FINE STRUCTURES OF MODULI SPACES
IN POSITIVE CHARACTERISTICS:
HECKE SYMMETRIES AND
OORT FOLIATION

§1. Elliptic curves and their moduli
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§1. Moduli of Elliptic curves

§1.1. **Def.** An elliptic curve over \( \mathbb{C} \) is the quotient of a one dimensional vector space \( V \) over \( \mathbb{C} \) by a lattice \( \Gamma \) in \( V \).

Concretely, we can take \( V = \mathbb{C} \), and a lattice \( \Gamma \) in \( \mathbb{C} \) has the form

\[
\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2
\]

\( \omega_1, \omega_2 \in \mathbb{C} \), linearly independent over \( \mathbb{R} \).

The quotient \( E(\Gamma) = \mathbb{C}/\Gamma \) of \( \mathbb{C} \) by a lattice \( \Gamma \) is a compact one-dimensional complex manifold, and is also an abelian group.
§1.2. The Weistrass $\wp$-function

$$\wp_\Gamma(u) = \frac{1}{u^2} + \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \left[ \frac{1}{(u - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right]$$

is a meromorphic function on $E(\Gamma)$.

Define complex numbers $g_2(\Gamma), g_3(\Gamma)$ by Eisenstein series

$$g_2(\Gamma) = 60 \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\omega_1 + n\omega_2)^4}$$

$$g_3(\Gamma) = 140 \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\omega_1 + n\omega_2)^6}$$
Let \( x_\Gamma = \wp_\Gamma(u), y_\Gamma = \frac{d}{du} \wp_\Gamma(u) \). Then the two
meromorphic functions \( x_\Gamma, y_\Gamma \) on \( E(\Gamma) \) satisfy the
polynomial equation

\[
y^2_\Gamma = 4x^3_\Gamma - g_2(\Gamma) x_\Gamma - g_3(\Gamma)
\]

This equation tells us that the pair \((x_\Gamma, y_\Gamma)\) defines a
map \( \iota \) from the elliptic curve \( E(\Gamma) \) to \textit{algebraic curve} in
\( \mathbb{P}^2 \) cut out by the cubic homogeneous equation

\[
Y^2Z = 4 X^3 - g_2(\Gamma) XZ^2 - g_3(\Gamma) Z^3;
\]

the map \( \iota \) turns out to be an isomorphism. (The affine
equation

\[
y^2 = 4 x^3 - g_2(\Gamma) x - g_3(\Gamma)
\]

with \( x = \frac{X}{Z}, y = \frac{Y}{Z} \) describes \( E(\Gamma) \setminus \{0\} \).)
§1.3 Moduli of elliptic curves

Two elliptic curves $E_1, E_2$ attached to lattices $\Gamma_1, \Gamma_2$ in $\mathbb{C}$ are isomorphic iff they are homothetic, i.e.

$$\exists \lambda \in \mathbb{C}^\times \text{ s.t. } \lambda \cdot \Gamma_1 = \Gamma_2$$

To parametrize lattices, for $\Gamma = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2$, write

$$(\omega_1, \omega_2) = \lambda(\tau, 1), \tau \in \mathbb{C} - \mathbb{R} =: X^\pm$$

The group $GL_2(\mathbb{Z})$ operates on the right of $X^\pm$ by

$$(\tau, 1) \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (c\tau + d) \cdot \left( \frac{a\tau + b}{c\tau + d}, 1 \right)$$

$a, b, c, d \in \mathbb{Z}, ad - bd = \pm 1$. 
Elliptic curves corresponding to lattices $\mathbb{Z} \tau_i + \mathbb{Z}$, $i = 1, 2$ are isomorphic iff $\tau_1 \cdot \gamma = \tau_2$ for some $\gamma \in \text{GL}_2(\mathbb{Z})$.

Algebraically, one can attach to every elliptic curve $E$ a complex number $j(E)$, such that $E_1$ and $E_2$ are isomorphic iff $j(E_1) = j(E_2)$. For an elliptic curve $E$ given by a Weistrass equation

$$y^2 = 4x^3 - g_2x - g_3, \quad \Delta := g_2^3 - 27g_3^2 \neq 0,$$

the $j$-invariant is $j(E) = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}$

For an elliptic curve defined by

$$y^2 = x(1 - x)(\lambda - x), \quad \lambda \neq 0, 1,$$

the $j$-invariant is $j(\lambda) = 2^8 \frac{(1-\lambda)(1-\lambda))}{\lambda^2 (1-\lambda)^2}$
§1.4 Hecke symmetries

(1) A Hecke correspondence on $X^\pm / \text{GL}_2(\mathbb{Z})$ is defined by a diagram

$$
\begin{array}{ccc}
X^\pm / \text{GL}_2(\mathbb{Z}) & \xleftarrow{\pi} & X^\pm \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
X^\pm / \text{GL}_2(\mathbb{Z})
\end{array}
$$

with $\gamma \in \text{GL}_2(\mathbb{Z})$.

(2) The Hecke orbit of an element $\pi(x)$ of $X^\pm / \text{GL}_2(\mathbb{Z})$ is the countable subset $\pi(x \cdot \text{GL}_2(\mathbb{Q}))$ of $X^\pm / GL_2(\mathbb{Z})$.

(3) Geometrically, the Hecke orbit of the modular point $[E]$ is the subset consisting of all $[E_1]$ such that there exists a surjective holomorphic homomorphism from $E \to E_1$ (called an isogeny.)
Another way to look at Hecke orbits:

Let $G(n)$ be the set of all $2 \times 2$ matrices in $\text{GL}_2(\mathbb{Z})$ which are congruent to $\text{Id}_2$ modulo $n$. We have a projective system

$$\tilde{X} := (X^\pm / G(n))_{n \in \mathbb{N} \geq 1}$$

of modular curves, with the indexing set $\mathbb{N} \geq 1$ ordered by divisibility. We have a large group $\text{GL}_2(\mathbb{A}_f)$ operating on the tower $\tilde{X}$, and $\tilde{X} / \text{GL}_2(\hat{\mathbb{Z}})$ is isomorphic to the $j$-line $X / \text{GL}_2(\mathbb{Z})$. (Here $\hat{\mathbb{Z}} := \lim_{\leftarrow}(\mathbb{Z}/n\mathbb{Z})$, $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.)

The Hecke correspondences on $\tilde{X} / \text{GL}_2(\hat{\mathbb{Z}})$ are induced by the action of $\text{GL}_2(\mathbb{A}_f)$ on $\tilde{X}$. 
§2. Moduli of abelian varieties

§2.1. **Def.** A *complex torus* is a compact complex group variety of the form $V/\Gamma$, where $V$ is a finite dimensional complex vector space and $\Gamma$ is a cocompact discrete subgroup of $\Gamma$;

$$\text{rank}(\Gamma) = 2 \dim_{\mathbb{C}}(V).$$

**Def.** A complex torus $V/\Gamma$ is an *abelian variety* if it can be holomorphically embedded in $\mathbb{P}^N$; this happens iff there exists a definite hermitian form on $V$ whose imaginary part induces a $\mathbb{Z}$-valued symplectic form on $\Gamma$. Such a form, called a *polarization*, is *principal* iff the discriminant of the symplectic form is 1.
§2.2. A lattice in $\mathbb{C}^g$ which admits a principal polarization can be written as

$$C \cdot (\Omega \cdot \mathbb{Z}^g + \mathbb{Z}^g)$$

for some $C \in \text{GL}_g(\mathbb{C})$ and some symmetric $\Omega \in \text{M}_g(\mathbb{C})$ with definite imaginary part. The set $X_g^\pm$ of all such period matrices $\Omega$’s is called the Siegel upper-and-lower half-space.

The group $GSp_{2g}(\mathbb{Q})$ operates on the right of $X_g^\pm$ by:

$$\Omega \cdot \begin{pmatrix} A & C \\ B & D \end{pmatrix} = (\Omega C + D)^{-1} (\Omega A + B)$$

Here $GSp_{2g}$ denotes the group of $2g \times 2g$ matrices which preserve the standard symplectic pairing up to scalars.
§2.3. The isomorphism classes of $g$-dimensional abelian principally polarized abelian varieties is parametrized by

$$X^\pm_g / \text{GSp}_{2g}(\mathbb{Z})$$

Just as in the elliptic curve case, we have a projective system

$$\tilde{X} = (X^\pm_g / G(n))_{n \in \mathbb{N}_{\geq 1}}$$

Here $G(n)$ consists of elements of $\text{GSp}_{2g}(\mathbb{Z})$ which are congruent to $\text{Id}_{2g}$ modulo $n$. Again the group $\text{GSp}_{2g}(\mathbb{A}_f)$ operates on the tower $\tilde{X}$. This action induces *Hecke correspondences* on $X^\pm_g / \text{GSp}_{2g}(\mathbb{Z})$. The Hecke orbit of a point $[A]$ is the countable subset consisting of all principally polarized abelian varieties which are *symplectically isogenous* to $A$. 
§3. Modular varieties with Hecke symmetries

§3.1. Generalizing §2, consider (a special class of) Shimura varieties \( \tilde{X} = \left( X_n \right)_{n \in \mathbb{N} \geq 1}, G \), where

- \( G \) is a connected reductive group over \( \mathbb{Q} \),
- \( \tilde{X} \) is a moduli space of abelian varieties with prescribed symmetries (of a fixed type)

The group \( G \) is the symmetry group of the “prescribed symmetries”, giving the “type” of the prescribed symmetries.
§3.2. Hecke symmetries on Shimura varieties

The group $G(\mathbb{A}_f)$ operates on the tower $\tilde{X}$:

$$ \bowtie G(\mathbb{A}_f) $$

$$ \tilde{X} = ( \cdots \to X_n \to \cdots X_0 = X) $$

On the “bottom level” $X = X_0$, the symmetries from $G(\mathbb{A}_f)$ induces Hecke correspondences; these correspondences are parametrized by $G(\mathbb{Z}) \backslash G(\mathbb{A}_f) / G(\mathbb{Z})$.

Remark: For a fixed finite level $X_n \to X_0$, the symmetry subgroup preserving the covering map is $G(\mathbb{Z}/n\mathbb{Z})$. 
§3.3. Modular varieties in characteristic $p$

Abelian varieties can be defined in purely algebraic terms (Weil), so are the modular varieties classifying them. In particular one can define these modular varieties over a field $k$ of characteristic $p > 0$.

In the case of elliptic curves, if $p \neq 2, 3$, then every elliptic curve is defined by a Weistrass equation

$$y^2 = 4 x^3 - g_2 x - g_3, \quad \Delta := g_2^3 - 27g_3^2 \neq 0,$$

the moduli is given by the $j$-invariant; the $j$-invariant is

$$j(E) = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2}$$
The diagram for Hecke symmetries in characteristic $p$ is

\[ \overset{\otimes}{G(\mathbb{A}_f^{(p)})} \]

\[ \tilde{X} = (\cdots \to X_n \to \cdots X_0 = X) \]

\[ G(\mathbb{Z}/n\mathbb{Z}) \]

The indices $n$ are relatively prime to $p$, and the Hecke correspondences come from prime-to-$p$ isogenies between abelian varieties.
§4. The Hecke orbit problem

**Problem** Characterize the Zariski closure of Hecke orbits in a modular variety $X$.

– The closed subsets for the Zariski topology of $X$ consists of algebraic subvarieties of $X$.

– Each Hecke orbit is a countable subset of $X$. 
§5. Solution in characteristic 0.

**Prop** In characteristic 0, every Hecke orbit is dense in the modular variety $X$.

*Proof in the Siegel case:* May assume that the base field is $\mathbb{C}$.

Claim: Every Hecke orbit is dense for the finer metric topology on $X$.

The Hecke symmetries come from the action of the group $\text{GSp}_{2g}(\mathbb{Q})$ on $X_g^\pm$. Conclude by

- $\text{GSp}_{2g}(\mathbb{Q})$ is dense in $\text{GSp}_{2g}(\mathbb{R})$
- $\text{GSp}_{2g}(\mathbb{R})$ operates transitively on $X_g^\pm$
§6. Fine structures in char. \( p \)

The base field \( k \) has char. \( p \) from now on.

§6.1. Elliptic curves have *Hasse invariant*; explicitly, for

\[ E : y^2 = x(1 - x)(\lambda - x), \ p \neq 2 \]

\[ j(\lambda) = 2^8 \frac{(1 - \lambda(1 - \lambda))^3}{\lambda^2 (1 - \lambda)^2}, \text{ then} \]

\[ A(\lambda) = (-1)^r \sum_{i=0}^{r} \binom{r}{i}^2 \lambda^i, \ r = \frac{1}{2}(p - 1) \]

gives the Hasse invariant of \( E \).

**Def.** The elliptic curve \( E \) is *supersingular* if its Hasse invariant vanishes, otherwise \( E \) is *ordinary*; \( E \) is ordinary iff \( E \) has \( p \) points which are killed by \( p \).

The (finite) set of supersingular elliptic curves is stable Hecke correspondences. So the Hecke orbit of \([E]\) is dense iff \( E \) is ordinary.
§6.2 From now on $X = A_g$ denotes the Siegel modular variety in char. $p$; it classifies $g$-dimensional principally polarized abelian varieties.

Source of fine structure on $X$ (or $\tilde{X}$): Every family of abelian varieties $A \to S$ gives rise to a Barsotti-Tate group

$$A[p^\infty]_S := \lim_{\longrightarrow} A[p^n]_S,$$

an inductive system of finite locally free group schemes $A[p^n] := \text{Ker}([p^n] : A \to A)$; the height of $A[p^\infty]$ is $2g = 2 \dim(A/S)$. The Frobenius $F_A : A \to A^{(p)}$ and Verschiebung $V_A : A^{(p)} \to A$ pass to $A[p^\infty]$.
§6.3. The slope stratification

The slopes of a Barsotti-Tate group $A[p^\infty]$ over a field $k/F_p$ is a sequence $2g$ of rational numbers

$$\lambda = (\lambda_j), \quad 0 \leq \lambda_1 \leq \cdots \leq \lambda_{2g} \leq 1,$$

such that $\lambda_j + \lambda_{2g+1-j} = 1$. The denominator of each $\lambda_j$ divides its multiplicity. The slopes are defined using divisibility properties of iterations of the Frobenius.

The slope sequence, a discrete invariant, defines a stratification

$$X = \bigsqcup_{\lambda} X_\lambda$$

The Zariski closure of each stratum $X_\alpha$ is equal to a union of (smaller) strata.
(a) $\mathcal{A}_1$ is the union of two strata.

(b) The open dense stratum of $\mathcal{A}_g$ corresponds to \textit{ordinary} abelian varieties, with slopes $(0, \ldots, 0, 1, \ldots, 1)$. The minimal stratum of $\mathcal{A}_g$ corresponds to \textit{supersingular} abelian varieties, with slopes $(\frac{1}{2}, \ldots, \frac{1}{2})$, and has dimension $\lfloor g^2/4 \rfloor$ (Li-Oort).

§6.4. Ekedahl–Oort stratification

The isomorphism type $A[p]$ of the $p$-torsion subgroup of a principally polarized abelian variety $A$, together with the \textit{Weil pairing} on it, turns out to be a discrete invariant and gives rise to a stratification of $X$. 
§7. Foliation and a conjecture of Oort

§7.1 Replacing the discrete invariants (such as slopes) of the Barsotti-Tate groups by their **isomorphism types**, one gets a much finer decomposition of the modular variety $X = \mathcal{A}_g$, introduced by Oort. One can also define the foliation structure for more general modular varieties.

**Def.** The locus of $X$ with a fixed isomorphism type of $(\mathcal{A}[p^{\infty}] + \text{polarization})$ is called a leaf.

- Each leaf is a locally closed subset of $X$, smooth over $\overline{\mathbb{F}}_p$.
- (With a one exception) there are infinitely many leaves on $M$. For instance the leaf containing a supersingular point in $\mathcal{A}_g$ is finite.
- The dense open slope stratum of $X$ is a leaf. For instance if there exists an ordinary fiber $\mathcal{A}_x$, then the ordinary locus in $X$ is a leaf.
§7.2. Characterize leaves by Hecke symmetries

Clearly the foliation structure of $M$ is stable under all prime-to-$p$ Hecke correspondences. A recent conjecture of Oort predicts that the leaves are determined by the Hecke symmetries.

**Conj.** (HO). The foliation structure is characterized by the prime-to-$p$ Hecke symmetries: For each point $x \in X$, the prim-to-$p$ Hecke orbit of $x$ is dense in the leaf containing $x$.

Note: Each Hecke orbit is a countable subset of $X$. 
§8. Hecke orbits of ordinary points

The first piece of evidence supporting Conj. (HO) is the case of the dense open leaf in $X$.

**Thm.** Conj. (HO) holds for the ordinary locus of $\mathcal{A}_g$. Every ordinary symplectic prime-to-$p$ isogeny class is dense in $\mathcal{A}_g$.

**Rmk:** The same holds for modular varieties of PEL-type C (CLC). But the density of Hecke orbits on the dense open leaf has not been established for all PEL-type modular varieties.
§9. Canonical coordinates for leaves.

**Thm.** (Serre-Tate) The formal completion of any closed point of the ordinary locus of $X$ has a natural structure as a formal torus over the base field $k$.

This classical result generalizes to every leaf in $X$:

**Thm.** The formal completion of every leaf is the maximal member of a finite projective system $(Y_\alpha)_{\alpha \in I}$ of smooth formal varieties, indexed by a finite partially ordered set $I$. The poset $I$ is the set of all segments of a linearly ordered finite set $S = \{1, \ldots, n\}$. Each map

$$\pi_{i,j} : X_{[a,b]} \to X_{[a+1,b]} \times_{X_{[a+1,b-1]}} X_{[a,b-1]}$$

has a natural structure as a torsor for a $p$-divisible formal group over $k$. 

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§10. Known cases of Conj. (HO):

§10.1. Examples

(1) \( A_g \) for \( g = 1, 2, 3 \) (with Oort)

(2) The HB varieties (work in progress with C.-F. Yu).

Write \( F \otimes_{Q} Q_p = \bigoplus_i F_{p_i} \),

\[ A_x[p^\infty] = \bigoplus A_x[p_i^\infty] =: B_i. \]

Each \( B_i \) has two slopes \( \frac{r_i}{g_i}, \frac{s_i}{g_i} \) with multiplicity \( g_i = [F_{p_i} : Q_p] \). Then the dimension of the leaf passing through \( x \) is \( \sum_i |r_i - s_i| \).

(3) The Hecke orbit of a “very symmetric” ordinary point of a modular variety of PEL-type is dense.

(4) PEL-type modular varieties attached to a quasi-split \( U(n, 1) \).
\section{10.2. Local Hecke orbits}

The following result is a local version of the Hecke orbit problem.

**Thm.** Let $k$ be an algebraically field of char. $p > 0$. Let $X$ be a finite dimensional $p$-divisible smooth formal group over $k$. Let $E = \text{End}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $G$ be a connected linear algebraic group over $\mathbb{Q}_p$. Let $\rho : G \to \mathbb{E} \times$ be a homomorphism of algebraic groups over $\mathbb{Q}_p$ such that the trivial representation $1_G$ is not a subquotient of $(\rho, E)$. Suppose that $Z$ is a reduced and irreducible closed formal subscheme of the $p$-divisible formal group $X$ which is closed under the action of an open subgroup $U$ of $G(\mathbb{Z}_p)$. Then $Z$ is stable under the group law of $X$ and hence is a $p$-divisible smooth formal subgroup of $X$.

**Rem.** It is helpful to consider first the case when $X$ is a formal torus and $G$ is $\widehat{\mathbb{G}_m}$. We sketch a proof.
**Prop.** Let $X$ be a finite dimensional $p$-divisible smooth formal group over $p$. Let $k$ be an algebraically closed field. Let $R$ be a topologically finitely generated complete local domain over $k$. In other words, $R$ is isomorphic to a quotient $k[[x_1, \ldots, x_n]]/P$, where $P$ is a prime ideal of the power series ring $k[[x_1, \ldots, x_n]]$. Then there exists an injective local homomorphism $\iota : R \hookrightarrow k[[y_1, \ldots, y_d]]$ of complete local $k$-algebras, where $d = \dim(R)$.

**Prop.** Let $k$ be a field of characteristic $p > 0$. Let $q = p^r$ be a positive power of $p$, $r \in \mathbb{N}_{>0}$. Let $F(x_1, \ldots, x_m) \in k[x_1, \ldots, x_m]$ be a polynomial with coefficients in $k$. Suppose that we are given elements $c_1, \ldots, c_m$ in $k$ and a natural number $n_0 \in \mathbb{N}$ such that $F(c_1^{q^n}, \ldots, c_m^{q^n}) = 0$ in $k$ for all $n \geq n_0$, $n \in \mathbb{N}$. Then $F(c_1^{q^n}, \ldots, c_m^{q^n}) = 0$ for all $n \in \mathbb{N}$; in particular $F(c_1, \ldots, c_m) = 0$. 

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**Prop.** Let $k$ be a field of characteristic $p > 0$. Let $f(u, v) \in k[[u, v]]$, $u = (u_1, \ldots, u_a)$, $v = (v_1, \ldots, v_b)$, be a formal power series in the variables $u_1, \ldots, u_a, v_1, \ldots, v_b$ with coefficients in $k$. Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ be two new sets of variables. Let $g(x) = (g_1(x), \ldots, g_a(x))$ be an $a$-tuple of power series without the constant term: $g_i(x) \in (x)k[[x]]$ for $i = 1, \ldots, a$. Let $h(y) = (h_1(y), \ldots, h_b(y))$, with $h_j(y) \in (y)k[[y]]$ for $j = 1, \ldots, b$. Let $q = p^r$ be a positive power of $p$. Let $n_0 \in \mathbb{N}$ be a natural number, and let $b'$ be a natural number with $1 \leq b' \leq b$. Let $(d_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers such that $\lim_{n \to \infty} \frac{q^n}{d_n} = 0$. Suppose we are given power series $R_{j,n}(v) \in k[[v]]$, $j = 1, \ldots, b$, $n \geq n_0$, such that $R_{j,n}(v) \equiv 0 \mod (v)^{d_n}$ for all $j = 1, \ldots, b$ and all $n \geq n_0$. 
For each \( n \geq n_0 \), let \( \phi_{j,n}(v) = v_j^{q_n} + R_{j,n}(v) \) if \( 1 \leq j \leq b' \), and let \( \phi_{j,n}(v) = R_{j,n}(v) \) if \( b' + 1 \leq j \leq b \). Let \\
\( \Phi_n(v) = (\phi_{1,n}(v), \ldots, \phi_{b,n}(v)) \) for each \( n \geq n_0 \). Assume that \( 0 = f(g(x), \Phi_n(h(x))) = \\
f(g_1(x), \ldots, g_a(x), \phi_{1,n}(h(x)), \ldots, \phi_{b,n}(h(x))) \) in \( k[[x]] \), for all \( n \geq n_0 \). Then \( 0 = \\
f(g_1(x), \ldots, g_a(x), h_1(y), \ldots, h_{b'}(y), 0, \ldots, 0) \) in \( k[[x, y]] \).