p-adic Modular Forms and Arithmetic
A conference in honor of
Haruzo Hida
on his 60th birthday

(I knew the truth in all I heard when I turned sixty. Confucious)
HECKE SYMMETRY, RIGIDITY AND THE LUBIN-TATE ACTION

Ching-Li Chai

Department of Mathematics
University of Pennsylvania
and
Institute of Mathematics
Academia Sinica

UCLA, June 22, 2012
Outline

1. Hecke symmetry on PEL moduli varieties
2. Local Hecke symmetry
3. The global rigidity problem
4. Local rigidity problems
5. Known results, obstacles and hope
6. A glimpse of a new approach
7. The Lubin-Tate action
8. Universal isomorphism $p$-typical formal group laws
9. Sketch of the steps
10. The first test
PEL type modular varieties

I. An overview

A PEL type modular variety $\mathcal{M}$ is the moduli space attached to a PEL input datum $\mathcal{D} = (D, \ast, \mathcal{O}_D, H, \langle \cdot, \cdot \rangle, h)$, whose points correspond to abelian varieties with imposed symmetry $(A, \rho : A \to A^t, \iota : \mathcal{O}_D \to \text{End}(A), \text{level structure})$ whose $H_1$ are modeled on the linear algebra structure $\mathcal{D}$.

Fix a prime number $p$, unramified for the PEL datum $\mathcal{D}$. We will focus on the equal characteristic $p$ situation unless otherwise specified: $\mathcal{M}$ is a moduli space over $\overline{\mathbb{F}}_p$.

Let $B = \text{End}_D(H)$, with involution $\ast_B$ induced by $\ast$. Let $G = \text{SU}(B, \ast_B)$ (or $\text{GU}(B, \ast_B)$). 

Hecke symmetry on PEL moduli varieties

Local Hecke symmetry

The global rigidity problem

Local rigidity problems

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Hecke symmetry

Let $\tilde{M}$ be the prime-to-$p$ tower for $M$; it is a profinite etale Galois cover of $M$ with group $G(\hat{\mathbb{Z}}(p))$. The group $G(\mathbb{A}_f^{(p)})$ operates on $\tilde{M}$, inducing Hecke correspondences on $M$.

**Example:** $M = \mathcal{A}_g$ = the moduli space classifying $g$-dimensional principally polarized abelian varieties, $G = \text{Sp}_{2g}$ (or $\text{GSp}_{2g}$).
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Local Hecke symmetry

Given a point $x \in \mathcal{M}(\overline{\mathbb{F}}_p)$, corresponding to a quadruple $(A_x, \rho_x : A_x \rightarrow A_x^t, \iota_x : \mathcal{O}_D \rightarrow \text{End}(A_x), \text{level structure})$.

Let $\mathcal{M}/x$ be the formal completion of $\mathcal{M}$ at $x$.

Let $H_x := \text{U}(\text{End}_D^0(A_x), *_{\text{Ros}})(\mathbb{Z}_p)$, and let $G_x := \text{U}(\text{End}_D^0(A_x[p^\infty]), *_{\text{Ros}})(\mathbb{Z}_p)$.

The Serre-Tate deformation theorem implies that there is a natural action of the compact $p$-adic group $G_x$ on $\mathcal{M}/x$, by “changing the marking”.

This action can be regarded as a local version of the global Hecke symmetries.
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Local stabilizer subgroups

We call $G_x$ the local stabilizer subgroup at $x$. The group $H_x$ can be thought of as the “intersection” of $G_x$ with the global Hecke symmetries on $\mathcal{M}$.

**Lemma.** If a closed subvariety $Z \subset \mathcal{M}$ is stable under all Hecke symmetries, then $Z/x \subset \mathcal{M}/x$ is stable under the action of the $p$-adic closure of $H_x$ in $G_x$.

**Examples.** For a “general” $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ (in particular $x$ is ordinary), the Zariski closure of $H_x$ is a $g$-dimensional torus, while the Zariski closure of $G_x$ is $\text{GL}_g$.

For a supersingular point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$, $H_x$ is $p$-adically dense in $G_x$, and the Zariski closure of $G_x$ is a twist of $\text{Sp}_{2g}$.
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The global rigidity problem

(Oort’s Hecke orbit conjecture)

**Prediction.** Let $Z \subset \mathcal{M}/\mathbb{F}_p$ be a reduced closed subset of $\mathcal{M}$ stable under all prime-to-$p$ Hecke correspondences. Then $Z$ contains the leaf $C(x)$ passing through $x$ for every point $x \in Z(\mathbb{F}_p)$.

(Every Hecke-invariant closed subset of $\mathcal{M}/\mathbb{F}_p$ is a union of leaves; the latter can be regarded as “generalized Shimura subvarieties in char. $p$”.)
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Definition and examples of leaves

- A leaf $C(x)$ in $\mathcal{M}/\overline{\mathbb{F}}_p$ is the locus in $\mathcal{M}/\overline{\mathbb{F}}_p$ where all $p$-adic invariants have the same “value” as those of $x$.

- The ordinary locus $\mathcal{A}_g^{\text{ord}} \subset \mathcal{A}_g/\overline{\mathbb{F}}_p$ is a leaf in $\mathcal{A}_g\overline{\mathbb{F}}_p$.

- The leaf passing through a supersingular point in $\mathcal{A}_g$ is finite.

- The leaf passing through a point in $\mathcal{A}_3$ corresponding to a 3-dimensional abelian variety with slopes $\{1/3, 2/3\}$ is two-dimensional. Such leaves form a one-dimensional family in the slopes $\{1/3, 2/3\}$ locus of $\mathcal{A}_3$.

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Strong forms of global rigidity problem

Remark. In application(s) to Iwasawa theory pioneered by Hida, certain strong versions of the global rigidity problem appear naturally:

- The assumption on $Z$ is weakened to: $Z$ is stable under the action of a “not-to-small” subset of Hecke correspondences.
- The desired conclusion is that $Z$ is a union of leaves in the reduction of certain Shimura subvarieties.

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Local rigidity problems

**Set-up.** $Z \subset \mathcal{M}/^x$ is a reduced closed formal subscheme of $\mathcal{M}/^x$, stable under the action of a “not-too-small” subgroup of $G_x$.

**Restricted** local rigidity problem (to make it easier):
Assume in addition that $Z \subset C(x)/^x$.

**Desired conclusion.** $Z$ has a (very) special form (e.g. defined by a finite collection of Tate cycles.)
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Results on the restricted local rigidity problem

II. Known results, obstacles and hope

Propostion. Restricted local rigidity holds for $A_g$, in the case when $A_x$ has only two slopes.

- $C(x)^/x$ has a natural structure as a torsor for an isoclinic $p$-divisible formal group $X_x$.
- If $Z \subset C(x)^/x$ is stable under a not-too-small subgroup of $G_x$, then $Z_x$ is a torsor for a $p$-divisible subgroup of $X_x$. 
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Restricted local rigidity: an example and consequences

**An example.** Let $Z$ be an irreducible formal subscheme of a formal torus $\hat{G}_m^r$ over $\overline{F}_p$. Suppose that $Z$ is closed under the action of $[1 + p^m]$ for some $m \geq 2$. Then $Z$ is a formal subtorus of $\hat{G}_m^r$. (exercise)

**Consequence of restricted local rigidity:** linearization of the global rigidity problem, helped by considerations of local and global $p$-adic monodromy.
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**Consequence** of restricted local rigidity: linearization of the global rigidity problem, helped by considerations of local and global $p$-adic monodromy.
**Theorem.** Global rigidity holds for $\mathcal{A}_g$.

**Remarks.** (1) Besides the restricted local rigidity and monodromy arguments, the proof uses a trick: Every point $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is contained in a Hilbert modular subvariety of $\mathcal{A}_g$. (Global rigidity is substantially easier for these “small” modular varieties; see below.)

(2) This “Hilbert trick” fails for PEL modular varieties of type A or D.

(3) A strong global rigidity statement holds for Hilbert modular varieties (and many other modular varieties associated to semisimple groups of $\mathbb{Q}$-rank one).
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A tantalizing dream

The **holy grail** for the rigidity problems (don’t have better leads):

To pry actionable intelligence out of the action of the local stabilizing subgroup.

**Main obstacle:** Our poor understanding of this action (so cannot deploy enhanced interrogation techniques).

- Don’t have helpful (exact or approximate) formulas (have tried Norman’s algorithm).
- Linearization via crystalline techniques leads to formulas with high powers of $p$ in denominators.
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To pry **actionable intelligence** out of the action of the local stabilizing subgroup.

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where $\mathcal{M}/x = \text{Def}(G_0)$ is the Lubin-Tate moduli deformation space for a one-dimensional formal group $G_0$ of finite height $h$ over $\overline{\mathbb{F}}_p$. 
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Let $h$ be a positive integer.

Let $G_1$ be the one-dimensional formal group over $\mathbb{Z}_p$ with logarithm

$$\sum_{j \in \mathbb{N}} p^{-j} x^{p^j} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots$$

(so it is a Lubin-Tate formal group for $W(\mathbb{F}_{p^h})$.)

Let $G_0$ be the base extension to $\overline{\mathbb{F}_p}$ of the closed fiber of $G_1$; it is a one-dimensional formal group over $\mathbb{F}_p$ of height $h$.

It is well-known that $\text{End}(G_0)$ is the maximal order of $\text{End}^0(G_0) = \text{a central division algebra over } \mathbb{Q}_p$ of dimension $h^2$. 

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Let $\mathcal{M}_h := \text{Def}(G_0)$; it is a smooth formal scheme over $W(\overline{\mathbb{F}_p})$ of relative dimension $h - 1$.

Let $G_{\text{univ}} \to \mathcal{M}_h$ be the universal formal group over $\mathcal{M}_h$.

The compact $p$-adic group $\text{Aut}(G_0) = \text{End}(G_0)^\times$ operates on $\mathcal{M}_h$ by functoriality, as follows.

$\forall \gamma \in \text{Aut}(G_0), \exists!$ formal scheme automorphism $\rho(\gamma)$ of $\mathcal{M}_h$ and a formal group isomorphism

$$\tilde{\rho}(\gamma) : G_{\text{univ}} \to \rho(\gamma)^* G_{\text{univ}}$$

such that $\tilde{\rho}(\gamma)|_{G_0} = \gamma$

Remark. This action $\gamma \mapsto \rho(\gamma)$ of $\text{Aut}(G_0)$ on the Lubin-Tate moduli space $\mathcal{M}_h$ was first studied by Lubin and Tate in 1966. It is also known as (the essential part of) the Morava stabilizer subgroup action in chromatic homotopy theory.
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The universal $p$-typical formal group law

Let $\mathcal{R} = \mathbb{Z}_p \{v\} = \mathbb{Z}_p \{v_1, v_2, v_3, \ldots\}$, and let $\sigma : \mathcal{R} \to \mathcal{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$.

Let $G_v(x) \in \mathcal{R}[x, y]$ be the one-dimensional $p$-typical formal group law over $\mathcal{R}$ whose logarithm

$$g_v(x) \in \mathcal{R}[1/p][x] = \sum_{n \geq 1} a_n(v) \cdot x^{p^n}$$

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$$g_v(x) = x + \sum_{i=1}^{\infty} \frac{v_i}{p} \cdot g_v^{\sigma^i}(x^{p^i})$$
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Let $\tilde{R} = \mathbb{Z}_p[[v]] = \mathbb{Z}_p[[v_1, v_2, v_3, \ldots]]$, and let $\sigma : \tilde{R} \to \tilde{R}$ be the ring homomorphism such that $\sigma(v_j) = v_j^p$ for all $j \geq 1$.

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Remarks on the formal group law $G_\nu$

**Remarks.** (1) The above “functional equation” is a recursive formula for the coefficients $a_n(\nu) \in p^{-n} \cdot \mathbb{Z}_p[v_1, v_2, \ldots, v_n]$ of $g_\nu(x)$.

(2) Explicitly:

$$a_n(\nu) = \sum_{i_1, i_2, \ldots, i_r \geq 1 \atop i_1 + \cdots + i_r = n} p^{-r} \cdot \prod_{s=1}^{r} v_{i_s}^{i_1 + i_2 + \cdots + i_{s-1}}$$

$$= \sum_{i_1, i_2, \ldots, i_r \geq 1 \atop i_1 + \cdots + i_r = n} p^{-r} \cdot v_{i_1} \cdot v_{i_2}^{i_1} \cdot v_{i_3}^{i_1 + i_2} \cdots v_{i_r}^{i_1 + \cdots + i_{r-1}}$$

Note that $a_n(\nu)$ is a homogeneous polynomial in $v_1, \ldots, v_n$ of weight $p^n - 1$ when $v_j$ is given the weight $p^j - 1 \ \forall j \geq 1$.

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The universal formal group over $\mathcal{M}_h$ made explicit

Let $R = R_h = W(\overline{\mathbb{F}}_p)[[w_1, w_2, \ldots, w_{h-1}]]$.

Let $\pi = \pi_h : \tilde{R} \to R$ be the ring homomorphism such that

$$\pi(v_i) = \begin{cases} 
  w_i & \text{if } 1 \leq i \leq h - 1 \\
  1 & \text{if } i = h \\
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\end{cases}$$

The classifying morphism $\text{Spf}(R) \to \mathcal{M}_h$ for the deformation $\pi_* G_{\nu}$ of $G_0$ is an isomorphism.

We will identify $\mathcal{M}_h$ with $\text{Spf}(R)$ and the universal deformation $G_{\text{univ}}$ of $G_0$ with the formal group underlying the formal group law $G_R := \pi_* G_{\nu}$.
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The universal strict isomorphism

Let $\mathbb{Z}_p[v, t] = \mathbb{Z}_p[v_1, v_2, v_3, \ldots; t_1, t_2, t_3, \ldots]$, and let $\sigma : \mathbb{Z}_p[v, t] \to \mathbb{Z}_p[v, t]$ be the obvious Frobenius lifting as before, with $\sigma(v_i) = v_i^p$ and $\sigma(t_i) = t_i^p \forall i \geq 1$.

Let $G_{v,t}(x, y)$ be the one-dimensional formal group law over $\mathbb{Z}_p[v, t]$ whose logarithm $g_{v,t}(x)$ satisfies

$$g_{v,t}(x) = x + \sum_{i=1}^{\infty} t_i \cdot x^{p^i} + \sum_{j=1}^{\infty} \frac{v_j}{p} \cdot g_{v,t}^{(\sigma^j)}(x^{p^j})$$
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The universal strict isomorphism, continued

It is known that $\alpha_{v,t} := g_{v,t}^{-1} \circ g_v \in \mathbb{Z}_p[v,t][[x]]$, and defines a strict isomorphism

$$\alpha_{v,t} : G_v \rightarrow G_{v,t}$$

between $p$-typical formal group laws over $\mathbb{Z}_p[v,t]$.

(A strict isomorphism is an isomorphism between formal group laws which is $\equiv x$ modulo higher degree terms in $x$.)

Moreover $\alpha_{v,t}$ is “the” universal strict isomorphism between one-dimensional $p$-typical formal group laws.
The universal strict isomorphism, continued

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Parameters of $G_{\nu,t}$

By the universality $G_{\nu}$ for $p$-typical formal group laws, there exists a unique ring homomorphism

$$
\eta : \mathbb{Z}(p)[\nu] \to \mathbb{Z}(p)[\nu, t]
$$

such that

$$
\eta_* G_{\nu} = G_{\nu,t}.
$$

The elements

$$
\bar{v}_n = \bar{v}_n(\nu, t) \in \mathbb{Z}(p)[\nu, t], \quad n \in \mathbb{N}_{\geq 1}
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are the parameters of the $p$-typical formal group law $G_{\nu,t}$. 
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A known recursive formula for the parameters of $G_{v,t}$

\[
\bar{v}_n = v_n + pt_n + \sum_{i+j=n \atop i,j \geq 1} (v_j t_i^p - t_i \bar{v}_j^p)
\]

\[
+ \sum_{j=1}^{n-1} a_{n-j}(v) \cdot (v_j^{p^n-j} - \bar{v}_j^{p^n-j})
\]

\[
+ \sum_{k=2}^{n-1} a_{n-k}(v) \cdot \sum_{i+j=k \atop i,j \geq 1} (v_j^{p^n-k} t_i^{p^n-i} - t_i^{p^n-k} \bar{v}_j^{p^n-j})
\]

(This formula contains high power of $p$ in the denominators. Consequently it is not very useful for our purpose.)
A known recursive formula for the parameters of $G_{v,t}$

$$
\overline{v}_n = v_n + pt_n + \sum_{i+j=n, i,j \geq 1} \left( v_j t_i^{p^j} - t_i \overline{v}_j^i \right)
$$

$$
+ \sum_{j=1}^{n-1} a_{n-j}(v) \cdot \left( v_j^{p^{n-j}} - \overline{v}_j^{p^{n-j}} \right)
$$

$$
+ \sum_{k=2}^{n-1} a_{n-k}(v) \cdot \sum_{i+j=k, i,j \geq 1} \left( v_j^{p^{n-k}} t_i^{p^{n-i}} - t_i^{p^{n-k}} \overline{v}_j^{p^{n-j}} \right)
$$

(This formula contains high power of $p$ in the denominators. Consequently it is not very useful for our purpose.)
An integral recursion formula for $\bar{v}_n(\nu, t)$

(useful for computing the Lubin-Tate action)

$$\bar{v}_n = v_n + pt_n - \sum_{j=1}^{n-1} t_j \cdot \bar{v}_{n-j}^p +$$

$$+ \sum_{l=1}^{n-1} \sum_{k=1}^{n-l-1} \frac{1}{p} \cdot a_{n-k-l}(\nu)(p^l) \cdot \left\{ \left( \bar{v}_k^p \right)^{p^{n-l-k}} - \left( \bar{v}_k^p \right)^{p^{n-l-k}} \right\}$$

$$+ \sum_{i+j=k, i,j \geq 1} t_j^{p^{n-k}} \left[ \left( \bar{v}_i^{p^l} \right)^{p^{n-l-i}} - \left( \bar{v}_i^{p^l} \right)^{p^{n-l-i}} \right] \}$$

$$+ \sum_{l=1}^{n-1} v_l \cdot \left\{ \frac{1}{p} \left( \bar{v}_{n-l}^{p^l} - \bar{v}_{n-l}^{p^l} \right) + \sum_{i+j=n-l, i,j \geq 1} t_j^{p^l} \cdot \frac{1}{p} \cdot \left[ \left( \bar{v}_i^{p^l} \right)^{p^j} - \left( \bar{v}_i^{p^l} \right)^{p^j} \right] \}$$

for every $n \geq 1$. 
Step 1

Given an element $\gamma \in \text{Aut}(G_0)$, construct

- a $p$-typical one-dimensional formal group law $F = F_\gamma$ over $R$ whose closed fiber is equal to $G_0$, and
- an isomorphism

$$\overline{\psi} = \overline{\psi}_\gamma : F_\overline{R} \rightarrow G_\overline{R}$$

over $\overline{R} := R/pR = \overline{\mathbb{F}}_p[[w_1, \ldots, w_{h-1}]]$ whose restriction to the closed fibers is

$$(\psi|_{G_0} : G_0 \rightarrow G_0) = \gamma.$$  

Here $F_\overline{R} = F \otimes_R \overline{R}$, $G_\overline{R} = G_R \otimes_R \overline{R}$.

Note that both the formal group law $F$ over $R$ and the isomorphism $\psi$ over $\overline{R}$ depends on the given element $\gamma \in \text{Aut}(G_0)$. 
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The formal group law $F_c, c \in W(\mathbb{F}_{p^h})^\times$

For $\gamma = [c] \in W(\mathbb{F}_{p^h})^\times = \text{Aut}(G_1)$, we can take $F_c$ to be the formal group over $R$ whose logarithm $g_c(x)$ satisfies

$$f_c(x) = x + \sum_{i=1}^{h} \frac{c^{-1+\sigma_i} \cdot w_i}{p} \cdot f_c(\sigma_i)^i \cdot (x^{p^i})$$

($w_h=1$ by convention).

Let

$$\psi_c(x) = \log^{-1}_{G_R} \circ (c \cdot f_c)$$

We have $\psi_c(x) \in R[[x]]$ and $\psi_c$ defines an isomorphism from $F_c$ to $G_R$ over $R$ (not just over $\overline{R}$!) with $\psi_c|_{G_0} = [c]$. 

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Local Hecke symmetry
The global rigidity problem
Local rigidity problems
Known results, obstacles and hope
A glimpse of a new approach
The Lubin-Tate action
Universal isomorphism
$p$-typical formal group laws
Sketch of the steps
The first test
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Step 2

Compute the parameters

\[(u_i = u_i(w_1, \ldots, w_{h-1}))_{i \in \mathbb{N} \geq 1}\]

for the \(p\)-typical group law \(F = F_\gamma\) over \(R\).

The above condition means that

\[\xi_\gamma(G_{\tilde{V}}) = F,\]

where

\[\xi = \xi_\gamma : \tilde{R} \rightarrow R\]

is the ring homomorphism such that

\[\xi(v_i) = u_i \quad \forall i \geq 1.\]
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\[\xi_* G_{\tilde{\nu}} = F,\]

where

\[\xi = \xi_\gamma : \tilde{R} \to R\]

is the ring homomorphism such that

\[\xi(v_i) = u_i \quad \forall i \geq 1.\]
Parameters for $F_c$, $c \in W(\mathbb{F}_{p^h})^\times$

In the case when $\gamma \in \text{Aut}(G_0)$ lifts to an element $[c]$ with $c \in W(\mathbb{F}_{p^h})^\times \simeq \text{Aut}(G_1)$, we have the following integral recursive formula for the parameters $u_n = u_n(c; w)$.

$$u_n(c; w) = c^{-1+\sigma^n}w_n$$

$$+ \sum_{j=1}^{n-1} c^{-1+\sigma^j} \cdot \frac{1}{p} \left[ u_{n-j}(c; w)^{(p^j)} - u_{n-j}(c; w)p^j \right] \cdot w_j$$

$$+ \sum_{j=1}^{n-1} \sum_{i=1}^{n-j-1} \frac{1}{p} a_{n-i-j}(w)^{(p^j)} \cdot c^{-1+\sigma^{n-i}} \cdot$$

$$\left[ (u_i(c; w)^{(p^j)})^{p^{n-i-j}} - (u_i(c; w)^{p^j})^{p^{n-i-j}} \right] \cdot w_j$$

where $w_h = 1$, $w_m = 0 \ \forall m \geq h + 1$ by convention.
Remark. The above recursive formula for the parameters $u_n(c; w)$ can be turned into an explicit “path sum” formula for $u_n(c, w)$, with terms indexed by “paths”.
Step 3

Find/compute the uniquely determined element

\[ \tau_n \in \mathfrak{m}_R, \quad n \in \mathbb{N}_{\geq 1} \]

and

\[ \hat{u}_1 \in \mathfrak{m}_R, \ldots, \hat{u}_{h-1} \in \mathfrak{m}_R, \hat{u}_h \in 1 + \mathfrak{m}_R \]

such that

\[ \bar{v}_n(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h, 0, 0, \ldots; \tau) = u_n \quad \forall n \geq 1. \]
**Remark.** (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over \( \tilde{\mathbb{R}} \), applied to the “vector-valued” function with components \( \bar{v}_n \) in the integral recursion formula discussed before.

(2) This step is a substitute for the operation *taking the quotient of the group “changes of coordinates”* in a space of formal group laws.

(3) The approximate solution coming from the linear term in the \( \tau_j \) variables is often good enough for our application.
Remark. (1) The existence and uniqueness statement above is an application the implicit function theorem for an infinite dimensional space over $\tilde{R}$, applied to the “vector-valued” function with components $\bar{v}_n$ in the integral recursion formula discussed before.

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(3) The approximate solution coming from the linear term in the $\tau_j$ variables is often good enough for our application.
A congruence formula for $\overline{v}_n$

The follow formula helps to explain the last remark.

$$\overline{v}_n \equiv v_n - \sum_{j=1}^{n} t_j \cdot v_{n-j}^p + \sum_{i,t,s_1,s_2,\ldots,s_t \geq 1 \atop s_1 + \ldots + s_t + i + j = n} (-1)^{t-1} t_i \cdot v_j^p \cdot v_1^{(p^{s_1} + p^{s_2} + \cdots + p^{s_t} - t)/(p-1)}$$

$$\cdot v_{n-s_1}^{p^{s_1}-1} \cdot v_{n-s_1-s_2}^{p^{s_2}-1} \cdots v_{n-s_1-\cdots-s_t}^{p^{s_t}-1} \mod (pt_a, t_a \cdot t_b)_{a,b \geq 1} \mathbb{Z}[v,t]$$
Step 4

**Rescale** $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_h$ as follows:

$\exists \tau_0 \in m_R$ such that

$$(1 + \tau_0)^{p^h - 1} \cdot \hat{u}_h = 1.$$

Let

$$\hat{v}_i := (1 + \tau_0)^{p^i - 1} \cdot \hat{u}_i \text{ for } i = 1, \ldots, h - 1.$$

Let $\omega : \tilde{R} \to R$ be the ring homomorphism such that

$$\omega(v_i) = \hat{u}_i \quad \forall i \geq 1.$$

Let $\rho : R \to R$ be the $W(\overline{F_p})$-linear ring homomorphism such that

$$\rho(w_i) = \hat{v}_i \quad \forall i \geq 1.$$
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The meaning of Steps 3 and 4

The universal strict isomorphism $\alpha_{\nu, t}$ specializes to a strict isomorphism

$$\alpha = \alpha_{\nu, t} : F \to \omega_* G_{\nu}$$

with $\alpha|_{G_0} = \text{Id}_{G_0}$.

The rescaling in step 4 gives an isomorphism (not necessarily a strict isomorphism)

$$\beta : \omega_* G_{\nu} \to \rho_* G_R$$

with $\beta|_{G_0} = \text{Id}_{G_0}$. 
The meaning of Steps 3 and 4

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Conclusion

Combined with $\overline{\psi}$, we obtain an isomorphism

$$\overline{\psi} \circ \overline{\alpha}^{-1} \circ \overline{\beta}^{-1} : \overline{\rho_* G_R} \to \overline{G_R}$$

whose restriction to the closed fiber $G_0$ is equal to the given element $\gamma \in \text{Aut}(G_0)$.
(Here $\overline{\alpha} = \alpha \otimes_R \overline{R}$ and $\overline{\beta} = \beta \otimes_R \overline{R}$.)

Conclusion. The given element $\gamma \in \text{Aut}(G_0)$ operates on the equi-characteristic deformation space $\text{Spf}(\overline{R})$ of $G_0$ via the ring automorphism $\rho$.
(Notice that $\overline{\psi}$, $\alpha$ and $\beta$ all depend on $\gamma$.)
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Local rigidity for the Lubin-Tate moduli space: the first non-trivial case

**Proposition.** Let $Z \subset \mathcal{M}_3_{\overline{F}_p} = \text{Spf}(\overline{F}_p[[w_1, w_2]])$ be an irreducible closed formal subscheme of $\mathcal{M}_3$ over $\overline{F}_p$ corresponding to a height one prime ideal of $\overline{F}_p[[w_1, w_2]]$. If $Z$ is stable under the action of an open subgroup of $W(F_{p^3})^\times$, then $Z = \text{Spf}(\overline{F}_p[[w_1, w_2]]/(w_1))$. 
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