Moduli of abelian varieties

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Preliminary version May 2017.
Next: corrections, suggestions, etc. by Ching-LI,
many more references should be included in the text,
then I will finish, and fill out all XX etc.
Add more questions?
Check bibliography, include references, or delete items.
We ask Rachel Pries for her comments.

Introduction

We discuss three Questions in the 1995 manuscript [101] and (partial) answers to these. Moreover we will describe other ideas connected with other problems on these topics.

0.1. [101], 8.3 Conjecture (irreducibility of Newton Polygon strata). Let β be a symmetric NP, with $\beta \neq \sigma$, i.e. W_β is not the supersingular locus. Then for principally polarized abelian varieties in the moduli space $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of principally polarized abelian varieties in positive characteristic we conjectured in 1995:

$$\textit{the NP stratum } W_{\beta,1} \textit{ is irreducible.} \qquad (?)$$

We knew at that moment that (for large g) the supersingular locus is reducible. In 1995 we had good reasons to believe that the non-supersingular Newton Polygon strata in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ should be *geometrically irreducible.* This is indeed the case, as was proved in [16], Theorem 3.1. For a sketch of a proof, for ingredients, and for other details, see Section 4.

0.2. [101], Section 12 (CM liftings). Tate proved that any abelian variety defined over a finite field is a CM abelian variety, see [139] XX. Honda and Tate proved that any abelian variety over $\overline{\mathbb{F}_p}$ is isogenous with an abelian variety that can be CM lifted to characteristic zero, see [42], basically contained in his Main Theorem, and [140], Th. 2 on page 102:

For any abelian variety A over a finite field $\kappa = \mathbb{F}_q$ there exists a finite extension $\kappa \subset K$ and an isogeny $A \otimes_\kappa K \ \sim_K \ B_0$ and a CM *lift of B_0 to characteristic zero.*

Is the field extension necessary? Is the isogeny necessary?

We knew, 1992, there exist abelian varieties defined over $\overline{\mathbb{F}_p}$ that cannot be CM lifted to characteristic zero (i.e. there are cases where an isogeny over $\overline{\mathbb{F}_p}$ is necessary to make a CM lift possible), see [98]: *an isogeny is necessary in general.*

Later, more precise questions were formulated:

0.2.1. Suppose A is an abelian variety over a finite field κ. Does there exists a κ-isogeny $A \sim B_0$ such that B_0 admits a CM lift to characteristic zero? I.e. *is a field extension necessary in the Honda-Tate theory*?

0.2.2. Suppose A is an abelian variety over a finite field κ. Does there exists a κ-isogeny $A \sim B_0$ and a CM lifting of B_0 to a *normal* mixed characteristic domain?

Complete answers to these questions are given in [12]. Below, in Section 5 we will describe more precise questions, methods and results.

0.3. [101], Section 13 (generalized canonical coordinates). Around a moduli point $x_0 = [X_0, \lambda_0]$ of an *ordinary* abelian variety over a perfect field the formal completion

$$(\mathcal{A}_{g,1})^{\wedge}_{x_0} \quad \cong \quad \left((\mathbb{G}_m)^{\wedge}\right)^{g(g+1)/2}$$

in mixed characteristic, is a formal torus, where the origin is chosen to be the canonical lift of (X_0, λ_0); see [65], Chapter V; as proved by Drinfeld [25], see [54], also see [46]. XXX

0.4. In [101], 13A we ask for

"canonical coordinates" around any moduli point in $\mathcal{A}_g \otimes \mathbb{F}_p$.

Right-fully we said in 1995: "This question should be made much more precise before it can be taken seriously." We will show results and limitations:

- We do not know how to make canonical coordinates in *mixed characteristic* around any point in the non-ordinary Newton Polygon locus, but
- we can construct the right concept around a point $y_0 \in \mathcal{A}_g \otimes \mathbb{F}_p$ on the *central leaf* passing through y_0.

Note that the central leaf through the moduli point of an *ordinary abelian variety* is dense-open in the moduli space, and there Serre-Tate show us how to find coordinates in the mixed characteristic case.

- For an *almost ordinary* abelian variety (i.e. the p-rank equals $\dim(A) - 1$) the central leaf passing through is dense in the related Newton Polygon stratum in characteristic p;
- we do construct a generalization of canonical coordinates on this stratum, however in characteristic p.
- Such a construction in mixed characteristic around the moduli point of an *almost ordinary* abelian variety in positive characteristic does not exist if we impose the condition the lifted stratum should be flat over the base and Hecke-invariant.
- For other moduli points in positive characteristic central leaves are smaller than the related Newton Polygon stratum; we do construct a generalization (analogue) of canonical coordinates on (the formal completion of) the central leaf in characteristic p. But we do not construct something like "canonical coordinates" on the whole Newton polygon stratum if the p-rank is smaller than $\dim(A) - 1$. We will explain why. For references see below. For much more information see [19].

1 p-divisible groups

In this section we briefly recall notations we are going to use.

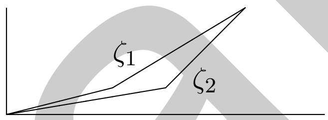
1.1. For given integers $d \in \mathbb{Z}_{>0}$ and $h \in \mathbb{Z}_{\geq d}$ a Newton Polygon ζ (of height h and dimension d) is a lower convex polygon starting at $(0,0)$ and ending at (h,d) and having break points in $\mathbb{Z} \times \mathbb{Z}$; equivalently: we have slopes $0 \leq s_1 \leq \cdots \leq s_h$ with $s_j \in \mathbb{Q}$ and $s_1 + \cdots + s_h = d$ and if $s_b < s_{b+1}$ we have $s_1 + \cdots + s_b \in \mathbb{Z}$; the rational numbers s_j are called the slopes of ζ. We only consider Newton Polygons with $0 \leq s_j \leq 1$ for every j. A Newton Polygon is called *isoclinic* if all slopes are equal; in this case $s_j = d/h$ for every j.

A *symmetric* Newton Polygon is a Newton Polygon ξ with $d = g$, and $h = 2g$ such that any slope $r \in \mathbb{Q}$ with $0 \leq r \leq$ and the slope $1 - r$ appear with the same multiplicity. An isoclinic symmetric Newton Polygon consists of $2g$ slopes $s_j = 1/2$ this is called the *supersingular* Newton Polygon, $\sigma = \sigma_g$.

For Newton Polygons ζ_1, ζ_2 of the same height and dimension we write

$$\zeta_1 \prec \zeta_2 \Longleftrightarrow \quad \text{no point of} \quad \zeta_1 \quad \text{is below} \quad \zeta_2$$

and we will say "ζ_1 is above ζ_2" (a bit strange? explanation: ζ_1 defines a smaller stratum than ζ_2 in this case).



For any field K, a p-divisible group X of height h over K is an inductive system of finite group schemes G_i over K, with maps

$$G_i \xrightarrow{\sim} G_{i+1}[p^i] \subset G_{i+1}, \quad \mathrm{rk}(G_i) = p^{ih},$$

$$G = \cup_i \, G_i, \quad \text{and} \quad \times p : G \twoheadrightarrow G \quad \text{is epimorphic.}$$

Crucial example: For any abelian variety A over a field K we see that $\cup_i \, A[p^i]$ is p-divisible group, denoted by $X = A[p^\infty]$.

The notion of a p-divisible group can be studied over any base field. If p is not the characteristic of the base field, the dual notion of a Tate p-group is an equivalent datum; in this case to notion of a p-divisible group over a perfect K, and the notion of a Tate p-group are determined by a Galois representation.

If however p is the characteristic of the base field K we should be careful using the notion of a "Tate p-group", but the notion of a p-divisible group is very useful. In this case the notion of a p-divisible group can be much more than just a Galois representation.

Over an arbitrary base scheme S one can define the notion of a p-divisible group over S; for details see [46]. For an abelian scheme $A \to S$ indeed $X = A[p^\infty]$ is a p-divisible group over S. This enables us to use this notion in mixed characteristic, even when the residue characteristic is p.

An isogeny of p-divisible groups $\psi : X \to Y$ is a homomorphism with finite kernel and $Y = X/\mathrm{Ker}(\psi)$. A p-divisible group X over a field is called *simple* if for every p-divisible group subgroup $Y \subset X$ either $Y = 0$ or $Y = X$ ("iso-simple" would be a better terminology).

1.2. For the theory of **Diedonné modules** we refer to existing literature, e.g. XXmore references [15], pp. 479–485. The result is that for any *perfect* field $\kappa \supset \mathbb{F}_p$ we have the Dieudonné ring R_κ generated over the ring $\Lambda = \Lambda_\infty(\kappa)$ with automorphism $\sigma \in \mathrm{Aut}(\Lambda_\infty(\kappa))$, the unique lift to Frobenius automorphism $x_0 \mapsto x_0^p$ on κ and operators $\mathcal{F}$ and $\mathcal{V}$ subject to relations

$$\mathcal{F}\cdot\mathcal{V} = p = \mathcal{V}\cdot\mathcal{F}, \quad \mathcal{F}\cdot x = x^\sigma\cdot\mathcal{F}, \quad x\cdot\mathcal{V} = \mathcal{V}\cdot x^\sigma, \quad x \in \Lambda.$$

(For the Witt ring the usual notation is W; however this symbol we use for Newton Polygon strata, see below, hence the notation Λ for the Witt ring.) The ring R_K is commutative if and only if $\kappa = \mathbb{F}_p$. For a finite commutative κ-group scheme N or for a p-divisible group X over κ we define a module $\mathbb{D}(N)$, respectively $\mathbb{D}(G)$ (see the literature for definitions). In particular $\mathbb{D}(G)$ is a left R_κ-module, that is free of rank $\mathrm{ht}(G) = h$ over $\Lambda_\infty(\kappa)$.

In [63] the contravariant theory is defined, used and developed. It has turned out that the covariant theory is easier in use, especially with respect to theories like Cartier theory and the theory of displays; up to duality these two different theories are the same, so for achieving results it does not make much difference which is used.

Note that the Frobenius morphisms $F : G \to G^{(p)}$ induces $\mathcal{V}$ on $\mathbb{D}(G)$, and the Verschiebung $V : G^{(p)} \to G$ induces $\mathcal{F}$ on $\mathbb{D}(G)$. For this reason we distinguish the morphisms F and V (on group schemes) and the operation $\mathcal{V}$ and $\mathcal{F}$ (on covariant Dieudonné modules) by using different symbols. XX see last section in Texel

Some examples. The p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ over $\mathbb{F}_p$ corresponds with $\Lambda\cdot e$ with $\mathcal{V}\cdot e = e$ and $\mathcal{F}\cdot e = pe$.

The p-divisible group μ_{p^∞} over $\mathbb{F}_p$ corresponds with $\Lambda\cdot e$ with $\mathcal{V}\cdot e = pe$ and $\mathcal{F}\cdot e = e$.

For a p-divisible group X of dimension d the module $\mathbb{D}(X)/R_\kappa\cdot V$ is a κ-vector space of dimension d:

$$\dim(X) \quad = \quad \dim_\kappa(\mathbb{D}(X)/R_\kappa\cdot V).$$

For coprime integers $m, n \in \mathbb{Z}_{\geq 0}$ we define the p-divisible group $G_{m,n}$ over $\mathbb{F}_p$ by the property

$$\mathbb{D}(G_{m,n}) = R_{\mathbb{F}_p}/R_{\mathbb{F}_p}\cdot(\mathcal{V}^n - \mathcal{F}^m); \quad \text{we see: } \dim(G_{m,n}) = m, \quad \mathrm{ht}(G_{m,n}) = m + n.$$

This group $G_{m,n}$ is simple over any extensions field. We write $G_{m,n}$ instead of $G_{m,n} \otimes K$ for any $K \supset \mathbb{F}_p$ if no confusion is possible.

1.3. Theorem (Dieudonné, Manin). *Suppose $k = \overline{k} \supset \mathbb{F}_p$ is an algebraically closed field. For any p-divisible group X over k there is an isogeny*

$$X \quad \sim_k \quad \oplus_{(m_i,n_i)\in S} \quad G_{m_i,n_i}$$

for some set S of pairs (repetitions are allowed).

For p-divisible groups of height h and dimension d over an algebraically closed field $k \supset \mathbb{F}_p$ and Newton polygons with the same invariants the set of isogeny classes of p-divisible groups and the set of Newton Polygons are "equal"

$$\{X\}/\sim \quad \xrightarrow{\sim} \quad \{\mathrm{NP}\}, \quad X \mapsto \mathcal{N}(X); \quad \text{height} = h, \quad \dim = d.$$

XXexact reference

This correspondence is given by defining $\mathcal{N}(G_{m,n})$ to be the isoclinic Newton Polygon of height h and slope $m/(m+n)$. For X as in the theorem we order all slopes $m_i/(m_i+n_i)$ with multiplicities m_i+n_i in non-decreasing order, resulting in $\mathcal{N}(X)$.

We mention that the slopes of the Newton Polygon are given basically by the p-adic valuations of the Frobenius of X (but we need Dieudonné module theory to make sense of this). For example, on $X = G_{m,n}$ over $\mathbb{F}_p$ we have $F^n = V^m$ (or, if you want, $(\mathcal{V}^n - \mathcal{F}^m)e = 0$ on the canonical generator of the Diedonné module); from $F^n{\cdot}F^m = V^m{\cdot}F^m = p^m$, and we see "the p-adic valuation of $F = F_X$ is $m/(m+n)$".

We illustrate this for a simple p-divisible X of height h over a finite field $\kappa = \mathbb{F}_q$; in this case, $q = p^r$, we have a Frobenius $\pi : X \to X$, as the r-times iterated absolute Frobenius; write $[\mathbb{Q}(\pi) : \mathbb{Q}] = d$; consider the Newton polygon of the eigenvalues of π, i.e. take the characteristic polynomial of π, and define the slopes by taking p-adic values of the zeros; by stretching this polynomial in horizontal and vertical direction by h/d and multiplying vertically by $1/r$ we obtain $\mathcal{N}(X)$. For the general case, we need the whole theory of Dieudonné modules to define $\mathcal{N}(X)$ in general. XXcheck in example

For the theory of Diedonné modules, see [63], [84], [23], see [33], [5]; for a discussion of the various approaches see [12], Appendix B3.

1.4. Minimal p-divisible groups. One can ask, when is a p-divisible group X determined by its BT_1? (Here BT_1 stands for "truncated Barsotti-Tate group at level one".) This is answered in [113]. For coprime $m, n\mathbb{Z}_{\geq 0}$ define $H_{m,n}$ over $\mathbb{F}_p$ by its Diedonné module $M = \mathbb{D}(H_{m,n})$ generated over $R_{\mathbb{F}_p}$ by a free Λ-basis $\{e_0, e_1, \cdots, e_{m+n-1}\} \subset M$ with relations

$$j = i + b(m+n) \text{ then } e_j = p^b e_i, \quad \mathcal{F}{\cdot}e_i = e_{i+n}, \quad \mathcal{V}{\cdot}e_i = e_{i+m};$$

for simplicity here we write $e_{i+m+n} = p{\cdot}e_i \in \mathbb{D}(H_{m,n})$. As $(\mathcal{F}^m - \mathcal{V}^n)(e_i) = 0$ for every $0 \leq i$, we see that

$$\mathbb{D}(G_{m,n}) = R_{\mathbb{F}_p}{\cdot}e_0 \subset \mathbb{D}(H_{m,n}), \quad \text{hence} \quad G_{m,n} \sim H_{m,n}.$$

This group $H_{m,n}$ is simple over any extension field. We write $H_{m,n}$ instead of $H_{m,n} \otimes K$ for any $K \supset \mathbb{F}_p$ if no confusion is possible, and we know $G_{m,n} \sim_K H_{m,n}$ over any extension field.

Note there exists an endomorphism $u \in \mathrm{End}(\mathbb{D}(H_{m,n})) = \mathrm{End}(H_{m,n})$ (already defined over $\mathbb{F}_p$) with the property $u(e_i) = e_{i+1}$. We see that

$$\mathrm{End}((H_{m,n})_k) \subset \mathrm{End}^0((H_{m,n})_k) = \mathrm{End}^0((G_{m,n})_k) \quad \text{is the maximal order}$$

in the division algebra $\mathrm{End}^0((G_{m,n})_k)$; this is central simple of degree $(m+n)^2$ over a field K that contains $\mathbb{F}_{p^{m+n}}$ checkXXX; over any algebraically closed field $k \supset \mathbb{F}_p$ this characterizes $(H_{m,n})_k$.

Suppose a Newton Polygon ζ is given by $\{(m_i, n_i)\}$; in this case we define

$$H_\zeta = \oplus_i \;\; H_{m_i,n_i}.$$

This is called the minimal p-divisible group in the isogeny class given by ζ.

Theorem ([113] , Theorem 1.2). *Let $k \supset \mathbb{F}_p$ be algebraically closed, and let X be a p-divisible group over k with $\mathcal{N}(X) = \zeta$. Then*

$$X[p] \cong H_\zeta[p] \quad \Longrightarrow \quad X \cong H_\zeta.$$

1.5. Duality. Suppose X is a p-divisible group over a field K, defined by $X[p^i] = G_i$. The exact sequence

$$0 \to G_i \longrightarrow G_{i+j} \longrightarrow G_j \to 0$$

using Cartier duality of finite group schemes gives the exact sequence

$$0 \leftarrow (G_i)^D \leftarrow (G_{i+j})^D \leftarrow (G_j)^D \leftarrow 0.$$

This defines a p-divisible group

$$G^t := \quad \cup \quad \left((G_j)^D \hookrightarrow (G_{j+1})^D\right) \quad = \quad \cup \quad (G_{j+1} \twoheadrightarrow G_j)^D,$$

called the Serre dual of G (please do do not call this the Cartier dual of G; please do not write G^D).

Example/Remark.

$$(G_{m,n})^t \quad = \quad G_{n,m}.$$

Theorem (duality theorem), see [89], Theorem 19.1. *For an isogeny ψ of abelian varieties over any base scheme, with $N = \mathrm{Ker}(\psi)$, the exact sequence*

$$0 \to N \longrightarrow A \xrightarrow{\psi} B \to 0$$

and duality theory of abelian varieties and Cartier duality for finite group schemes give

$$0 \to N^D \longrightarrow B^t \xrightarrow{\psi^t} A^t \to 0.$$

Corollary. *For an abelian scheme A over any base scheme we have*

$$A^t[p^\infty] = (A[p^\infty])^t.$$

For any abelian variety A over $K \supset \mathbb{F}_p$ its Newton Polygon $\mathcal{N}(A)$ is symmetric.

2 Newton Polygon strata

In this section we briefly recall notations we are going to use. These strata can be defined in deformation spaces, but we will not recall definitions in those case. All base schemes are in characteristic p.

2.1. For any abelian scheme $A \to S$, or for any p-divisible group $X \to S$ and a Newton Polygon ξ (respectively ζ) we define

$$\mathcal{W}^0_\xi(A \to S) = \{x \in S \mid \mathcal{N}(A) = \xi\}.$$

Inside $\mathcal{A}_{g,d}$ we write $W_{\xi,d} = \mathcal{W}^0_\xi(\mathcal{A}_{g,d})$. For principally polarized abelian varieties we write

$$W^0_\xi = W^0_{\xi,1} = \mathcal{W}^0_\xi(\mathcal{A}_{g,1}).$$

Discussion. This is a "point-wise" definition. We do not have a good functorial approach to these strata.

Grothendieck and Katz showed these sets are locally closed in $\cup_d \ \mathcal{A}_{g,d}$, see [40], [52], 2.3.2. We consider these definitions either over $\mathbb{F}_p$, or over a perfect field κ, and we consider these

as a scheme with the *reduced scheme structure* on this locally closed set.
Discussion. The proof by Grothendieck and by Katz does give a (non-canonical) scheme structure (in the proof equations are given, but these depend on choices, and clearly we obtain in general many more nilpotents than we want). It is possible that inside $\mathcal{A}_{g,1}$, or more generally for p not dividing d, we do obtain reduces schemes, and on can expect that some other components (when a high power of p divides d) will give a non-reduced scheme, once a good functorial description is available; part of this will follow from [19].

2.2. Here we consider closed Newton Polygon strata. There are two ways of doing this. We can consider the Zariski closure of $\mathcal{W}^0_\xi(A \to S)$ or we can consider

$$\mathcal{W}^0_\xi(A \to S)^{\mathrm{Zar}} \quad \subset \quad \{x \in S \mid \mathcal{N}(A) \prec \xi\}$$

(inclusion because, by Grothendieck, we know that under specialization, Newton Polygons go up).

Fact. *In the deformation space of unpolarized p-divisible groups, and in $\mathcal{A}_{g,d}$ where p does not divide d we have equality:*

$$W_\xi := \mathcal{W}^0_\xi(\mathcal{A}_{g,1})^{\mathrm{Zar}} = \{x \in \mathcal{A}_{g,1} \mid \mathcal{N}(A) \prec \xi\}$$

This follows from the Grothendieck conjecture (for p-divisible groups, for principally polarized p-divisible groups, for principally polarized abelian varieties); see 7.4.

However, there are many examples where

$$\left(\mathcal{W}^0_\xi(\mathcal{A}_{g,d})\right)^{\mathrm{Zar}} \quad \subsetneqq \quad \{x \in \mathcal{A}_{g,d} \mid \mathcal{N}(A) \prec \xi\}$$

Here is an easy example, see [51]XXX; consider $g = 3$, and $d = XX$. Inside $\mathcal{A}_{3,d}$ the supersingular locus has a component V of dimension three; this is one of the irreducible components of the locus inside $\mathcal{A}_{3,d}$ of abelian varieties with p-rank equal to zero; consider $\xi = (2,1) + (1,2)$; the closure $(\mathcal{W}^0_\xi(\mathcal{A}_{g,1}))^{\mathrm{Zar}}$ does not contain V, and we see inequality in this case. Below we discuss many other cases, see Section 3, and a systematic way to find them.

2.3. Supersingular abelian varieties. We say an elliptic curve E is *supersingular* if $E[p](k) = \{0\}$. An elliptic curve is supersingular if and only if $\mathcal{N}(E) = \sigma_1 = (1,1)$. As there only two symmetric Newton Polygons for $g = 1$ this is clear. For every p there is at least one elliptic curve in characteristic p.

We define

$$a(G) := \dim_\kappa \mathrm{Hom}(\alpha_p, G),$$

where κ is a perfect field, and G a commutative group scheme over κ.
Theorem. *Let $k \supset$ be an algebraically closed field.*
(1) *For an abelian variety A of dimension g the following statements are equivalent*

- $\mathcal{N}(A) = \sigma_g := g\cdot(1,1)$;
- $A[p^\infty] \sim (G_{1,1})^g$;
- *for any supersingular elliptic curve E there exists an isogeny $E^g \sim A$;*
- **Defintion.** *The abelian variety is* supersingular.

(2) *Suppose $a(A) = g > 1$ then for every supersingular elliptic curve E there is an isomorphism $E^g \cong A$;* in this case the abelian variety is called *superspecial.*
XXX? give steps in proof

Note the fact that every supersingular abelian variety of dimension at least two is *not geometrically simple*, but for every other Newton Polygon we can find a geometrically simple abelian variety having that Newton Polygon, see HWLFOXX *[60]*

2.4. Description of components of ss locus

2.5. EO strata XX

3 Stratifications and foliations

In this section we briefly recall notations and constructions we are going to use. All base schemes are in characteristic p. These constructions were suggested by the Hecke orbit problem: try to describe the Hecke orbit $\mathcal{H}(x)$ of a moduli point. In characteristic p the Newton Polygon does not change under such actions. We are interested in the Zariski closure of the full Hecke orbit. Foliations described here "separate" the two qui different cases, quasi-isogenies of degree prime to p on the one hand ("moving in a central leaf") and α-isogenies (kernels have filtrations where successive quotients are isomorphic with α_p, "moving in an isogeny leaf").

3.1. Central leaves. Suppose $[(B, \mu)] = x \in \mathcal{A}_{g,d}(\kappa)$, where κ is a *perfect* field. We write

$$\mathcal{C}(x)_d = \{[(A, \nu)] = \{y \in \mathcal{A}_{g,d} \mid (B, \mu)[p^\infty] \otimes \Omega \cong (A, \nu) \otimes \Omega\};$$

here $\Omega \supset \kappa$ is an algebraically closed field over which (A, ν) is defined; this is called the *central leaf* passing through x; for $d = 1$ we simply write $\mathcal{C}(x) = \mathcal{C}(x)_1$. Theory about these constructions and objects we find in [110]. The set $\mathcal{C}(x)_d \subset \mathcal{A}_{g,d} \otimes \kappa$ is locally closed, in fact $\mathcal{C}(x)_d \subset (W_{\xi,d})_\kappa$ is closed, where $\xi = \mathcal{N}(B)$. We consider this as a κ-subscheme with the reduced scheme structure. WE attache the index d in order to remind reader we are working in $\mathcal{A}_{g,d}$.
Discussion. This is a "point-wise" definition. At the time of writing [110] we did not have a better way of approach. Now we develop a functorial approach, see [19]. It will turn out that there is a canonical scheme structure on these $\mathcal{C}(x)_d$. In case $d = 1$ (the principally polarized case) the previous $\mathcal{C}(x)$ and the new leaves coincide. However there are cases (where p divides d), where the new structure gives a scheme with nilpotents, over a perfect field having $\mathcal{C}(x)_d$ as reduced scheme structure. Moreover, over non-perfect fields subtleties appear: the new structure can give a reduced scheme, which over the perfection of the base field does have nilpotents (the case of "hidden nilpotents"); hence the "old" definition and construction of $\mathcal{C}(x)_d$ is full of difficulties over non-perfect fields. These phenomena, with many examples will be discussed in [19].

3.2. Theorem. see [110], Theorem 2.3.

$$\mathcal{C}_X(S) \subset \mathcal{W}^0_{\mathcal{N}(X)}(S)$$

is a closed set.

3.3. Theorem. Isogeny correspondences, unpolarized case. *Let $\psi : X \to Y$ be an isogeny between p-divisible groups. Then the isogeny correspondence contains an integral scheme T with two finite surjective morphisms*

$$\mathcal{C}_X(D(X)) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_Y(D(Y))$$

such that T contains a point corresponding with ψ.

3.4. For any two points $x \in \mathcal{A}_{g,d}$, $y \in \mathcal{A}_{g,e}$ belonging to the same Newton Polygon ξ, we have

$$\dim(\mathcal{C}(x)_d) \quad = \quad \dim(\mathcal{C}(y)_e),$$

all central leaves in the same Newton Polygon stratum have the same dimension. This number, depending only on $\xi = \mathcal{N}(A_x)$ will be denoted by $c(\xi)$.

3.5. Isogeny correspondences, polarized case. Let $\psi : A \to B$ be an isogeny, and let λ respectively μ be a polarization on A, respectively on B, and suppose there exists an integer $n \in \mathbb{Z}_{>0}$ such that $\psi^*(\mu) = n{\cdot}\lambda$. Then there exist finite surjective morphisms

$$\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [110], 3.16.

3.6. The dimension of $\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$ only depends on the isogeny class of (X, λ).

Remark/Notation. *In fact, this dimension depends only on the isogeny class of X. We write*

$$c(\xi) := \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)\right), \quad X = A[p^\infty], \quad \xi := \mathcal{N}(X);$$

this is well defined: all irreducible components have the same dimension.

3.7. The central stream. Suppose $[(A, \lambda)] = x \in \mathcal{A}_{g,1}$ such that $A[p^\infty]$ is the minimal p-divisible group associated with $\xi = \mathcal{N}(A)$, i.e. $A[p^\infty] \otimes k \cong H_\xi \otimes k$. In this case we write

$$\mathcal{C}(x) =: \mathcal{Z}_\xi \subset W^0_\xi \subset \mathcal{A}_{g,1}, \quad \text{called the } \textit{central stream} \text{ in } \ W^0_\xi.$$

We will see this central leaf plays an important role in many considerations.

3.8. Isogeny leaves. We say $A \cdots \to B$ is a quasi-isogeny if there is an integer q such that

$$A \xrightarrow{\times q} A \cdots \to B$$

an isogeny. We say it is an α-quasi-isogeny if (perhaps over some extension field) kernels have filtrations where successive quotients are isomorphic with α_p. Suppose $[(B, \mu)] = x \in \mathcal{A}_{g,d}(\kappa)$, where κ is a *perfect* field. We write

$$I(x)_d = \{[(A, \nu)]y \in \mathcal{A}_{g,d} \mid (A, \nu) \otimes \Omega \ \ \text{q}\!\sim \ \ (B, \mu) \otimes \Omega\}.$$

This is a closed set in $W_{\xi,d}$, we give it the reduced scheme structure, it is proper over $\mathrm{Spec}(\kappa)$.
Discussion. This is a “point-wise” definition.

Remark. As we see in Katz XXYYY complete this remark on functorial def of isogeny leaf too big.

The almost product structure on $W_{\xi,d}$. For any $W_{\xi,d}$ over k, with ξ not the ordinary Newton Polygon, there exist C' and I' and a surjective, finite, flat morphism $C' \to \mathcal{C}(x)$, a surjective, finite morphism $I' \to \mathcal{I}(x)$, and a finite surjective morphism $\Phi : C' \times I' \to W_{\xi,d}$ such that for any $x' \in C'$, with $x' \mapsto x$, the image

$$\mathcal{I}(x) = \Phi(\{x'\} \times I' \subset W_{\xi,d}$$

is the isogeny leaf $\mathcal{I}(x)$, and for any $x'' \in C'$, with $x'' \mapsto x$ the image

$$\mathcal{C}(x) = \Phi(C' \times \{x''\}) \subset W_{\xi,d}$$

is the central leaf $\mathcal{C}(x)$. This is the reason we use the word "foliation" in both cases in the sense of a disjoint union of (sometimes) lower dimensional closed sets (although in some theories in mathematics this word is used in a more strict sense). Note that on any irreducible component we have that $\dim(W'_{\xi,d})$ and $\dim(I(x)_d) = \dim(W'_{\xi,d}) - c(\xi)$ can depend on the choice of the component $W'_{\xi,d}$.

We define $i(\xi) = \dim W^0_\xi - c(\xi)$; we see $i(\xi) = \dim(I(\xi))$ in $\mathcal{A}_{g,1}$. We will see $i(\xi) \leq \dim(I(\xi)_d)$, and we will see (many) cases where the inequality sign holds.

Some examples. For the *ordinary* symmetric Newton polygon $\rho = (g, 0) + (0, g)$ isogeny leaves are empty and $W_{\rho,d}$ is a central leaf.

For the *almost ordinary* symmetric Newton Polygon $\xi = (g-1, 0) + (1, 1) + (0, g-1)$ isogeny leaves are zero-dimensional, and $W_{\xi,d}$ is a central leaf.

For the *supersingular* symmetric Newton Polygon $\sigma = g{\cdot}(1, 1)$ central leaves in $W_{\sigma,d}$ are finite, and $W_{\sigma,d}$ is an isogeny leaf.

For $\xi = (g-1, 1) + (1, g-1)$ isogeny leaves are rational curves, and $c(\xi) = g(g+1)/2 - g - 1$; in this case all components of the whole Newton polygon stratum have the same dimension. This example can be used to illustrate interesting phenomena.

3.9. For every Newton Polygon the dimension of the related stratum (in the deformation space of a p-divisible group) can be computed from the data defining the polygon; also the dimension of W_ξ can be seen; also the possible dimensions of $W_{\xi,d}$ and of $\mathcal{I}(x)_d$ can be computed; see [118]. Here are the results.

Notation. Let ζ be a Newton polygon, and $(x, y) \in \mathbb{Q} \times \mathbb{Q}$. We write
$(x, y) \prec \zeta$ if (x, y) is on or above ζ,
$(x, y) \precneqq \zeta$ if (x, y) is strictly above ζ,
$(x, y) \succ \zeta$ if (x, y) is on or below ζ,
$(x, y) \succneqq \zeta$ if (x, y) is strictly below ζ.

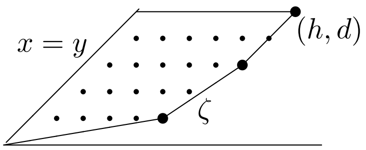
Notation (the unpolarized case). We fix integers $h \geq d \geq 0$, and we write $c := h - d$. We consider Newton polygons ending at (h, d). For such a Newton polygon ζ we write

$$\diamondsuit(\zeta) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < d, \quad y < x, \quad (x, y) \prec \zeta\},$$

and we write

$$\boxed{\dim(\zeta) := \#(\diamondsuit(\zeta)).}$$

Example:



$\zeta = 2 \times (1,0) + (2,1) + (1,5) =$

$= 6 \times \frac{1}{6} + 3 \times \frac{2}{3} + 2 \times \frac{1}{1}; \quad h = 11.$

Here $\dim(\zeta) = \#(\Diamond(\zeta)) = 22$.

3.10. Notation. We write

$$\Diamond(\zeta;\zeta^*) := \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid (x,y) \prec \zeta, \quad (x,y) \not\precneq \zeta^*\},$$

$$\boxed{\mathrm{cdu}(\zeta) := \#\left(\Diamond(\zeta;\zeta^*)\right)};$$

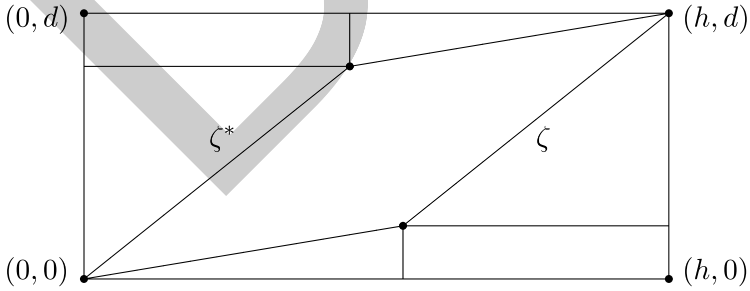
"cdu" = dimension of central leaf, unpolarized case.

We suppose $\zeta = \sum_{1 \leq i \leq u} \mu_i{\cdot}(m_i, n_i)$, written in such a way that $\gcd(m_i, n_i) = 1$ for all i, and $\mu_i \in \mathbb{Z}_{>0}$, and $i < j \Rightarrow (m_i/(m_i + n_i)) > (m_j/(m_j + n_j))$. Write $d_i = \mu_i{\cdot}m_i$ and $c_i = \mu_i{\cdot}n_i$ and $h_i = \mu_i{\cdot}(m_i + n_i)$; write $\nu_i = m_i/(m_i + n_i) = d_i/(d_i + c_i)$ for $1 \leq i \leq u$. Note that the slope ν_i equals $\mathrm{slope}(G_{m_i,n_i}) = m_i/(m_i + n_i) = d_i/h_i$: this Newton polygon is the "Frobenius-slopes" Newton polygon of $\sum (G_{m_i,n_i})^{\mu_i}$. Note that the slope ν_i appears h_i times; these slopes with these multiplicities give the set $\{\beta_j \mid 1 \leq j \leq h := h_1 + \cdots + h_u\}$ of all slopes of ζ.

3.11. Combinatorial Lemma, the unpolarized case. *The following numbers are equal*

$$\#\left(\Diamond(\zeta;\zeta^*)\right) =: \mathrm{cdu}(\zeta) \quad = \quad \sum_{i=1}^{i=h} (\zeta^*(i) - \zeta(i)) =$$

$$= \sum_{1 \leq i < j \leq u} (d_i c_j - d_j c_i) = \sum_{1 \leq i < j \leq u} (d_i h_j - d_j h_i) = \sum_{1 \leq i < j \leq u} h_j{\cdot}h_i{\cdot}(\nu_i - \nu_j).$$



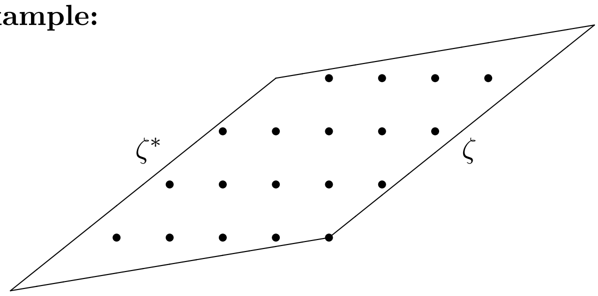
$h = h_1 + \cdots + h_u$

$d = d_1 + \cdots + d_u$

3.12. Theorem. (Dimension formula, the unpolarized case.) *Let X_0 be a p-divisible group, $D = D(X_0)$; let $y \in D$, let Y be the p-divisible group given by y with $\beta = \mathcal{N}(Y) \succ \mathcal{N}(X_0)$;*

$$\dim(\mathcal{C}_Y(D)) \quad = \quad \mathrm{cdu}(\beta).$$

Example:



$\dim(\mathcal{C}_X(D)) = \#\left((\Diamond(\zeta;\zeta^*)\right); \quad (\frac{4}{5} - \frac{1}{6}){\cdot}5{\cdot}6 = 19,$

$d_1h_2 - d_2h_1 = 4{\cdot}6 - 1{\cdot}5 = 19; \quad d_1c_2 - d_2c_1 = 4{\cdot}5 - 1{\cdot}1 = 19.$

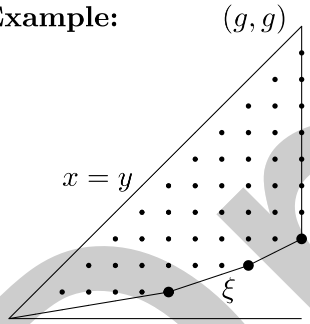
3.13. The dimension of central leaves, the polarized case.
Notation. We fix an integer g. For every *symmetric* Newton polygon ξ of height $2g$ we define

$$\triangle(\xi) = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid y < x \leq g, \ \ (x,y) \ \text{ on or above } \ \xi\},$$

and we write

$$\boxed{\mathrm{sdim}(\xi) := \#(\triangle(\xi)).}$$

Example:



$\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1}\otimes\mathbb{F}_p)) = \#(\triangle(\xi)),$
$\xi = (5,1) + (2,1) + 2{\cdot}(1,1) + (1,2) + (1,5)$, $g = 11$;
slopes: $\{6\times\frac{5}{6}, 3\times\frac{2}{3}, 4\times\frac{1}{2}, 3\times\frac{1}{3}, 6\times\frac{1}{6}\}$.
This case: $\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1}\otimes\mathbb{F}_p)) = \mathrm{sdim}(\xi) = 48$.

3.14. Let ξ be a symmetric Newton polygon. For convenience we adapt notation to the symmetric situation:

$$\xi = \mu_1{\cdot}(m_1,n_1) + \cdots + \mu_s{\cdot}(m_s,n_s) + r{\cdot}(1,1) + \mu_s{\cdot}(n_s,m_s) + \cdots + \mu_1{\cdot}(n_1,m_1)$$

with

$m_i > n_i$ and $\gcd(m_i,n_i) = 1$ for all i,
$1 \leq i < j \leq s \Rightarrow (m_i/(m_i+n_i)) > (m_j/(m_j+n_j))$,
$r \geq 0$ and $s \geq 0$.
We write $d_i = \mu_i{\cdot}m_i$, and $c_i = \mu_i{\cdot}n_i$, and $h_i = d_i + c_i$. Write $g := \left(\sum_{1\leq i\leq s}(d_i+c_i)\right) + r$ and write $u = 2s+1$.

We write

$$\triangle(\xi;\xi^*) := \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid (x,y) \prec \xi, \quad (x,y) \gneqq \xi^*, \quad x \leq g\},$$

$$\boxed{\mathrm{cdp}(\xi) := \#\left(\triangle(\xi;\xi^*)\right);}$$

"cdp" = dimension of central leaf, polarized case.

Write $\xi = \sum_{1 \leq i \leq u} \mu_i \cdot (m_i, n_i)$, i.e., $(m_j, n_j) = (n_{u+1-j}, m_{u+1-j})$ for $s < j \leq u$ and $r(1,1) = \mu_{s+1}(m_{s+1}, n_{s+1})$. Write $\nu_i = m_i/(m_i + n_i)$ for $1 \leq i \leq u$; hence $\nu_i = 1 - \nu_{u+1-i}$ for all i.

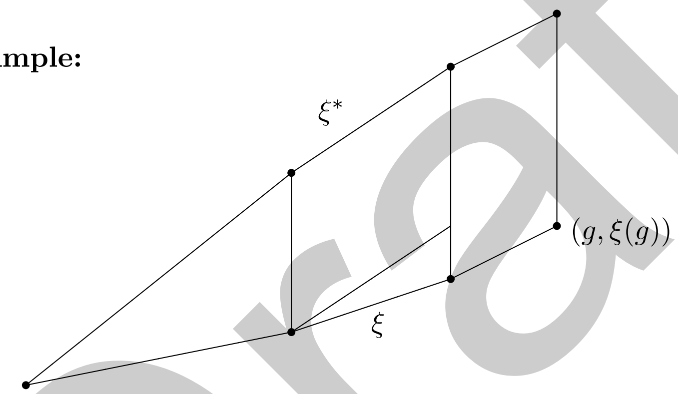
3.15. Combinatorial Lemma, the polarized case. *The following numbers are equal*

$$\#\left(\triangle(\xi;\xi^*)\right) =: \mathrm{cdp}(\xi) = \frac{1}{2}\mathrm{cdu}(\xi) + \frac{1}{2}(\xi^*(g) - \xi(g)) = \sum_{1 \leq j \leq g} (\xi^*(j) - \xi(j)) =$$

$$= \sum_{1 \leq i \leq s} \left(\frac{1}{2} \cdot d_i(d_i+1) - \frac{1}{2} \cdot c_i(c_i+1)\right) + \sum_{1 \leq i < j}^{j \leq s} (d_i - c_i)h_j + \left(\sum_{i=1}^{i=s} (d_i - c_i)\right) \cdot r =$$

$$= \frac{1}{2} \sum_{1 \leq i \leq s} (2\nu_i - 1)h_i(h_i+1) + \frac{1}{2} \sum_{1 \leq i < j \neq u+1-i} (\nu_i - \nu_j)h_i h_j.$$

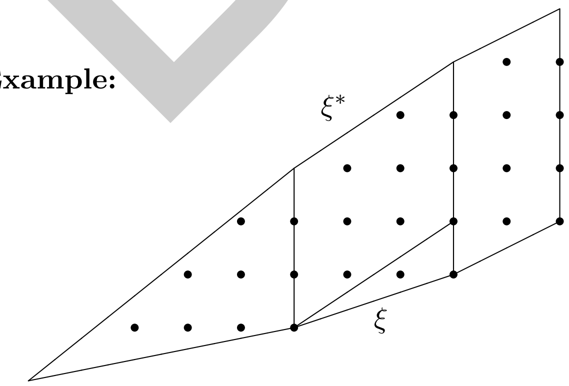
Example:



3.16. Theorem (Dimension formula, the polarized case). *Let (A, λ) be a polarized abelian variety. Let $(X, \lambda) = (A, \lambda)[p^\infty]$; write $\xi = \mathcal{N}(A)$; then*

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A} \otimes \mathbb{F}_p)\right) \quad = \quad \mathrm{cdp}(\xi).$$

Example:



$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \sum_{0 < i \leq g} (\xi^*(i) - \xi(i))$,
slopes $1/5, 4/5, h = 5$: $\quad \frac{1}{2}4 \cdot 5 - \frac{1}{2}1 \cdot 2 = 9$,

slopes $1/3, 2/3, h = 3$: $\quad \frac{1}{2}2{\cdot}3 - \frac{1}{2}1{\cdot}2 = 2$,
$(d_1 - c_1)h_2 = 3{\cdot}3 = 9$,
$(d_1 + d_2 - c_1 - c_2)r = 4{\cdot}2 = 8$,
$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \#\left(\triangle(\zeta;\zeta^*)\right) = 28$.

3.17. We know that $\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1})) = \mathrm{sdim}(\xi)$, see 3.13. We like to know what the dimension could be of an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g)$. Note that isogeny correspondences blow up and down in general, hence various dimensions a priori can appear.

Write $\mathcal{V}_f(\mathcal{A}_g)$ for the moduli space of polarized abelian varieties having p-rank at most f; this is a closed subset, and we give it the induced reduced scheme structure. By [83], Theorem 4.1 we know that every irreducible component of this space has dimension exactly equal to $(g(g+1)/2) - (g-f) = ((g-1)g/2) + f$ (it seems a miracle that under blowing up and down this locus after all has only components of exactly this dimension).

Let ξ be a symmetric Newton polygon. Let its p-rank be $f = f(\xi)$. This is the multiplicity of the slope 1 in ξ; for a symmetric Newton polygon it is also the multiplicity of the slope 0. Clearly

$$\mathcal{W}^0_\xi(\mathcal{A}_g) \quad \subset \quad \mathcal{V}_{f(\xi)}(\mathcal{A}_g).$$

Hence for every irreducible component

$$T \subset \mathcal{W}^0_\xi(\mathcal{A}_g) \quad \text{we have} \quad \dim(T) \leq \frac{1}{2}(g-1)g + f.$$

In [?], 5.8, we find the conjecture that

for any ξ we expect there would be an irreducible component
T of $\mathcal{W}^0_\xi(\mathcal{A}_g)$ with $\dim(T) = ((g-1)g/2) + f(\xi)$.

In this section we settle this question completely by showing that this is true for many Newton polygons, but not true for all. The result is that a component can have the maximal possible (expected) dimension: for many symmetric Newton polygons the conjecture is correct (for those with $\delta(\xi) = 0$, for notation see below), but for every $g > 4$ there exists a ξ for which the conjecture fails (those with $\delta(\xi) > 0$); see 3.19 for the exact statement.

3.18. Notation. Consider $\mathcal{W}^0_\xi(\mathcal{A}_g)$ and consider every irreducible component of this locus; let $\mathrm{minsd}(\xi)$ be the minimum of $\dim(T)$, where T ranges through the set of such irreducible components of $\mathcal{W}^0_\xi(\mathcal{A}_g)$, and let $\mathrm{maxsd}(\xi)$ be the maximum. Write

$$\delta = \delta(\xi) := \lceil(\xi(g))\rceil - \#\left(\{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid f(\xi) < x < g, \quad (x,y) \in \xi\}\right) - 1,$$

where $\lceil b\rceil$ is the smallest integer not smaller than b. Note that $\xi(g) \in \mathbb{Z}$ if and only if the multiplicity of $(1,1)$ in ξ is even. Here δ stands for "discrepancy." We will see that $\delta \geq 0$. We will see that $\delta = 0$ and $\delta > 0$ are possible.

3.19. Theorem.

$$\mathrm{sdim}(\xi) = \mathrm{minsd}(\xi),$$

and

$$\mathrm{maxsd}(\xi) = \mathrm{cdp}(\xi) + \mathrm{idu}(\xi) = \frac{1}{2}(g-1)g + f(\xi) - \delta(\xi).$$

3.20. Corollary/Examples. *Suppose* $\xi = \sum(m_i, n_i)$ *with* $gcd(m_i, n_i) = 1$ *for all* i. *Then*

$$\delta(\xi) = 0 \quad \Longleftrightarrow \quad \min(m_i, n_i) = 1, \quad \forall i.$$

We see that $0 \le \delta(\xi) \le \lceil g/2 \rceil - 2$*. We see that*

$$\mathrm{maxsd}(\xi) = \frac{1}{2}(g-1)g/2 + f \quad \Longleftrightarrow \quad \delta(\xi) = 0.$$

We see that $\delta(\xi) > 0$ for example in the following cases:
$g = 5$ and $\delta((3,2)+(2,3)) = 1$, $\quad g = 8$ and $\delta((4,3)+(1,1)+(3,4)) = 2$,
more generally, $g = 2k+1$, and $\delta((k+1,k)+(k,k+1)) = k-1$,
$g = 2k+2$, and $\delta((k+1,k)+(1,1)+(k,k+1)) = k-1$.

Knowing this theorem, one can construct many examples of pairs of symmetric Newton polygons $\zeta \precneqq \xi$ such that

$$\mathcal{W}^0_\zeta(\mathcal{A}_g) \quad \not\subset \quad \left(\mathcal{W}^0_\xi(\mathcal{A}_g)\right)^{\mathrm{Zar}}, \text{ i.e. } \{(B,\mu) \mid \mathcal{N}(B) \prec \xi\} \subsetneqq \left(\mathcal{W}^0_\xi(\mathcal{A}_g)\right)^{\mathrm{Zar}}.$$

For proofs and more information, see [110], [118], [119].

4 Irredicibility

We sketch proofs of:

4.1. Theorem. *Any Newton polygon stratum* $W_\xi \subset \mathcal{A}_g$ *with* $\xi \neq \sigma$ *is geometrically irreducible.*
See [101], Conjecture 8.3.
Here $\mathcal{A}_g = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is the moduli space of principally polarized abelian varieties, ξ is a symmetric Newton Polygon not equal to the supersingular Newton Polygon σ and W_ξ is the related Newton stratum

$$W_\xi = \mathcal{W}_\xi(\mathcal{A}) = \{(A,\lambda) \in \mathcal{A} \mid \mathcal{N}(A) \prec \xi\} \subset \mathcal{A}_{g,1}.$$

Note that the locus $W^0_\xi \subset \mathcal{A}_{g,1}$ where the Newton Polygon equals ξ has W_ξ as Zariski closure in the principally polarized case, as follows by the Grothendieck conjecture.

4.2. Theorem. *Let* $x \in \mathcal{A}_g = \mathcal{A}_{g,d} \otimes \mathbb{F}_p$*, not in the supersingular locus. The central leaf* $\mathcal{C}(x)$ *containing* x *is geometrically irreducible.* Th. 4.1

4.3. Theorem. *Suppose* $k \supset \mathbb{F}_p$ *is algebraically closed,* $x \in \mathcal{A}_{g,d}(k)$ *not in the supersingular locus; the* p*-adic monodromy on* $\mathcal{C}(x)$ *is surjective on* $\mathrm{Aut}(A_x, \lambda_x)[p^\infty]$.
For precise formulation of these theorems, see [16], 3.1, 4.1, 5.6.

4.4. Step 1. Claim. *For every symmetric Newton polygon* ξ *and every irreducible component* W *of* W_ξ *we have* $W \cap W_\sigma \neq \emptyset$*, i.e.* W *contains a supersingular point.*

4.5. Tool 2: EO strata. For details, see [103]. We make use of the fact that finite, commutative group schemes of given rank annihilated by p are finite in number (up to isomorphism over an algebraically closed field); this was proved by Kraft and by Oort; indeed we need more, namely also the polarization has to be taken into account, see [103], Section 9. On $\mathcal{A}$ we can consider for a given geometric isomorphism class φ of $(A,\lambda)[p]$ the set S_φ thus defined. In this way we obtain the Ekedahl-Oort stratification of $\mathcal{A}$. Some of the basic facts:

- *the isomorphism types are characterized by "elementary sequences"; for every value of g these form a partially ordered, finite set (in fact there are two orderings, see* [103], 14.3);
- *for every φ the locus $S_\varphi \subset \mathcal{A}$ is locally closed and quasi-affine;*
- *the superspecial $S_{E^g[p]}$ has dimension zero, and all other strata have positive dimension;*
- *consider $\mathcal{A}^*$, the "minimal compactification" (or, the Satake compactification); for every φ there is a naturally defined $T_\varphi \subset \mathcal{A}$, see* [103], 6.1; *for every φ such that the dimension of S_φ is positive (i.e. φ not superspecial), the Zariski closure of S_φ contains a point in $\mathcal{A}$ not in S_φ.*

4.6. Tool 3: Finite Hecke orbits (Chai).
Proposition. *A Hecke-ℓ orbit $\mathcal{H}_\ell{\cdot}x$ is finite iff A_x is supersingular.*
See [6], Prop. 1 on page 448.

Proof of Step one 4.4. Consider

$$\Gamma := \cup_{W\subset W_\xi, W\cap W_\sigma=\emptyset} \;\; W,$$

the union over all irreducible components of W_ξ which contain no supersingular point (and we show that Γ is empty). From the fact that any component H' of a Hecke-ℓ^i-correspondence $\mathcal{A}_g \leftarrow H' \rightarrow \mathcal{A}_g$ is finite-to-finite outside characteristic ℓ we see that Γ as defined above is Hecke-ℓ stable.

Consider all EO-strata meeting Γ; let φ be an elementary sequence appearing on Γ which is minimal for the "$\subset$" ordering. Let $x \in \Gamma \cap S_\varphi$, hence $\mathrm{ES}(A_x) = \varphi$. Note that $x \notin W_\sigma$. Hence $\mathcal{H}_\ell{\cdot}x$ is not finite by 4.6. Note that $\mathcal{H}_\ell{\cdot}x \subset \Gamma \cap S_\varphi$. Hence $S_\varphi \cap \Gamma$ has positive dimension. By 4.5 this implies that there is point y in the closure of $\Gamma \cap S_\varphi \subset \mathcal{A}$ which is not in S_φ. This is a contradiction with minimality of φ. $\square$4.4

4.7. Step 2.
Claim. *For every symmetric Newton polygon ξ and every component, $W \subset W_\xi$ there is a component $T \subset W_\sigma$ such that $T \subset W$.*

For every $\zeta \prec \xi$, ordering by inclusion of components gives a well-defined map $\pi_0(W_\zeta) \to \pi_0(W_\xi)$ which moreover is surjective: every component of W_ζ is contained in a unique component of W_ξ and every component of W_ξ contains at least one of W_ζ.

4.8. Tool 4: Purity. See [48], 4.1: *In a family, if the Newton polygon jumps, it already jumps in codimension one.*

4.9. Tool 5: Deformations with constant Newton polygon. *For any* principally *polarized abelian variety (A, λ) there exists a deformation with generic fiber (A', λ') with $\mathcal{N}(A) = \mathcal{N}(A')$ and $a(A') \leq 1$.*
See [48], (5.12), and [104], (3.11) and (4.1).

4.10. Tool 6: Cayley-Hamilton. See [102], 3.5 and use 4.9. *In particular this shows that around any point $x \in \mathcal{A}$, with $\mathcal{N}(A_x) = \zeta$ and $a(A_x) = 1$ the Newton strata $\{W_\xi^{/x} \mid \xi \succ \zeta\}$ are formally smooth and nested like the graph of all Newton polygons below ζ. The dimension of $W_\xi \subset \mathcal{A}$ equals* $\mathrm{sdim}(\xi)$*, and the generic point of any component of W_ξ has a-number at most one.* See 7.11.

Remark. Using 4.9 and 4.10 we can prove a conjecture by Grothendieck, see [102] and [104].

For the definition of $\mathrm{sdim}(\xi)$ see [102], 3.3, or [104], 1.9. Note that different components of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ may have different dimensions: our formula using $\mathrm{sdim}(\xi)$ works for *principally polarized* abelian varieties.

Proof of Step two 4.7. We have seen by 4.4 that W contains a supersingular point. By Purity, 4.8, we know that Newton polygons jump in codimension one; by 4.10 we know what the dimensions are of all strata defined by symmetric Newton polygons; in particular the difference of the codimensions of W_σ and of W_ξ in $\mathcal{A}$ is precisely the length of the longest chain of symmetric Newton polygons between ξ and σ. Combining these two we conclude: *W contains a component of W_σ*: by purity a component of $W_\xi \cap W_\sigma$ has at least codimension $\mathrm{sdim}(\xi) - \mathrm{sdim}(\sigma)$ in W_ξ, hence dimension at least $\mathrm{sdim}(\sigma)$ and, being contained in W_σ which is pure of dimension $\mathrm{sdim}(\sigma)$ the statement follows.

For an inclusion $T \subset W$ we choose a point $x \in T$ with $a(A_x) = 1$, which exists, by 4.10, or by [61], (4.9.iii). Around this point $x \in T \subset W$ we apply 4.10. This ends the roof of Step 2. $\square$4.7

Notation. For $g \in \mathbb{Z}_{>1}$ and $j \in \mathbb{Z}_{\geq 0}$ we write $\Lambda_{g,j}$ for the set of isomorphism classes of polarizations μ on the superspecial abelian variety $A = E^g \otimes k$ such that $\mathrm{Ker}(\mu) = A[F^j]$; here E is a supersingular elliptic curve defined over $\mathbb{F}_p$. Note that $\Lambda_{g,j} \xrightarrow{\sim} \Lambda_{g,j+2}$ under $\mu \mapsto F^t{\cdot}\mu{\cdot}F$.

4.11. Tool 7: Characterization of components of W_σ. *There is a canonical bijective map*

$$\pi_0(W_\sigma) \quad \xrightarrow{\sim} \quad \Lambda_{g,g-1}.$$

See [61], 3.6 and 4.2; this uses [87], 2.2 and 3.1.

4.12. Tool 8: Transitivity. *The action of $\mathcal{H}_\ell$ on $\pi_0(W_\sigma)$ is transitive.*
By 4.11 the problem is translated into a question of transitivity of the set of isomorphism classes of certain polarizations on a superspecial abelian variety. Use [29], pp. 158/159 to describe the set of isomorphism classes of such polarizations. Use the strong approximation theorem, see [125], Theorem 7.12 on page 427.

4.13. Tool 9: Theorem (C.-L. Chai). *Choose notation as above. Let $Z \subset \mathcal{B}$ be a locally closed subscheme, smooth over* $\mathrm{Spec}(k)$, *such that:*

Z is Hecke-ℓ-stable, and

the Hecke-ℓ-action on the set $\pi_0(Z)$ is transitive, and

$\eta \notin W_\sigma$ (equivalently: Z contains a non-supersingular point).

Then:

$$\rho_{A,\ell} : \pi_1(Z^0, \overline{\eta}) \longrightarrow \mathrm{Sp}(T_\ell, <,>_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$$

is surjective, *and*

$$Z \textit{ is irreducible, i.e. } Z = Z^0.$$

See [6], 4.4.

The end of the proof of 4.1. We show that $\mathcal{W}_\xi(\mathcal{A}_{g,1,N} \otimes \mathbb{F}_p)$ is geometrically irreducible for $\xi \neq \sigma$; from this the conclusion of 4.1 clearly follows. Write $\mathcal{B} = \mathcal{A}_{g,1,N} \otimes \mathbb{F}_p$. Indeed, by Step 2 we know that $\pi_0(W_\sigma) \to \pi_0(W_\xi)$ is surjective, and the same we conclude for $\pi_0(\mathcal{W}_\sigma(\mathcal{B})) \to \pi_0(\mathcal{W}_\xi(\mathcal{B}))$. By 4.11 and 4.12 we conclude that the action of Hecke-ℓ on $\pi_0(\mathcal{W}_\xi(\mathcal{B}))$ is transitive. Hence by 4.13 we conclude that $\mathcal{W}_\xi(\mathcal{B})$ is geometrically irreducible for $\xi \neq \sigma$. This ends the proof of 4.1. $\square$

For more precise formulation of 4.2, and 4.3 and for proofs, see [16], 4.1, 5.6.

5 CM liftings

We recall questions and results as stated in [12]. We start with an abelian variety A_0 or a p-divisible group X_0 over a finite field $\kappa = \mathbb{F}_q$, and we ask for a CM lift. In general we know that after a field extension and an isogeny a CM lift does exist (as Honda and Tate showed).

5.1. An isogeny is necessary. In [98] we see *the existence of p-divisible groups over $\overline{\mathbb{F}_p}$ that do not allow a* CM *lift.* In fact, in =this paper we can choose any $g \geq 3$, and any symmetric Newton polygon of "p-rank" at most $g-3$ and find examples for this Newton Polygon.Hence the analogous statement holds for abelian varieties over $\overline{\mathbb{F}_p}$. In [12], Chapter 3 this method and result have been generalized.

Here is the basic idea of the proof. Start with an abelian variety A over $\overline{\mathbb{F}_p}$ with $a(A) = 2$ such that there exists $\alpha_p \cong N \subset A$ with $a(A/N) = 1$ (for every Newton polygon of "p-rank" at most $g-2$ such an abelian variety does exist). The $\mathbb{P}^1$-family of (α_p)-quotients of A is studied, and we see, using CM theory in characteristic zero, that at most finitely many of these quotients admit a CM lift; for details see [98]. A proof not using any characteristic zero methods can be found in [12], Chapter 3.

5.2. Theorem. *For any $\kappa = \mathbb{F}_q$ and any abelian variety A over κ there exists a κ-isogeny $A \sim_\kappa B_0$ such that B_0 admits a* CM *lift.* See [12], Theorem 4.1.1.

This is the central result of the book: "*a field extension is not necessary*". The proof we gave is far from trivial.

5.3. The residual reflex condition. In the result of the preceding theorem one can ask whether the CM lift of B_0 can be achieved over a *normal* domain in mixed characteristic. The answer is negative in general. On can test this by the *residual reflex condition*, see [12], 2.1.5. Here is an easy example, see [12], 2.3 (and many more are given in the book).

Example. *Choose a prime number p that is either* 2 *or* $3 \pmod 5$. *Choose $\pi = \zeta_5 \cdot p$. Clearly π is a p^2-Weil number. Hence* (by Honda and Tate) *there exists an abelian variety A over $\kappa = \mathbb{F}_{p^2}$ having π as κ-Frobenius. Then any B_0 with $A \sim_\kappa B_0$ does not admit a* CM *lift to a mixed characteristic* normal *domain.*
Note that we know that we can choose B_0 such that it does allow a CM lift to a mixed characteristic domain, as we have seen in the previous theorem.

We give (an easy) proof of this claim. Note that the condition on p implies that p is inert in $\mathbb{Q}(\zeta_5)/\mathbb{Q}$. We easily see that A with π as Weil number is a κ-simple abelian surface, $g = 2$, and any CM lift of B_0 is defined over a field containing the reflex field L of the CM-type of $\mathbb{Q}(\zeta_5)$. As L is a CM field, moreover contained in $\mathbb{Q}(\zeta_5)$, we have $L = \mathbb{Q}(\zeta_5)$ hence B_0 can be defined over a field containing $L = \mathbb{Q}(\zeta_5)$. Suppose this lifting of B_0 takes place over a normal domain R (with field of fractions containing L). Then we know that the residue class of R at p contains $\mathbb{F}_{p^4}$, and does not contain $\mathbb{F}_{p^2}$; this is a contradiction because B_0 is over $\mathbb{F}_{p^2}$. $\square$
Note that B_0 does admit a CM lift to a mixed characteristic domain R'; hence it does admit a lifting to the normalization of R'; however under normalization the residue class field may extend, and that is typically what happens in this situation.

5.4. In general in lifting problems, a formal lifting need not to give an actual lifting, as algebraization might not be possible. In the problem of CM lifting however this problem does not present itself, if the residual reflex condition is satisfied, as CM formal liftings automatically admit algebraization, see [12], 2.1.6. YYYplease checkChing-Li. This makes lifting easier in the CM-lifting case

However another tool of lifting theory is not available here: the Mumford method, see 7.6, does not apply as any non-trivial deformation over an integral domain in characteristic p may destroy CM.

6 Generalized Serre-Tate coordinates

6.1. We start with the obvious remark that Hecke correspondences in characteristic zero translate one point in $\mathcal{A}_g(\mathbb{C})$ to a dense set of points (dense in the classical topology, and also dense in the Zariski topology). However for $k \supset \mathbb{F}_p$ one point $x \in \mathcal{A}_g(k)$ corresponding to $A = A_x$ every Hecke correspondence between A_x and A_y gives an equality of the Newton Polygons $\xi = \mathcal{N}(A_x) = \mathcal{N}(A_y)$; i.e. Hecke orbits of x stay inside he Newton Polygon stratum W_ξ. Conclusion: if ξ is not the ordinary Newton Polygon there is no subscheme in mixed characteristics of the moduli scheme, flat over $\Lambda_{p^\infty}(k)$ (the ring of inifinite Witt vectors) such that this subscheme is Hecke-stable, and its chracteristic p fiber is inside W_ξ.
We conclude there is *no hope for a generalization for Serre-Tate coordinates in mixed characteristic around $x \in \mathcal{A}_g(k)$ for non-ordinary $\xi = \mathcal{N}(A_x)$.*

Then we can try to find a generalization for Serre-Tate coordinates around $x \in \mathcal{A}_g \otimes \mathbb{F}_p$ with $\xi = \mathcal{N}(A_x)$ on the formal scheme $W_\xi^{/x}$ in characteristic p.

Thinking about Serre-Tate coordinates we see they arise as extensions over $\mathbb{Q}_p/\mathbb{Z}_p$ with kernel μ_{p^∞}. Hence to form such extensions in a more general situation *we need p-divisible groups with a slope filtration.* After [159] and [124] we can construct slope filtrations (and hence extension classes) in certain situations. This can be done over the ordinary locus, and that is why Sere-Tate coordinates exist. Soon we realized this can only be done over central leaves in Newton Polygon strata in $\mathcal{A}_g \otimes \mathbb{F}_p$. Therefore the correct question seems:

find generalized Serre-Tate coordinates for the formal scheme $(\mathcal{C}(x))^{/x}$.

Here we encounter the difficulty that the definition of a central leaf $\mathcal{C}(x)$ is a "pointwise definition"; this does not lend to good functorial considerations. We had to find a functorial description of the notion of central leaves. In order to perform such a construction, and to find a structure of generalized Serre-Tate coordinates, we developed the theory of *sustained p-divisible groups.* In [19] such a theory is developed, and we describe the related result in that paper. In order to obtain an idea of what could come out, we describe here one very special example. This illustrates the general idea.

6.2. Example. We study the following special case:

$$g = 3, \text{ and a Newton Polygon given by } \xi = (2,1) + (1,2),$$

i.e. slopes 2/3 and 1/3 both with multiplicity 3, and a moduli point $x \in (\mathcal{A}_{3,1} \otimes \mathbb{F}_p)(k)$ such that

$$A_x[p^\infty] \cong G_{2,1} \times G_{1,2}; \quad \mathcal{C}(x) = \mathcal{Z}_\xi,$$

which we called the central stream in W_ξ^0. We know that $\dim(\mathcal{C}(x)) = 2$ and $\dim(W_\xi) = 3$. In this case:

Result (generalized Serre-Tate coordinates, in this special case). *The formal scheme* $(\mathcal{Z}_\xi)^{/x}$ *over* k *canonically has the structure of a* p*-divisible group of slope* $(2/3) - (1/3) = 1/3$ *and height* 9.

We indicate where this structure comes from, and we explain why this construction gives a result on $\mathcal{C}(x)$ but cannot be extended over W_ξ.

We note that on W_ξ we have two foliations, the central leaves and the isogeny leaves. For $x \in \mathcal{Z}_\xi$ the isogeny leaf $\mathcal{I}(x)$ has two nonsingular branches: one is obtained by deforming $G_{2,1} \to X_0 \to (G_{1,2}/\alpha_p)$, the other by deforming $(G_{2,1}/\alpha_p) \to X_0 \to G_{1,2}$. This geometric structure (a 3-fold W_ξ with a normal crossing singularity along a non-singular surface $\mathcal{Z}_\xi$) is reflected in the group structures we are going to describe.

General remarks.

- This structure can be studied in characteristic p for unpolarized p-divisible groups, and for polarized p-divisible groups.
- In case the Newton Polygon has two slopes this is just as in the result above.
- If $X = A_x[p^\infty]$ is not a split extension of the two isoclinic parts, i.e. x is not on the central stream, we do not obtain a group structure, but a structure on $(\mathcal{C}(x))^{/x}$ principal homogeneous under the related structure on the central stream.
- For more than two slopes in the Newton polygon a different structure exists. For details see [19].

An indication of a proof in this special case. We only sketch some of the results. For proofs we refer to [19]. The example discussed here indicates the possibilities and some of the limitations of the general theory of *sustained p-divisible groups* and generalized Serre-Tate coordinates.

- Write $X_0 = A_x[p^\infty]$; consider the formal deformation space
$$\mathcal{D} = \mathrm{Def}(G_{2,1} \to X_0 \to G_{1,2}, \lambda).$$
By the theory of extensions we see this is a formal group, of dimension three at hand. In this case of a principal polarization it is smooth.
- This group has a maximal p-divisible subgroup $G' \subset \mathcal{D}$, and a maximal unipotent subgroup $G'' \subset \mathcal{D}$.
- On shows that $G' = (\mathcal{Z}_\xi)^{/x}$, and this is the group structure we are looking for.
- We note that G'' is the completion of that branch of the isogeny leaf on which the slope filtration extends.
- Remark that through x there is another branch of the isogeny leaf; on that branch the filtration $G_{1,2} \to X_0 \to G_{2,1}$ extends; on that branch there does not exist a slope filtration, it does exist outside the closed point x, but does not extend over that point; the choice of the generalized Serre-Tate coordinates does not extend on W_ξ outside $\mathcal{Z}_\xi \subset W_\xi$.

7 Some historical remarks

We discuss some aspect of the time line for these topics. We see that our considerations have their roots in earlier questions and results.

7.1. From Gauss to CM liftings. An expectation by Gauss preludes to the Weil conjectures, and an attempt by Hasse and theory by Deuring led to CM liftings.
Notation. We write RH for the classical Riemann hypothesis, and we write pRH for the analogous conjectures and theorems about the zeta function attached to a variety in characteristic p.

7.1.1. Carl Friedrich Gauss considered solutions of equations modulo p; in his "Last Entry" (1814) Gauss discussed (as we would say now) the number of points on an elliptic curve over the prime field of characteristic p, see [35].

Interesting aspect: in writing his question Gauss writes an equality sign, adding "modulo p" (actually Gauss wrote $\pmod{a+bi}$ which amounts to the same). Felix Klein in his edition of the Tagebuch, see [55], "corrects" this to an equivalence sign; in many other instances we see that Gauss very well knows the difference between "$=$" and "$\equiv$", no correction was necessary. One could give an interpretation of this Tagebuch notation that Gauss (in some sense) considered an object in characteristic p (as we would do no, instead of solutions in characteristic zero, reduced modulo p). We will see that this point of view was only slowly accepted in the history of arithmetic algebraic geometry.

For the nonsingular curve E over $\mathbb{F}_p$ with $p \equiv 1 \pmod 4$ defined as the normalization of the completed plane curve given by

$$1 - X^2 + Y^2 + X^2Y^2, \quad \text{Gauss expects} \quad \#(E(\mathbb{F}_p)) = (a-1)^2 + b^2,$$

where $p = a^2 + b^2$ and $a - 1 \equiv b \pmod 4$. This is the way Gauss did write his expectation, and the way we would like to express this now. Equivalently:

$$\#\left(\{(x,y) \in (\mathbb{F}_p)^2 \mid x^2 + y^2 + x^2y^2 = 1\}\right) \;\; = \;\; (a-1)^2 + b^2 - 4; \;\; p \equiv 1 \pmod 4.$$

It is clear from his text that Gauss knew very well that the projective plane model of this curve has two points at infinity, and that for $p \equiv 1 \pmod 4$ the nonsingular branches through these nodes are rational over $\mathbb{F}_p$.

This expectation by Gauss (later proved by Herglotz, and many others) is a special case of conjectures made and solved by E.Artin, Hasse, Weil and many others. For the fascinating story of the Last Entry see [35], [55], [36], [120].

For another instance where Gauss approaches such a topic see § 358 of his 1801 "Disquisitiones Arithmeticae" [34], see [64]; for an explanation and references see [138], Theorem 2.2.

7.1.2. Emil Artin in his PhD-thesis (1921/1924) discussed the number of rational points on an elliptic curve in positive characteristic, 40 special cases were computed, and Artin conjectured what the outcome should be. An analogy with the classical Riemann Hypothesis was noted. At that moment many people thought that solving this analogous pRH of the Riemann Hypothesis would be as difficult as proving the classical RH about the classical Riemann zeta function.

7.1.3. Helmut Hasse (and F.K.Schmidt and several others) tried to prove pRH for elliptic curves in positive characteristic in the period of time $1933 - 1937$ (in the language of function fields in characteristic p). In this attempt we see an interesting difference between two approaches. In the first a lift of a special kind of an elliptic curve (we would say now, a CM lift) from characteristic p to characteristic zero was studied; here the difference between an ordinary and a supersingular elliptic curve was noted by Hasse; this proof was not completed by Hasse, as the full liftability result was not proved. In a second approach Hasse basically found the characteristic p approach, proving that the Frobenius endomorphism π of an elliptic curve over a field with q elements satisfies $\pi{\cdot}\overline{\pi} = q$, a profound basis for later proofs of the pRH / Weil conjectures. The "Frobenius operator" was already constructed by Hasse in 1930.

For $F : E \to E^{(p)}$ one shows that the dual map F^t satisfies $F^t{\cdot}F = p$; for $q = p^n$ and $\pi = \pi_{E,\mathbb{F}_q}$ this results in

$$\overline{\pi}{\cdot}\pi = F^t \times \cdots \times F^t \times F \times \cdots \times F = p^n = q.$$

From this we see

$$\mid \pi \mid = \sqrt{q} \quad \text{and} \quad \#(E(\mathbb{F}_q)) = \#(\mathrm{Ker}(\pi - 1 : \mathrm{E} \to \mathrm{E})) = \mathrm{Norm}(\pi - 1) = 1 - (\pi + \overline{\pi}) + q.$$

Here we see a modern proof for the expectation in the Last Entry of Gauss, and a prelude to the Weil conjectures. Basically this proof is contained in the second proof by Hasse.

7.1.4. André Weil discussed this on several occasions with his German colleagues, insisting that one should consider these properties as *geometric aspects* of objects in positive characteristic. For example, see the letter of Weil to Hasse on 17 July 1936, see [122], page 619. This resulted later (1949) in the formulation of the Weil conjectures (using a hypothetical analogue of the Lefschtz fixed point formula, indeed proved much later). An interesting aspect of mathematics: the difference between aspects of *number theory*, as in the German school 1930-1940, and the *geometric approach* to this problem. For discussions an surveys e.g. see [64], [68], [122]. We do not discuss later developments in algebraic geometry around the Weil conjectures (Serre, Dwork, Grothendieck, Deligne and many others).

7.1.5. In 1941 Max Deuring took up these questions raised by Hasse, see [24]. In classical language an elliptic curve with complex multiplication over the complex numbers was said to have a *singular j-invariant.* Deuring discussed the case, already noted by Hasse, then denoted by "the Hasse invariant is zero" that even more endomorphisms can be present in positive characteristic.

The Frobenius $\pi : E \to E$ of an elliptic curve over a finite field can be induced on the space of differentials on E; if this map is non-zero (respectively zero) we say the Hasse invariant of E equals 1 (respectively equals 0). Hasse found out that elliptic curves with Hasse invariant zero behave quite differently from those with Hasse invariant equal to one. Indeed: the following statements are equivalent:

- The Hasse invariant of E equals zero;
- $\mathrm{rank}_{\mathbb{Z}}\left(\mathrm{End}(E) \otimes \overline{\mathbb{F}_p})\right) = 4.$
- There are no points of order p on $E(\overline{\mathbb{F}_p})$.
- **Definition.** The elliptic curve is *supersingular.*

Note that a more correct wording here would be "an elliptic curve with supersingular j-invariant". This terminology "supersingular" by Deuring is now applied to abelian varieties in general, see 2.3.

In Deuring's paper we find the full lifting theorem for CM elliptic curves:

every (E_0, β_0) *with an endomorphism* $\beta_0 \in \mathrm{End}(E_0)$ *can be lifted to characteristic zero.*

For another proof see FOBowdoinXXX2. This result by Deuring finished the first proof of the pRH by Hasse, although at that time we already had much better of the pRH for elliptic curves. These considerations of Deuring obtained their natural place later in history in work by Tate, [139], in the Honda-Tate theory, [42], [140], finished by the full theory of CM liftings as completed and surveyed in [12]:

a question raised by Hasse in 1933*, solved by Deuring for elliptic curves, developed into a whole theory of* CM *liftings for abelian varieties, completed* 70 *years later,*

and

theory developed for elliptic curves over a finite field by Deuring (1941), the Weil conjectures for abelian varieties and a suggestion by Mumford to Tate gave rise to the theorem (Tate, 1966) that any abelian variety over a finite field is a CM *abelian variety, and to many theorems and conjectures about ℓ-adic representations.*

7.2. Moduli spaces. In 1857, discussing what we now call Riemann surfaces of genus p, Riemann wrote: "*... und die zu ihr behörende Klasse algebraischer Gleichungen von 3p-3 stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.*" See [130], Section 12. Therefore, we use the word "moduli" as the number of parameters on which deformations of a given geometric object depend.

There were several attempts to have a solid foundation for the concept of a "moduli space". Grothendieck and Mumford both were working on this concept. To many it came as a surprise that actually such a space for curves of genus 2 over an arbitrary base does exist, see [44]. For some time it confused mathematicians who were carrying out constructions of "universal objects", under the technical term of "representable functors". In 1961 Samuel wrote in [131]: "Signalons aussitôt que le travail d'Igusa ne résoud pas, pour les courbes de genre 2, le "problème des modules" tel qu'il a été posé par Grothendieck à diverses reprises dans de Séminaire."

After the difference between "fine moduli schemes" (representing a functor) and "coarse moduli schemes" (where the points correspond to isomorphism classes of the objects considered) was clear we can work with these concepts, especially thanks to the pioneering work by Mumford, see [75]. This enables us to work with moduli spaces in all characteristics, and over arbitrary base rings. These methods are crucial for all considerations in this note.

7.3. Manin. In the influential paper [63], 1963, we see a new field of research opened. Almost all results discussed in this note in some way are connected with this paper. Earlier work by Barsotti and Dieudonné find their natural place in this paper. The central consideration of the paper gives a classification of p-divisible groups over an algebraically closed field of positive characteristic, and relations with the theory of abelian varieties in characteristic p. We find here the formulation of the "Manin conjecture": *for every prime number p any symmetric Newton can be realized by an abelian variety in characteristic p* (several proofs of this conjecture were given). In this paper we find the first step in describing strata where the Newton Polygon is constant. Above we have described results on stratifications and foliations building on this work by Manin. – The way the Newton Polygon strata can fit together (under

specialization) is discussed in the following development.
XXXSay more on Manin paper

7.4. A conjecture by Grothendieck. We work in positive characteristic. Under a specialization of an abelian variety (or of a p-divisible group) the Newton Polygon "goes up" (no point of the NP of the special fiber is strictly below the NP of the generic fiber, this partial order is indicated by $\prec$) as Grothendieck proved Montreal 70Katz....XXX [**?**], 2.3.2.

In 1970 Grothendieck conjectured the converse. In [40], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: *"$\cdots$ The wishful conjecture I have in mind now is the following: the necessary conditions $\cdots$ that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular deformation $\cdots$ we want to know if every sequence of rational numbers satisfying $\cdots$ these numbers occur as the sequence of slopes of a fiber of G as some point of S."*

We will discuss a proof of this below, see 7.6.2.

7.5. Lifting problems. Suppose given an object G_0 in algebraic geometry over a field $\kappa \supset \mathbb{F}_p$; one can think of: an algebraic curve, an algebraic curve with an automorphism, a higher dimensional algebraic variety, a finite group scheme, a polarized abelian variety, a CM abelian variety, a CM Jacobian, and many more situations.

7.5.1. Unobstructed problems. In many cases (local) moduli functors (in mixed characteristic) are (pro)-representable. If one shows moreover this functor to have a characteristic zero fiber, e.g. the functor is unobstructed, e.g. the functor is flat over $W_\infty(\kappa)$ for a perfect κ, it follows that a lifting is possible.

This approach was successful for complete, non-singular curves, and for principally polarized abelian varieties (Grothendieck), in which cases one shows that obstructions vanish. As an application Grothendieck determined the structure of the prime-to-p part of the etale fundamental group of a curve in characteristic p, using the characteristic zero result via a lifting to characteristic zero.

An interesting and crucial case is the Serre-Tate lifting theory of ordinary abelian varieties, see [133] (1964), giving rise also to Serre-Tate canonical coordinates in mixed characteristic around the moduli point of an ordinary polarized abelian variety; for an explanation see [54]. These have many applications.

7.5.2. Counter examples. In a second class of situations, where an obstruction to lifting cannot be shown to vanish, the "obstructed case", the moduli problem is not (formally) smooth, and different ideas have be invented. One can try to construct a counter example to the lifting problem.

In some cases this is easy, e.g. for a curve C with an automorphism group $G = \mathrm{Aut}(G)$ that violates the Hurwitz bound the pair (C, G) cannot be lifted to characteristic zero. Many more examples of a curve with a group of automorphisms can be handled along the same lines.

Serre (1981) gave an example of an algebraic variety that cannot be lifted to characteristic zero, see [132]. Many more examples can be given, e.g. of algebraic surfaces, see the appendix of [121], reproducing a letter by Deligne.

This inspired [123] (1986) were we did find an explicit example of a Jacobian of an ordinary curve where the canonical lifting of its Jacobian is not a Jacobian (at the same time Dwork and Ogus gave a more general approach, with a general answer for $g \geq 4$, but asking for an explicit example; see below).

7.5.3. Infinitesimal mod p^2 methods. Suppose you want to prove or to disprove a certain claim in characteristic zero. Of course, a very old method consists of reducing modulo p; if then you obtain a negative answer in characteristic p, you conclude the answer is negative in characteristic zero.

There is an even more subtle version of this, comparing "infinitesimal deformations modulo p and modulo p^2". In this situation we have a kind of differential calculus, with $p = \epsilon$ and $\epsilon^2 = 0$. We know several instances where this method was succesful. Also this idea can be used in lifting problems.

Raynaud's proof of the Manin-Mumford conjecture. The Manin-Mumford conjecture is a fact about curves and abelian varieties over the complex numbers: *for a curve $C \subset A$ inside an abelian variety with* $\mathrm{genus}(C) \geq 2$ *the number of torsion points of A situated on C is finite.*

This does not look as a theorem in positive characteristic; for a curve C_0 over a finite field, and $C_0 \subset A_0$ any point in $C_0(\overline{\mathbb{F}_p})$ is a torsion point in A_0: *the statement corresponding to the Manin-Mumford conjecture over an algebraically closed field of positive characteristic is not true.* However a beautiful (first) proof of this conjecture in characteristic zero was given by Raynaud in 1983, by the "mod p^2 / mod p" method, see [128]: first reducing to a number field, and then choosing carefully a reduction mod p^2, showing that only finitely many of the torsion points in $C_0(\overline{\mathbb{F}_p})$ have survived as torsion points mod p^2, magic!). For a second (different) proof see [129], and many other publications (Hrushovski, Pink, Roessler).

Dwork-Ogus theory for canonical liftings. The question by Katz whether the canonical lifting of an ordinary Jacobian from positive to zero characteristic again is a Jacobian was studied (for every $g \geq 4$) by Dwork and Ogus (1986). In their paper [26] we find a "mod p^2 / mod p" method showing that outside a smaller closed set of ordinary Jacobians (of genus at least 4) even a canonical lifting to $p^2 = 0$ the canonical lift is not a Jacobian. This method has important consequences en applications, e.g. see [47], [72].

Deligne-Illusie and degeneration of the Hodge spectral sequence. The degeneration of the Hodge spectral sequence for a smooth and projective algebraic variety over a field of characteristic zero is a basic fact in algebraic geometry. In [22] we find a beautiful proof (1987) by Deligne and Illusie of this theorem by "mod p^2 / mod p" methods.

7.6. The Mumford method. Suppose you have an obstructed lifting problem (or a deformation problem in equi-characteristic), how do you try to show that however a lifting does exist? In other words: there is a general theory, but that does not provide either a negative nor a positive answer to your lifting problem; how do you proceed? We explain a method, initiated by David Mumford, in the special case it was used as suggested by Mumford.

7.6.1. Theorem (Mumford, Norman-Oort). *Any polarized abelian variety (A_0, λ_0) over $\kappa \supset \mathbb{F}_p$ can be lifted to characteristic zero.*

In the Mumford method you first deform the situation in equi-characteristic-p to a "better" situation (usually, a non-canonical process), and you hope that the first step arrives at a situation where you know a lifting does exist. In the case at hand we know that an ordinary polarized abelian variety can be lifted (by the Serre-Tate theory). In order to study the situation we consider the Newton Polygon of A_0. A quite non-trival computation shows that you can deform (A_0, λ_0) to a new polarized abelian variety with a strictly lower Newton

Polygon, see [83]XXX for details. After a finite number of such deformations you arrive at an ordinary polarized (B_0, μ_0); this finishes the first (computational, non-canocial) *first step.* Then, Serre-Tate theory tells you that (B_0, μ_0) can be lifted to characteristic, and this *second step* finishes the proof.
Remark. The computation showing the deformation as in the first step uses the *theory of displays*, invented by Mumford for this situation. This theory was used and further developed, e.g. see [157], XXXWindows?.

7.6.2. A conjecture by Grothendieck. See 7.4.

Suppose given a p-divisible group X_0 and Newton polygons $\mathcal{N}(X_0) = \zeta \prec \tau$; does there exist a deformation $X \to S$ with $0 \in S$ and $\mathcal{N}(X_\eta) = \tau$ realizing $\zeta \prec \tau$ as Newton Polygons of a special, respectively the generic fiber?

One can also study this question for polarized p-divisible groups.

Theorem. *For p-divisible groups, for principally polarized p-divisible groups and for principally polarized abelian varieties this conjecture by Grothendieck holds.* XXX references.
Remark. There are many examples showing that this result is not correct in general for non-principally polarizations.XXX exa., general reference

The proofs follow the Mumford method (as we realized much later). We define

$$a(G) := \dim_\kappa \mathrm{Hom}(\alpha_p, G),$$

where κ is a perfect field, and G a commutative group scheme over κ. The crucial case in the theorem is that of (unpolarized) p-divisible groups, and $a(X) > 0$.

7.7. Methods: deformations to $a \leq 1$.
Theorem (the "Purity theorem"). *If in a family of p-divisible groups (say, over an irreducible scheme) the Newton polygon jumps, then it already jumps in codimension one.*
See [48] Th. 4.1. This very non-trivial result will be one of the main tools.

7.8. Catalogues. Let us fix a prime number p, and coprime $m, n \in \mathbb{Z}_{>0}$. We try to "classify" all p-divisible groups isogenous with $G_{m,n}$.

In general there is no good theory of moduli spaces for p-divisible groups (and there are various ways to remedy this). We use the (new) notion of a "catalogue". In our case this is a family $\mathcal{G} \to S$, i.e. a p-divisible group over some base scheme S, such that every isogenous $G \sim G_{m,n}$ defined over an algebraically closed field appears as at least one geometric fiber in $G \to S$. You can rightfully complain that this is a rather vague notion, that a catalogue is not unique (e.g. the pull back by a surjective morphism again is a catalogue), etc. However this notion has some advantages:

Theorem (catalogues). *Suppose given p, m, n as above. There exists a catalogue $\mathcal{G} \to T$ over $\mathbb{F}_p$ for the collection of p-divisible groups isogenous with $G_{m,n}$ such that T is geometrically irreducible.* See [48], Theorem 5.11.

7.9. Theorem. *Suppose G_0 is a p-divisible group; there exists a deformation to G_η such that*

$$\mathcal{N}(G_0) = \mathcal{N}(G_\eta) \quad \textit{and} \quad a(G_\eta) \leq 1.$$

We sketch a proof of the theorem on catalogues, using the Purity Theorem, see [48]. We write

$H = H_{m,n}$. We write $r := (m-1)(n-1)/2$. We see that for every $G \sim G_{m,n}$ there exists an isogeny $\varphi : H \to G$ of degree exactly $\deg(\varphi) = p^r$, see [48], 5.8. We construct $\mathcal{G} \to T$ as the representing object of isogenies $\varphi : H \times S \to G/S$ of this degree (it is easy to see that such a functor is representable).

Using this definition we see that the formal completion at $[(G_0, \varphi)] = s \in T$ embeds in $\mathrm{Def}(G_0)$, i.e. $T^{\wedge s} \hookrightarrow \mathrm{Def}(G_0)$. Furthermore we compute the longest chain of Newton polygons between $\mathcal{N}(G_{m,n})$ and the ordinary one: this equals $mn - r$ (an easy combinatorial fact). From these two properties, using 7.7, we deduce: *every component of T has dimension at least r.*

We make a stratification of T (using combinatorial data, such a thing like "semi-modules"). We show (using explicit equations) that every stratum is geometrically irreducible, and that there is one stratum, characterized by $a(G) = 1$, of dimension r, and *that all other strata have dimension less than r.* These considerations do not contain deep arguments, but the proofs are rather lengthy and complicated.

From these two aspects the proof follows: any component of T on which generically we would have $a > 1$ would have dimension strictly less than r, which contradicts "Purity". Hence the locus where $a = 1$ is dense in T, and we see that T is geometrically irreducible.

7.10. We sketch a proof of 7.6.2, see [104]. By 7.9 we conclude this deformation property 7.6.2. for simple groups. Then we study groups filtered by simple subfactors, and deformation theory of such objects. By the previous result we can achieve a deformation where all simple subfactors are deformed within the isogeny class to $a \leq 1$. Then we write down an explicit deformation ("making extensions between simple subfactors non-trivial") in order to achieve $a(G_\eta) \leq 1$, see [104], Section 2 for details.

Remark. This method of catalogues for p-divisible groups works fine for simple groups. However the use of "catalogues" for non-isoclinic groups does not seem to give what we want; it is even not clear that nice catalogues exist in general. Note that we took isogenies of the form $\varphi : H \times S \to G/S$; however over a global base scheme monodromy groups need not be trivial, and this obstructs the existence of one catalogue which works in all cases (to be considered in further publications).

7.11. Methods: Cayley-Hamilton This is taken entirely from [102]. In general it is difficult to read off from a description of a p-divisible group (e.g. by its Dieudonné module) its Newton polygon. However in the particular case that its a-number is at most one this can be done. This we describe here. The marvel is a new idea which produces for a given element in a given Dieudonné module a polynomial (in constants and in F) which annihilates this element (but, in general, is does not annihilate other elements of the Dieudonné module). This idea for constructing this polynomial comes from the elementary theorem in linear algebra: every endomorphism of a vector spaces is annihilated by its characteristic polynomial. As we work in our case with an operator which does not commute with constants, things are not that elementary. The method we propose works for $a(G_0) = 1$, but it breaks down if $a(G_0) > 1$ in an essential way.

7.12. Theorem (of Cayley-Hamilton type). *Let G_0 be a p-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$ with $a(G_0) \leq 1$. In $\mathcal{D} = \mathrm{Def}(G_0)$ there exists a coordinate system $\{t_j \mid j \in \diamond(\rho)\}$ and an isomorphism $\mathcal{D} \cong \mathrm{Spf}(k[[t_j \mid j \in \diamond(\rho)]])$ such that for any $\gamma \succ \mathcal{N}(G_0)$ we have*

$$\mathcal{W}_\gamma(\mathcal{D}) = \mathrm{Spf}(R_\gamma), \quad \textit{with} \quad R_\gamma := k[[t_j \mid j \in \diamond(\gamma)]] = k[[t_j \mid j \in \diamond(\rho)]]/(t_j \mid j \notin \diamond(\gamma)).$$

Corollary. *Let G_0 be a p-divisible group over a field K with $a(G_0) \leq 1$. In* $\mathrm{Def}(G_0)$ *every Newton polygon $\gamma \succ \mathcal{N}(G_0)$ is realized.*

These methods allow us to give a **proof for the Grothendieck conjecture**. In fact, starting with G_0 we use 7.9 in order to obtain a deformation to a p-divisible group with the same Newton polygon and with $a \leq 1$ (the first step in the Mumford method). For that group the method 7.11 of Cayley-Hamilton type can be applied, which shows that it can be deformed to a p-divisible group with a given lower Newton polygon (the second step in the Mumford method). Combination of these two specializations shows that the Grothendieck conjecture 7.6.2 is proven. $\square$

7.13. Lifting an automorphism of an algebraic curve. In [101]XXX we see the

Conjecture. *A pair (C_0, φ_0) of an algebraic curve C*
and $\varphi_0 \in \mathrm{Aut}(C_0)$ can be lifted to characteristic zero

for any (C_0, φ_0) over a field of characteristic $p > 0$. In general (for automorphisms of order divisible by p) this problem is obstructed; the general moduli approach does not give an answer, but also no counter example was known. After proofs for special cases, and several attempts, the final result is a proof for this conjecture, see [85] and [126]; see the survey paper [86] for details. We just mention that the proof for the "favorable" situation, needed for the *second step*, is contained in [85] (here finding the good condition, and the proof that lifting does work in such cases is far from obvious). The deformation argument needed in the *first step* of the Mumford method is contained in [126] (also a non-trivial step).

Some notations used in this paper.
We write k for an algebraically closed field, κ for a field in characteristic p;
for a perfect κ we write $\Lambda = \Lambda_\infty(\kappa)$ for the ring of infinite Witt vectors.
We write $\mathcal{A}_g$ either for $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$, or for $\cup_d \mathcal{A}_{g,d} \otimes \mathbb{F}_p$. However in Section 5 and in the beginning of Section 6 moduli spaces in mixed characteristic will be used.
For $\zeta_1 \prec \zeta_2$ see 1.1.
Notations $\mathcal{W}_\zeta$, W^0_ζ, $\mathcal{W}_\zeta$ will be used in the theory of Newton Polygon strata.
We use $\mathcal{C}(x)$ for the central leaf passing through x, $\mathcal{Z}_\xi$ for the central stream in $W^0_\xi \subset \mathcal{A}_{g,1}$,
XX EO
For a p-divisible group X we will say that $X[p^n]$ is a BT_n, a n-truncated Barsotti-Tate group.
Let $0 \leq f \leq g$; we write $\mathcal{V}_f(\mathcal{A}_g)$ for the (Zariski closed) locus of polarized abelian varieties of p-rank at most f.

8 Some Questions.

8.1. Is there a good functorial definition of Newton Polygon strata?

8.2.

8.3.

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