substack{n\in\mathbb{Z} \\ n\neq 0 \text{ if } m=0}} \dfrac{1}{(m\tau+n)^2} = -\dfrac{\pi^2}{3} P, \\ \eta_1(\tau) = -\sum\limits_{n\in\mathbb{Z}} \sum\limits_{m\neq 0 \text{ if } n=0} \dfrac{\tau}{(m\tau+n)^2} \\ \eta_2(-\frac{1}{\tau}) = \tau\eta_1(\tau) = \tau^2 \cdot \eta_2(\tau) + 2\pi\sqrt{-1}\tau \end{cases}$$

where $P = 1 - 24 \sum\limits_{n\geq 1} \sigma_1(n) q^n$ $\quad (q = e^{2\pi\sqrt{-1}\tau})$

equiv. $P(-\frac{1}{\tau}) = \tau^2 P(\tau) - \dfrac{6\sqrt{-1}\tau}{\pi}$

"almost a weight-two modular form"

Rewrite the Gauss-Manin connection for Tate(q):

$$\nabla(\theta) \begin{pmatrix} \omega_{can} \\ \eta_{can} \end{pmatrix} = \begin{pmatrix} -\frac{P}{12} & 1 \\ -\frac{(P^2 - 12\theta(P))}{144} & \frac{P}{12} \end{pmatrix} \cdot \begin{pmatrix} \omega_{can} \\ \eta_{can} \end{pmatrix}$$

$\theta = q\dfrac{d}{dq} \quad \left(= \dfrac{1}{2\pi\sqrt{-1}} \dfrac{d}{d\tau}\right)$

$\omega_{can} = 2\pi\sqrt{-1}\,\omega = \dfrac{dt}{t}$

$\eta_{can} = \dfrac{1}{2\pi\sqrt{-1}} \eta$

$\omega = \text{"}dz\text{"}, \quad \omega_{can} = \text{"}dw\text{"}$

$t = e^{2\pi\sqrt{-1}z} = e^{w}$

Kodaira-Spencer

$E_{/S/T} \qquad D \in \mathrm{Der}(S/T) \rightsquigarrow D \longmapsto \left(\omega^{\otimes 2} \longmapsto \langle \omega, \nabla(D)\omega \rangle_{DR}\right)$

$\mathrm{Der}(S/T) \xrightarrow{\sim} \underline{\omega}^{\otimes -2}$

$\Rightarrow \quad \omega_{can}^{\otimes 2} \xrightarrow{K.S.} \dfrac{dq}{q} \qquad \because \langle \omega_{can}, \nabla(\theta)\omega_{can} \rangle_{DR} = 1$

§6 The Gauss-Manin connection as a differential operator

$E/S/T \rightsquigarrow$

$t \in \mathbb{N}_{\geq 0}$

$$\begin{array}{ccc} \mathrm{Symm}^t\, \underline{H}^1_{DR}(E/S) & \longrightarrow & \mathrm{Symm}^t\, \underline{H}^1_{DR}(E/S) \otimes \Omega^1_{S/T} \cong \mathrm{Symm}^t\, \underline{H}^1_{DR}(E/S) \otimes \underline{\omega}^{\otimes 2} \\ \wr\wr & & \wr\wr \\ \bigoplus_{j=0}^{t} \underline{\omega}^{t-2j} & \xrightarrow{\nabla} & \bigoplus_{j=0}^{t} \underline{\omega}^{t+2-2j} \end{array}$$

Suppose: $a, b \in \mathbb{N}$, $0 \leq a, b \leq t$

f: modular form of wt $k = a-b$, canonically $f \cdot \omega_{can}^{\otimes a} \otimes \eta_{can}^{\otimes b} \in \Gamma(\mathrm{Symm}^t\, \underline{H}^1_{DR}(\mathrm{Tate}(q)))$

$$\boxed{\begin{aligned} \nabla(f) = {} & \left[\theta(f) - (a-b)\cdot \frac{P}{12} f\right] \cdot \omega_{can}^{\otimes a+2} \cdot \eta_{can}^{\otimes b} \\ & + (af) \cdot \omega_{can}^{\otimes a+1} \cdot \eta_{can}^{\otimes b+1} + \left[b \cdot \frac{-(P^2 - 12\theta(P))}{144} \cdot f\right] \cdot \omega_{can}^{\otimes a+3} \cdot \eta_{can}^{\otimes b-1} \end{aligned}}$$

$\theta = q\frac{d}{dq} = \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau}$

Consequences

1) $\boxed{P^2 - 12\theta(P) = \tilde{E}_4 \left(= 12\, g_2(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau)\right) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n}$

[Apply formula to $f=1$] The unique $\Gamma(1)$-form of wt 4 with const. term 0

2) $\boxed{P = \frac{\theta(\Delta)}{\Delta}}$ where $\Delta \left(= (\tilde{E}_4^3 - \tilde{E}_6^2)/1728\right)$, the unique normalized cusp form for $\Gamma(1)$ of wt. $=12$

$$\tilde{E}_6 = -216 \cdot g_3(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = 1 - 504 \sum_{n \geq 1} \sigma_5(n) \cdot q^n$$

3) f is a modular form of wt k

$\Longrightarrow$ $\theta(f) - k \cdot \frac{P}{12} \cdot f$ is a modular form of wt $k+2$

$$\boxed{\begin{array}{l} \text{Serre's operator } \partial \\ \partial: f \mapsto 12\theta(f) - k \cdot P \cdot f \end{array} = 12 \cdot \left(\text{the } \underline{\omega}^{\otimes k+2}\text{-component of}\right) \nabla}$$

"middle"

§7 How to think about "the" weight 2 Eisenstein series

Let $A_2(\omega_1,\omega_2) := \sum_{m\in\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0 \text{ if } m=0}} \frac{1}{m\omega_2+n\omega_1}$ $\quad (\omega_1,\omega_2)\in GL_+$ i.e. $\mathrm{Im}(\omega_2/\omega_1)>0$

$$= \left(\frac{2\pi\sqrt{-1}}{\omega_1}\right)^2 \cdot \Big(\underbrace{-\frac{1}{12} + 2\sum_{n\geq 1}\sigma_1(n)\, q^n}_{-\frac{1}{12}\cdot P(q)}\Big) \qquad q = e^{2\pi\sqrt{-1}\tau} = e^{2\pi\sqrt{-1}(\omega_2/\omega_1)}$$

$$\eta(\omega_1;\, \mathbb{Z}\omega_1+\mathbb{Z}\omega_2) = -\omega_1\cdot A_2(\omega_1,\omega_2)$$
$$\eta(\omega_2;\, \mathbb{Z}\omega_1+\mathbb{Z}\omega_2) = -\omega_2\, A_2(-\omega_2,\omega_1)$$

Classical notation
$A_2(\omega_1,\omega_2) = -\omega_1^2\cdot\eta_2(\tau)$
$\tau = \omega_2/\omega_1$

$$\boxed{\begin{aligned}\underline{\text{Define}}:\ -\frac{1}{12}S(\omega_1,\omega_2) &= A_2(\omega_1,\omega_2) - \frac{2\pi\sqrt{-1}\,\overline{\omega_1}}{\omega_1(\overline{\omega_1}\omega_2-\omega_1\overline{\omega_2})}\\ &\underset{\text{fact}}{=} \lim_{\varepsilon\to 0^+}\sum_{(n,m)\neq(0,0)}\frac{1}{(n\omega_1+m\omega_2)^2\cdot|n\omega_1+m\omega_2|^{2\varepsilon}}\end{aligned}}$$

Note: $\mathrm{Im}(\overline{\omega_1}\,\omega_2)$
$= \frac{1}{2\sqrt{-1}}(\overline{\omega_1}\omega_2-\omega_1\overline{\omega_2})$
$= \mathrm{area}(\mathbb{C}/\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$

$$-\frac{1}{12}S(2\pi\sqrt{-1},\, 2\pi\sqrt{-1}\tau) = -\frac{1}{12}P(q) + \frac{1}{4\pi y} = 2\cdot(\text{Shimura's } E_2(\tau))$$

This is a non-holomorphic Eisenstein series

$-\frac{1}{12}S(\omega_1,\omega_2)$ is invariant under $SL_2(\mathbb{Z})$

$\Longleftrightarrow$ (Legendre's period relation) $\boxed{A_2(\omega_1,\omega_2) - A_2(-\omega_2,\omega_1) = \frac{2\pi\sqrt{-1}}{\omega_1\omega_2}}$

period relation

$$\Longrightarrow\quad -\langle\overline{\omega},\eta\rangle_{top} = \underbrace{(\overline{\omega_1}\omega_2-\omega_1\overline{\omega_2})}_{\langle\overline{\omega},\omega\rangle_{top}}\cdot A_2(\omega_1,\omega_2) - \frac{2\pi\sqrt{-1}\,\overline{\omega_1}}{\omega_1}$$

$$\Longleftrightarrow\quad -\frac{1}{12}S(\omega_1,\omega_2) = -\frac{\langle\overline{\omega},\eta\rangle_{top}}{\langle\overline{\omega},\omega\rangle_{top}}$$

i.e. $\boxed{\overline{\omega} = a\omega + b\eta \quad \text{with } a/b = -\frac{1}{12}S(\omega_1,\omega_2)}$

§8 Ramanujan's series $P(q) = 1 - 24 \sum_{n\geq 1} \sigma_1(n)\, q^n$ as a p-adic modular form of weight 2 19

General construction: R: p-adically complete ring

$E_{/R}$ elliptic curve, $E \pmod p$ ordinary, $H \subseteq E[p]$ "canonical subgroup of order p"

$\pi : E_{/R} \twoheadrightarrow E/H =: E'$

Apply this to the universal situation: (on $\mathcal{M}_{n/W(\mathbb{F}_q)}^{ord}$)

$$R = \left\{ \begin{matrix} p\text{-adic modular functions of} \\ \text{level } n \text{ over } W(\mathbb{F}_q) \end{matrix} \right\}, \quad q \in p^{\mathbb{N}_{\geq 1}},\ q \equiv 1 \pmod n$$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\pi"] \arrow[dr] & \mathcal{E}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[d] \\ & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} R \\ & R & R \arrow[l, "\varphi"'] \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} H^1_{DR}(\mathcal{E}/R) & H^1_{DR}(\mathcal{E}'_{/R}) \arrow[l, "\pi^*"'] \simeq H^1_{DR}(\mathcal{E}/R)^{(\varphi)} & H^1_{DR}(\mathcal{E}/R) \arrow[l] \arrow[ll, bend left, "F(\varphi)"] \end{tikzcd}$$

$F(\varphi)$: φ-linear

for $0 \to \underline{\omega} \to H^1_{DR}(\mathcal{E}/R) \to H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}/R}) \to 0$

$F(\varphi)|_{\underline{\omega}} = P$. $F(\varphi)$ induces 1 on $H^1_{DR}(\mathcal{E}/R)/\underline{\omega} \cong H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}/R})$

Key 1) $F(\varphi)$ respects the Gauss-Manin connection (functoriality) 20

2) $\nabla(\theta)(\omega_{can}) = -\frac{P}{12}\omega_{can} + \eta_{can}$ is killed by $\nabla(\theta)$

on $\mathrm{Tate}(q)$ $\quad\therefore\ \nabla(\theta)$ is unipotent on this $rk = 2$ vector bundle

$$\Rightarrow F(\varphi)\big(\nabla(\theta)(\omega_{can})\big) = a \cdot \nabla(\theta)(\omega_{can})$$

$$\rightsquigarrow F(\varphi)(\eta_{can}) = \left(\frac{p \cdot \varphi(P) - P}{12}\right)\omega_{can} + \eta_{can}$$

$$\begin{pmatrix} F(\varphi)\ \omega_{can} \\ F(\varphi)\ \eta_{can} \end{pmatrix} = \begin{pmatrix} P & 0 \\ \frac{p\cdot\varphi(P) - P}{12} & 1 \end{pmatrix} \begin{pmatrix} \omega_{can} \\ \eta_{can} \end{pmatrix},$$

$$\begin{pmatrix} F(\varphi)\ \omega_{can} \\ F(\varphi)\big(\nabla(\theta)(\omega_{can})\big) \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_{can} \\ \nabla(\theta)(\omega_{can}) \end{pmatrix}$$

Frobenius splitting over $\mathcal{M}_n^{ord}/W(\mathbb{F}_q)$:

$$0 \to \underline{\omega} \longrightarrow \underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_n^{ord}/W(\mathbb{F}_q)) \longrightarrow \underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_n^{ord}/W(\mathbb{F}_q), \mathcal{O}_\mathcal{E}) \simeq \underline{\omega}^{-1} \to 0$$

$$\cup\!\!\!| \qquad\qquad U \xrightarrow{\sim} \underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_n^{ord}/W(\mathbb{F}_q), \mathcal{O}_\mathcal{E})$$

Both $\underline{\omega}$ and U are stable under $F(\varphi)$, of rank 1

"unit root part"

<u>Define</u>, for any $(E,\omega)/R$, R: p-adically complete, $E \pmod p$ ordinary

$$\boxed{-\frac{1}{12}\widetilde{P}((E,\omega)/R) \underset{\text{def}}{=} -\frac{\langle u,\eta\rangle_{DR}}{\langle u,\omega\rangle_{DR}} \quad \forall \text{ base } u \text{ of } U}$$

$-\frac{1}{12}\widetilde{P}$ is a p-adic modular form of weight 2

Evaluate $\widetilde{P}$ on $\mathrm{Tate}(q)$:

$$-\frac{1}{12}\widetilde{P}(\mathrm{Tate}(q),\omega_{can}) = \frac{-\langle\nabla(\theta)(\omega_{can}), \eta_{can}\rangle_{DR}}{\langle\nabla(\theta)(\omega_{can}), \omega_{can}\rangle_{DR}} = \frac{-\langle\frac{-P}{12}\omega_{can}+\eta_{can}, \eta_{can}\rangle_{DR}}{\langle\frac{-P}{12}\omega_{can}+\eta_{can}, \omega_{can}\rangle_{DR}}$$

$$= -\left(-\frac{P}{12}\right)\Big/ -1 = -\frac{P}{12}$$

<u>Conclude</u>: $-\frac{1}{12}\widetilde{P} = -\frac{1}{12}P$

Compare with: $-\frac{1}{12}S(\omega_1,\omega_2) = -\frac{\langle\bar{\omega},\eta\rangle_{DR}}{\langle\bar{\omega},\omega\rangle_{DR}}$

Compare notations (Shimura's notation)

$k \geq 4$, k even
$$E_k(\tau) = \frac{(k-1)!}{(2\pi\sqrt{-1})^k}\zeta(k) + \sum_{n\geq 1} \sigma_{k-1}(n)\, q^n \qquad q = e(\tau) = e(2\pi\sqrt{-1}\tau)$$

$$g_2(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = 20 \cdot \left(\frac{1}{240} + \sum_{n\geq 1}\sigma_3(n) q^n\right) = 20 \cdot E_4$$

$$g_3(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = \frac{7}{3}\left(-\frac{1}{504} + \sum_{n\geq 1}\sigma_5(n) q^n\right) = \frac{7}{3}E_6$$

$$E_2(\tau) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{n\geq 1}\sigma_1(n) q^n = -\frac{1}{24}S(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau)$$

$$-\frac{1}{12}S = -\frac{\langle \bar{\omega}, \eta\rangle}{\langle \bar{\omega}, \omega\rangle}, \quad -\frac{1}{12}P = \frac{\langle u, \eta\rangle}{\langle u, \omega\rangle} \qquad P = 1 - 24\sum_{n\geq 1}\sigma_1(n) q^n$$

u: base of $U \leftarrow$ unit root part of H^1_{DR}

§9 Nearly holomorphic modular forms

24

$$D_k f = (2\pi\sqrt{-1})^{-1}\cdot\left(\frac{\partial}{\partial\tau} + \frac{k}{2\sqrt{-1}y}\right)f \qquad f: \mathfrak{h} \to \mathbb{C},\ C^\infty\text{-function}$$

$$D_k^t := \underbrace{D_{k+2t-2}\circ\cdots\circ D_{k+2}\circ D_k}_{t\text{-times}} \quad \text{is } GL_2(\mathbb{R})_+\text{-invariant}:$$

$$D_k^t(f \|_k \gamma) = (D_k^t f)\|_{k+2t}\gamma \quad \forall \gamma \in GL_2(\mathbb{R})_+$$

Recall: $(f\|_k \gamma)(\tau) = \det(\gamma)^{k/2}\cdot(c\tau+d)^{-k} f(\gamma\cdot\tau)$, $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{R})$

Def (Shimura): $N_k^t(\Gamma) := \left\{ f: \mathfrak{h} \to \mathbb{C} \text{ smooth} \;\middle|\; f\|_k\gamma = f\ \forall\gamma\in\Gamma, \text{ and } \forall \xi \in GL_2(\mathbb{Q})\cap GL_2(\mathbb{R})_+\ (f\|_k\xi)(\tau) = \sum_{m=0}^{t}(\tau-\bar{\tau})^{-m}\cdot\sum_{n=0}^{\infty} c_{m,\xi,n}\, e(n\tau/N_\xi) \right\}$

$\Gamma(1)$ w/ cong. subgrp; $N_\xi > 0$

$\cup\!|$

$N_k^t(K, \Gamma) := \left\{ \text{Those } f: \mathfrak{h}\to\mathbb{C} \text{ in } N_k^t(\Gamma) \;\middle|\; f(\tau) = \sum_{a=0}^{t}(2\pi y)^{-a}\cdot\sum_{n=0}^{\infty} c_{a,n}\, e(n\tau/N) \right\}$ with all $c_{a,n} \in K$

For any subfield $K \subseteq \mathbb{C}$ (Use the standard cusp)

Facts about nearly holomorphic forms:

(1) $D_k^p : N_k^t(K,\Gamma) \longrightarrow N_{k+2p}^{t+p}(K,\Gamma)$ $\quad \forall K \subset \mathbb{C}$ subfield

(2) $E_2 = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)\, q^n \in N_2^1(\mathbb{Q},\Gamma)$

(3) $N_k(K,\Gamma) = \bigoplus_{0 \le b \le \frac{k}{2}} E_2^b \cdot M_{k-2a}(K,\Gamma)$

In particular $N_k^t(K,\Gamma) = 0 \quad \forall t > \frac{k}{2}$

(4) $D_k f + 2k E_2 f \in M_{k+2}(\bar{\mathbb{Q}},\Gamma) \quad \forall f \in M_k(\bar{\mathbb{Q}},\Gamma)$

(5) $N_k^t(K) = \bigcup_\Gamma N_k^t(K,\Gamma)$ is stable under $f \mapsto f\|_k \xi \quad \forall \xi \in GL_2(\mathbb{Q}) \cap GL_2(\mathbb{R})_+$

(6) $D_{ak+bl}(f^a \cdot g^b) = a f^{a-1} \cdot g^b \cdot (D_k f) + b f^a \cdot g^{b-1} \cdot (D_l g)$ $\quad f$: wt k $\quad g$: wt l

Ex. $D_2 E_2 = (\frac{5}{6}) \cdot E_4 - 2 \cdot E_2^2$ (p. 61 of Shimura, Elementary Dirichlet Series)

Modular interpretation of $N_k^t(\Gamma)$ $2t \le k$

Recall: $\bar{\omega}_{can} = a\omega_{can} + b\eta_{can}$ with $\frac{a}{b} = -\frac{1}{12} S(2\pi\sqrt{-1}, 2\pi\sqrt{-1}\tau) = 2 \cdot E_2$

$\leadsto \eta_{can} = -2E_2 \cdot \omega_{can} + (*) \cdot \bar{\omega}_{can}$

Consider: $\underline{\omega}^{\otimes k-t} \otimes \mathrm{Symm}^t(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma)) \cong \underline{\omega}^{\otimes k-t} \otimes \bigoplus_{b=0}^{t} (\underline{\omega}^{\otimes t-b} \otimes \underline{\omega}^{-b})$

modular curve

$\cong \bigoplus_{b=0}^{t} \underline{\omega}^{k-2b}$

$\Gamma(\underline{\omega}^{\otimes k-t} \otimes \mathrm{Symm}^t(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma))) \ni s = \sum_{b=0}^{t} f_b \cdot (\omega_{can}^{\otimes k-b} \cdot \eta_{can}^{\otimes b})$ $\quad f_b \in M_k(\Gamma)$

Let $\Pi_{Hdg}^{(k,0)}$ = projection to the Hodge $(k,0)$-component for $\mathrm{Symm}^k(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma))$

$j_{k,t}: \underline{\omega}^{\otimes k-t} \otimes \mathrm{Symm}^t(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma)) \longrightarrow \mathrm{Symm}^{\otimes k}(H^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma))$

$$\Pi^{(k,0)}_{Hdg} \circ j_{k,t} : \underset{\substack{\uparrow \\ M_{k-2b}(\Gamma)}}{f_b} \cdot (\omega_{can}^{\otimes k-b} \cdot \eta_{can}^{\otimes b}) \longmapsto (-2E_2)^b \cdot f_b \cdot \omega_{can}^{\otimes k} \qquad 0 \leq b \leq t$$

$$\rightsquigarrow \Pi^{(k,0)}_{Hdg} \circ j_{k,t} : \Gamma\left(\widetilde{M}_\Gamma, \underline{\omega}^{\otimes k-t} \otimes \mathrm{Symm}^t\left(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma)\right)\right) \xrightarrow{\sim} \mathcal{N}^t_k(\Gamma)$$

compactified modular curve — canonical ext^n of this

$\forall t \leq k/2$

Equiv.

$$\Pi^{(k,0)}_{Hdg} : \Gamma(\widetilde{M}_\Gamma, \mathrm{Symm}^k(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma))) \xrightarrow{\sim} \mathcal{N}_k(\Gamma) = \mathcal{N}^{\lfloor k/2 \rfloor}_k(\Gamma)$$

preserving the rationality structures $\forall$ subfield $K \subset \mathbb{C}$

(q-expansion principle for nearly holomorphic forms) rational structure from $\underline{H}^1_{DR}$ $\longleftrightarrow$ rational structure from q-expansion

Proposition (modular interpretation of Shimura's operator D_k)

$$\begin{array}{ccc} \Gamma(\widetilde{\mathcal{M}}_\Gamma, \mathrm{Symm}^k(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma))) & \xrightarrow{\nabla} & \Gamma(\widetilde{\mathcal{M}}_\Gamma, \mathrm{Symm}^k(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma)) \otimes \underline{\omega}^{\otimes 2}) \\ \Pi^{(k,0)}_{Hdg} \downarrow \simeq & & \simeq \downarrow \Pi^{(k+2,0)}_{Hdg} \\ \mathcal{N}_k(\Gamma) & \xrightarrow[D_k]{} & \mathcal{N}_{k+2}(\Gamma) \end{array}$$

where $D_k f = (2\pi\sqrt{-1})^{-1}\left(\frac{\partial}{\partial\tau} + \frac{k}{2\sqrt{-1}y}\right) f = \theta(f) - \frac{k}{4\pi y} f, \quad \theta = q\frac{d}{dq}$

Rmk For $f \in M_k(\Gamma)$, $D_k f + 2k\, E_2 f = \theta(f) - k \cdot \frac{P}{12} f \in M_{k+2}(\Gamma)$

(Revisit Fact 4 about nearly holomorphic forms) $\left(= \frac{1}{12} \cdot \partial f\right)$ — Serre's operator

Pf: $\Gamma(\widetilde{\mathcal{M}}_\Gamma, \mathrm{Symm}^k(\underline{H}^1_{DR}(\mathcal{E}/\mathcal{M}_\Gamma))) \cong \bigoplus_{b=0}^{\lfloor k/2 \rfloor} \Gamma(\widetilde{\mathcal{M}}_\Gamma, \underline{\omega}^{k-2b})$

$a = k-b$

For any $f_b \in \Gamma(\widetilde{\mathcal{M}}_\Gamma, \underline{\omega}^{k-2b})$, the corresponding section is $f_b \cdot (\omega_{can}^{k-b} \cdot \eta_{can}^{b})$

Know $\nabla : f_b (\omega_{can}^{a} \cdot \eta_{can}^{b}) \longmapsto \left[\theta(f_b) - (a-b)\frac{P}{12} f_b\right] \omega_{can}^{a+2} \cdot \eta_{can}^{b} + a f_b \cdot \omega_{can}^{a+1} \cdot \eta_{can}^{b+1}$

$+ \left(b \cdot \frac{-(P^2 - 12\theta(P))}{144} \cdot f_b\right) \omega_{can}^{a+3} \cdot \eta_{can}^{b-1}$

$\pi^{(k,0)}(LHS) = f_b \cdot (-2E_2)^b$

$\pi^{(k+2,0)}(RHS) = \left[\theta(f_b) - (a-b)\frac{P}{12} f_b\right] \cdot (-2E_2)^b + a \cdot f_b \cdot (-2E_2)^{b+1} + b \cdot f_b \cdot \left(-\frac{P^2 - 12\theta(P)}{144}\right) \cdot (-2E_2)^{b-1}$

$= \left\{\theta(f_b) + (a-b) \cdot \underbrace{\left[\frac{-P}{12} + (-2E_2)\right]}_{-\frac{1}{4\pi} y} \cdot f_b\right\} \cdot (-2E_2)^b + b \cdot f_b \cdot \underbrace{\left[(-2E_2)^2 - \frac{(P^2 - 12\theta(P))}{144}\right]}_{4E_2^2 - \frac{5}{3}E_4 = D_2(-2E_2)} \cdot (-2E_2)^{b-1}$

$\because \frac{P^2 - 12\theta(P)}{144} = \frac{1}{144} + \frac{240}{144} \sum_{n \geq 1} \sigma_3(n) q^n$

$= (D_{a-b} f_b) \cdot (-2E_2)^b + f_b \cdot D_{2b}(-2E_2)^b$

$= D_k(f_b \cdot (-2E_2)^b)$

QED

§10 Modular interpretation of some holomorphic Eisenstein series of wt ≥ 2

$k = \underline{wt \geq 3}$: Let $A_k(L) := \sum_{0 \neq \ell \in L} \frac{1}{\ell^k}$ $(= 0$ if k odd$)$ $\quad L \in \mathcal{L}$

& $\forall \ell_0 \in L_{\mathbb{Q}} \setminus L$. $\quad A_k(L, \ell_0) := \sum_{\ell \in L} \frac{1}{(\ell + \ell_0)^k}$

$$\begin{cases} \wp(z, L) = \frac{1}{z^2} + \sum_{n \geq 1} (n+1)!\, A_{n+2}(L) \frac{z^n}{n!} \\ A_k(L, \ell_0) = \frac{(-1)^k}{(k-1)!} \left(\frac{d}{dz}\right)^{k-2} \wp(z, L)\Big|_{z=\ell}, \end{cases}$$

$\frac{d}{dz}$ = the translation invariant derivation dual to ω

$k \geq 3$

Note

$\frac{1}{2} \cdot (-1)^k (k-1)! \cdot A_k(2\pi\sqrt{-1}\mathbb{Z} + 2\pi\sqrt{-1}\tau) = \frac{(k-1)!}{(2\pi\sqrt{-1})^k} \zeta(k) + \sum_{n \geq 1} \sigma_{k-1}(n) q^n = E_k(\tau)$

$\underline{wt = 2}$: $\ell_0 \in L_{\mathbb{Q}} \setminus L$

$A_2'(L, \ell_0) := \frac{1}{\ell_0^2} + \sum_{\ell \in L \setminus \{0\}} \left[\frac{1}{(\ell + \ell_0)^2} - \frac{1}{\ell^2}\right] = \wp(z, L)\Big|_{z = \ell_0}$

31

§11 Modular interpretation of some Eisenstein series of weight one

$$\zeta(z;L) = \frac{1}{z} + \sum_{\ell\in L-\{0\}}\left[\frac{1}{(z+\ell)} - \frac{1}{\ell} + \frac{z}{\ell^2}\right] \quad \text{Weistrass } \zeta$$

$$\eta = x\frac{dx}{y} = \wp(z;L)\,dz = -d\zeta(z,L)$$

Hecke: $A_1(L,\ell_0) := \zeta(\ell_0;L) + \frac{1}{N}\underbrace{\langle \eta, N\ell_0\rangle}_{\int_{N\cdot\ell_0}\eta = \zeta(z;L)-\zeta(z+N\ell_0)}$ Choose $\overset{\mathbb{N}}{\underset{0}{N}}$ s.t $N\cdot\ell_0\in L$

$\ell\in L_{\mathbb{Q}}\setminus L$

The modular interpretation uses

$$0\to \omega_{E/S} \to E^\dagger \xrightarrow[\pi]{} E \to 0,$$

$\omega_{E/S}$: vector group scheme $\omega_{E/S}(T) = H^0(E\times_S T, \Omega^1_{E\times_S T/T})$

$E^\dagger: T \rightsquigarrow \left\{\begin{array}{l}\text{isom. classes of } (\mathcal{L},\nabla) \\ \text{on } E\times_S T,\ \nabla = \text{integrable conn.} \\ \mathcal{L} = \text{invertible sheaf, } \deg \mathcal{L}_t = 0\ \forall t\in T\end{array}\right\}$

32

Recall: Lemma 1 $C/$ smooth curve, $D = \sum_i n_i P_i$ divisor, $\{P_i\}$ disjoint sections

Connections $\nabla: \mathcal{O}(D) \longrightarrow \mathcal{O}(D)\otimes_{\mathcal{O}_C}\Omega^1_{C/S}$ are of the form

$$g\cdot f \longmapsto f\,dg + g\cdot f\left(\frac{df}{f}+\omega\right) \qquad \begin{array}{l} f:\text{ local generator of } \mathcal{O}(D)\\ g\in\mathcal{O}_C\end{array}$$

where $\omega = \omega_\nabla$ = meromorphic 1-form with $\mathrm{Res}_{P_i}(\omega) = n_i\ \forall i$.

Write: $\nabla = \nabla_\omega$

Lemma 2 $(E,\omega)/_{S/\mathbb{Z}[\frac{1}{6}]}$ (ω generator), $P\in E^{\mathrm{aff}}$, then $\omega_P := \frac{1}{2}\frac{y+y(P)}{x-x(P)}\cdot\frac{dx}{y}$

satisfies $\mathrm{Res}_P(\omega_P) = 1$, $\mathrm{Res}_{\infty=0}(\omega_P) = -1$

This ω_P defines a point $[(\mathcal{O}(P-0), \nabla_{\omega_P})]\in\pi^{-1}(E^{\mathrm{aff}})$

$$P\longmapsto [(\mathcal{O}(P-0),\nabla_{\omega_P})] = \xi(P)\in\pi^{-1}(E^{\mathrm{aff}})$$

defines a section $\xi: E^{\mathrm{aff}}\to\pi^{-1}(E^{\mathrm{aff}})$ of $\pi: \pi^{-1}(E^{\mathrm{aff}})\to E^{\mathrm{aff}}$

$$\begin{array}{ccc}\pi^{-1}(E^{\mathrm{aff}}) & \to & E^{\mathrm{aff}}\\ \cap & & \cap \\ E^\dagger & \longrightarrow & E\end{array}$$

Lemma 3 : $\forall$ non-zero torsion point P of $E_{/S}$ of order N invertible in $\mathcal{O}_S$,
$\exists!$ point $P^{can} \in E^t$ of order N. $\left(\because [N] : \omega_{E/S} \xrightarrow{\sim} \omega_{E/S} \right)$

Definition / Construction

$$\boxed{(E_{/S}, \omega, \underset{\uparrow \text{non-trivial torsion point}}{P}) \longmapsto \frac{\xi(P) - P^{can}}{\omega}}$$

Note : $\xi(P) - P^{can} \in \underline{\omega}_{E/S}(S) = \mathcal{O}_S \cdot \omega$

"is" the transcendentally defined weight one form
$$(\underset{\underset{\mathfrak{L}}{\Uparrow}}{L}, \underset{\underset{L_{\mathbb{Q}} \setminus L}{\Uparrow}}{\ell_0}) \longmapsto A_1(L, L_0)$$