THE HECKE ORBIT CONJECTURE

CHING-LI CHAI & FRANS OORT

1. Introduction

Entry 15 of [37], *Hecke orbits*, contains the following prediction.

> *Given any $\overline{\mathbb{F}}_p$-point $x = [(A_x, \lambda_x)]$ in the moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties in characteristic $p > 0$, the Hecke orbit of x, which consists of all points $[(B, \nu)] \in \mathscr{A}_g$ related to $[(A_x, \lambda_x)]$ by symplectic isogenies, is Zariski dense in the Newton polygon stratum of $\mathscr{A}_g$ which contains x.*

The obvious generalization of this prediction, for (good) modular varieties of PEL-type in positive characteristic p or more generally for reduction modulo p of Shimura varieties, has since become known as (part of) the Hecke orbit conjecture. In this chapter we will survey results as well as new structures and methods which have been developed since 1995, in response to the above conjecture. We will also formulate several rigidity-type questions related to Hecke symmetries on reductions of Shimura varieties. These questions arose naturally in the pursuit of, and are similar in spirt to, the Hecke orbit conjecture.

1.1. Hecke orbits: why, what and how

It is fair to ask
(a) whether the above problem on Hecke orbits is of any interest or importance,
(b) where to place it among well-established programs and conjectures, and
(c) how to think about the Hecke orbit problem?

We will not attempt to answer the question (a), or to promote the intrinsic virtue of this problem on Hecke orbits. As far as we know none of the great conjectures will follow from the validity of the general Hecke orbit conjecture. Instead we offer a few bullet points, on several aspects related to the question (c).

(1) The Hecke orbit conjecture *is a rigidity statement.*

It asserts that there are not "too many" subvarieties of the reduction modulo p of a modular variety of PEL-type which are stable under all Hecke correspondences: they are specified by suitable p-adic invariants (of members of the universal abelian scheme plus PEL structure).

(2) Foliation on moduli spaces underlying Hecke symmetries.

Hecke symmetries are generated by prime-to-p Hecke symmetries and p-power Hecke symmetries. It is natural to ask analogous questions for subvarieties stable

under all prime-to-p Hecke symmetries (respectively all p-power Hecke symmetries). These questions lead naturally to the notions of *central leaves* and *isogeny leaves* in $\mathcal{M}$.

- The isogeny leaves are orbits of the family of all p-power Hecke correspondences on $\mathcal{M}$.

  For instance given an $\overline{\mathbb{F}_p}$-point $x_0 = [(A_0, \lambda_0)]$ in $\mathscr{A}_g$, the isogeny leaf $\mathcal{I}(x_0)$ in $\mathscr{A}_g$ passing through $x_0 =$ in $\mathcal{A}_g(\overline{\mathbb{F}_p})$ is the reduced closed subvariety of $\mathcal{A}_g$ over $\overline{\mathbb{F}_p}$ whose $\mathbb{F}_p$-points consists of all isomorphism classes $[(B, \mu)]$ of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}_p}$ such that there exists an isogeny $\beta : B \to A_0$ and $\beta^*\lambda_0 = p^m\mu$ for some positive integer m.
- The central leaves are locally closed subvarieties of $\mathcal{M}$ stable under all prime-to-p Hecke correspondences, obtained by *fixing all p-adic invariants* on the natural families of crystals over $\mathcal{M}$ with imposed endomorphism structure. Since these families of crystals come from the universal p-divisible group (with imposed endomorphism structure), the above description of central leaves can be made explicit for Siegel modular varieties as in the paragraph below.

  Given an $\overline{\mathbb{F}_p}$-point $\mathcal{C}(x_0)$ in $\mathcal{A}_g$, the *central leaf in $\mathscr{A}_g$ passing through* x_0 is the reduced locally closed subvariety of $\mathscr{A}_g$ over $\overline{\mathbb{F}_p}$ whose $\mathbb{F}_p$-points consists of isomorphism classes $[(B, \mu)]$ principally polarized g-dimensional abelian varieties over $\overline{\mathbb{F}_p}$ such that there exists an isomorphism

  $$\beta : B[p^\infty] \xrightarrow{\sim} A_0[p^\infty]$$

  of p-divisible groups satisfying $\beta^*(\lambda_0[p^\infty]) = \mu[p^\infty]$.

  The above definition of central leaves in $\mathscr{A}_g$ is based on the notion of *geometrically fiberwise constant p-divisible groups* introduced in [41]. In 3.1 and 3.3 we will explain the notion of *sustained p-divisible groups* which can be regarded as an "upgraded version" of geometrically fiberwise constained p-divisible groups. The definition of central leaves in PEL type modular varieties given in 3.6 is based on the notion of sustained p-divisible groups.

(3) The general Hecke orbit conjecture:

Each central leaf is minimal among subvarieties of $\mathcal{M}$ which are stable under all prime-to-p Hecke correspondences. In other words, the Zariski closure of the prime-to-p Hecke orbit of a point $x \in \mathcal{M}(\mathbb{F}_p)$ contains the central leaf passing through x.

Remark. The Hecke orbit conjecture has been proved in the case when $\mathcal{M}$ is the moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}_p}$. The case when the leaf is the ordinary locus of $\mathscr{A}_g$ was proved in [1]. See 1.5 for more information and §8 for a sketch of the proof.

(4) Further probes into central leaves

- The notion of *sustained p-divisible groups* provides a scheme theoretic definition of central leaves. This new concept also helps to elucidates the

properties of central leaves, such as the existence of slope filtrations on families of p-divisible groups over central leaves.

- The formal completion $\mathcal{C}^{/x}$ of a central leaf $\mathcal{C}$ at a closed point $x \in \mathcal{C}$ is "built up" from p-divisible formal groups via a web of fibrations. They can be thought of as "generalized Serre-Tate local coordinates" on a central leaf. (However these structures live entirely inside a central leaf.)

(5) New methods were developed for the Hecke orbit problem.
These will be explained in later sections with more details. Some of the tools have been applied to questions outside the immediate surrounding of the Hecke orbit problem.

(6) More rigidity questions.

- (prototype of global rigidity question)
  Suppose that Z is a subvariety of a modular variety $\mathcal{M}$ of PEL-type in characteristic p which is stable under a "not-to-small" subset of Hecke symmetries on $\mathcal{M}$. Does Z necessarily come from a Shimura subvariety $\mathcal{S}$ of $\mathcal{M}$ by imposing suitable conditions on p-adic invariants (on families of crystals over $\mathcal{S}$)?
- (prototype of local rigidity question)
  Suppose that V is a formal subvariety of the formal completion $\mathcal{M}^{/x}$ of a modular variety $\mathcal{M}$ of PEL-type in characteristic p at a point $x \in \mathcal{M}(\overline{\mathbb{F}_p})$, such that V is stable under the action of a "not-to-small" subset of Hecke symmetries on $\mathcal{M}$ having x as a fixed point. How much constraint does this condition impose on V?

1.2. General info about a Hecke invariant subvariety of a leaf

Let $\mathcal{C}$ be a central leaf over $\overline{\mathbb{F}_p}$ in a PEL-type modular variety $\mathcal{M}$ over $\overline{\mathbb{F}_p}$. Suppose that $Z \subset \mathcal{C}$ is a closed subvariety of $\mathcal{C}$ which is stable under all prime-to-p Hecke correspondences on $\mathcal{M}$. An example of such a Hecke-invariant subvariety Z is the Zariski closure of the prime-to-p Hecke orbit of a point $x_0 \in \mathcal{C}(\overline{\mathbb{F}_p})$.

Below are two general properties of Z. The first property is the combined out put of (a) the local structure of central leaves and (b) local rigidity result for formal subvarieties of the formal completion of a central leaf stable under the action of a p-adic Lie group. It gives a strong a priori constraint on subvarieties of central leaves which are stable under prime-to-p Hecke correspondences: they have to The second property, about the ℓ-adic monodromy of a subvariety stable under the action of all ℓ-adic Hecke correspondences, comes from group theory.

A. Linearization of the Hecke orbit problem. Let $z_0 \in Z(\overline{\mathbb{F}_p})$ be a closed point of a reduced closed subscheme $Z \subseteq \mathcal{C}$ stable under all prime-to-p Hecke correspondences as above.

(A1) Local structure of central leaves
The formal completion $\mathcal{C}^{/z_0}$ of $\mathcal{Z}$ at z_0 has a natural "linear structure", to the effect that it is assembled from a finite collection of p-divisible formal groups Y_i over $\overline{\mathbb{F}_p}$ through a web of fibrations.

In the case when $\mathcal{M}$ is $\mathscr{A}_g$ and $\mathcal{C}$ is the open subset of $\mathscr{A}_g$ corresponding to ordinary principally polarized abelian varieties, $\mathcal{C}^{/z_0}$ has a natural structure as a formal torus according to Serre and Tate.

(A2) Local structure of Hecke-invariant subvarieties of leaves
The local stabilizer principle, to be explained in 1.3, says that there formal completion $Z^{/z_0}$ of a Hecke-invariant subvariety Z of a central leaf $\mathcal{C}$ at an $\overline{\mathbb{F}}_p$-point z_0 of Z is stable under the natural action of a sizable p-adic Lie group G_{z_0} on $\mathcal{C}^{/z_0}$. It is expected that the methods used in [11], can be successfully applied to the linear structure of $\mathcal{C}^{/z_0}$. The expected result is that the formal completion $Z^{/z_0}$ of Z at z_0, as a closed formal subscheme of $\mathcal{C}^{/z_0}$, is a "linear subvariety" of $\mathcal{C}^{/z_0}$, in the sense that it is built up from a family of p-divisible subgroups $X_i \subset Y_i$ by the fibrations in the linear structure of $\mathcal{C}^{/z_0}$.

In the case when $\mathcal{M}$ is $\mathscr{A}_g$ and $\mathcal{C}$ is the ordinary locus $\mathscr{A}_g^{\mathrm{ord}}$ in $\mathscr{A}_g$, local rigidity for p-divisible groups tells us that $Z^{/z_0}$ is a formal subtorus of the Serre-Tate torus $\mathcal{C}^{/z_0}$, which is stable under the natural action of the local stabilizer subgroup G_{z_0}. This makes the Hecke orbit problem more tractable. For instance if the abelian variety A_{z_0} corresponding to the point $z_0 \in \mathscr{A}_g$ is isogenous to E^g for an elliptic curve E over $\overline{\mathbb{F}}_p$, the fact that the second symmetric product of the standard representation of $\mathrm{GL}_g(\mathbb{Q}_p)$ on $\mathbb{Q}_p^g$ is irreducible implies that the dimension of this formal subtorus $Z^{/z_0}$ is $g(g+1)/2$ and Z is equal to $\mathscr{A}_g^{\mathrm{ord}}$.

B. Maximality of ℓ-adic monodromy. The method in [2], based on group theory, generalize to all PEL-type modular varieties in characteristic p. For every prime number $\ell \neq p$, the ℓ-adic monodromy group for the Hecke-invariant subvariety Z is "as large as possible". No doubt careful readers will object that the phrase "as large as possible" is too vague.

In the case when $\mathcal{M} = \mathscr{A}_g$ and the central leaf $\mathcal{C}$ is not supersingular, the ℓ-adic monodromy of the Hecke-invariant subvariety $Z \subseteq \mathcal{C}$ is GSp_{2g}; see [2].

1.3. The local stabilizer principle

This is a general method to extract *local* information about a given subvariety Z of a central leaf $\mathcal{C}$ stable under all prime-to-p Hecke symmetries. It is analogous to the following fact in the context of group actions: suppose that an algebraic group G operates on an algebraic variety X, Y is a subvariety of X stable under G, and y is an $\overline{\mathbb{F}}_p$-point of Y, then $Y^{/y} \subset X^{/y}$ is stable under the action of the stabilizer subgroup $G_y = \mathrm{Stab}_G(y)$ at y. See 5.1 for the precise statement. Here we will only explain the general idea.

We need to recall that the modular variety $\mathcal{M}$ is part of a prime-to-p tower $\tilde{\mathcal{M}}$ of modular varieties attached to a PEL input data. There is a connected reductive group G over $\mathbb{Q}$ attached to the PEL input data, such that the locally compact group $G(\mathbb{A}_f^{(p)})$ of all prime-to-p adelic points of G operates on the tower $\tilde{\mathcal{M}}$, and the given modular varieties $\mathcal{M}$ is the quotient of this tower by a compact open subgroup $K_f^{(p)}$ of the tower $\tilde{\mathcal{M}}$. The prime-to-p Hecke correspondences on $\mathcal{M}$ is induced by the action of $G(\mathbb{A}_f^{(p)})$ on the tower $\tilde{\mathcal{M}}$; see §2.

Let z be an $\overline{\mathbb{F}}_p$-point of a closed subvariety $Z \subset \mathcal{M}$ over $\overline{\mathbb{F}}_p$ which is stable under all prime-to-p Hecke correspondences. The set $\mathscr{H}_z$ of all prime-to-p Hecke

correspondences which have z as a fixed point is closely related to a compact open subgroup G_z of the group of all $\mathbb{Q}_p$ points of an algebraic subgroup U_z of G over $\mathbb{Q}_p$: This p-adic group G_z represents the p-adic closure of the intersection of $U_z(\mathbb{Q})$ with $\mathscr{H}_z$, and it operates naturally on the formal completion $\mathcal{M}^{/z}$ of $\mathcal{M}$. The local stabilizer principle asserts that the formal completion $Z^{/z}$ at z of the Hecke-invariant subvariety Z is stable under the action of G_z on $\mathcal{M}^{/z}$.

The local stabilizer principle is a major source of information about Hecke-invariant subvarieties of $\mathcal{M}$. There have been two types of successful applications of this general principle.

(a) local rigidity results when $z \in Z(\overline{\mathbb{F}}_p) \subset \mathcal{C}(\overline{\mathbb{F}}_p)$, for instance the linearization in A2 of 1.2.

(b) splitting at supersingular points where $z \in \overline{Z}(\overline{\mathbb{F}}_p) \subset \overline{Z}(\overline{\mathbb{F}}_p)$, where $\overline{Z|}$ is the Zariski closure the closure of Z and $\overline{\mathcal{C}}$ is the Zariski closure of $\mathcal{C}$ in $\mathcal{M}$; see [**?**].

Local rigidity at points where z is a supersingular $\overline{\mathbb{F}}_p$-point of the given Hecke-invariant subvariety Z poses greater challenges (and offers high potential rewards), because both the local structure of the Zariski closure $\overline{\mathcal{C}}$ of $\mathcal{C}$ and the action of the local stabilizer subgroup on $\overline{\mathcal{C}}^{/z}$ are poorly understood. See 9.1 and 9.2 for some samples.

1.4. The rest of this chapter is organized as follows.

(1) In §2 we describe the construction of the tower of modular varieties in positive characteristic p with prime-to-p level structures from a given PEL input data, together with the action of the group $G(\mathbb{A}_f^{(p)})$ on the tower and the algebraic correspondences induced by the $G(\mathbb{A}_f^{(p)})$-action on individual members of the tower, where G is the group of symmetries of the given PEL input data, a reductive algebraic group over $\mathbb{Q}$.

(2) In §3 we explain the definition of sustained p-divisible groups, possibly with prescribed endomorphisms and polarization. Then we define a *central leaf* in a modular variety $\mathscr{M}$ as a maximal locally closed subscheme over which the universal p-divisible group with extra symmetry is sustained. The general Hecke orbit conjecture for modular varieties of PEL type is formulated in 3.8.

(3) In §4 we describe the structure of the formal completion of a leaf $\mathcal{C}(x_0)$ in $\mathscr{A}_g$ when the abelian variety corresponding to x_0 has at most three slopes. This illustrations the general phenomenon 1.2 (A1) that the formal completion of a leaf at a closed point has a natural "linear structure".

The formal completion $\mathcal{C}(x_0)^{/x_0}$ of $\mathcal{C}(x_0)$ is a closed formal subscheme of the local structure of the central leaf in the equi-characteristic-p deformation space $\mathcal{D}ef(A_{x_0}[p^\infty])$ of the p-divisible formal group $A_{x_0}[p^\infty]$, and the linear structure of $\mathcal{C}(x_0)^{/x_0}$ is compatible with the linear structure of the central leaf in $\mathcal{D}ef(A_{x_0}[p^\infty])$. We refer to [10] for the linear structure of the central leaf in the deformation space of a p-divisible group.

(4) The local stabilizer principle is formulated in §5. In §6 we explain local rigidity results for irreducible closed formal subschemes of p-divisible formal groups or subvarieties of bi-extensions of p-divisible formal groups which are stable under a *strictly non-trivial* action of a p-adic Lie group. The combination of the examples in §5 and §5 illustrate the general phenomenon 1.2 (A2) that the every Hecke-invariant closed subvariety of a central leaf $\mathcal{C}(x_0)$ is a "linear subvariety" of $\mathcal{C}(x_0)$.
(5) In §7 we explain results on the maximality of prime-to-p monodromy of Hecke-invariant subvarieties in characteristic p. These properties follow from group theoretic arguments, applied to semisimple simply connected algebraic groups over finite extension fields of $\mathbb{Q}_\ell$ with $\ell \neq p$. It is well-known that in Galois theory that large monodromy means that étale covers in the tower being considered do not have many connected components. So it is not a surprise that these results on maximality of monodromy have been applied to prove the irreducibility of central leaves in many situations; see 7.2 (2). When a Hecke-invariant subvariety Z of a central leaf contains a point with large stabilizer subgroup, the local stabilizer principle can be combined with the prime-to-p monodromy via a geometric form of the product formula to show that Z is large and its p-adic monodromy is also large; see 7.4 (3)–(4).
(6) In §8 we outline a proof of the Hecke orbit conjecture for the Siegel modulare variety $\mathscr{A}_g$. We will see the general tools developed for the Hecke orbit problem work together in this example. Full proofs are expected to appear in [12].
(7) Several open problems are presented in §9, with comments at the beginning of that section. We hope some of them will spawn new developments.

1.5. It should be noted that a special property of $\mathscr{A}_g$ is critical for the proof of the Hecke orbit conjecture for $\mathscr{A}_g$ through every $\overline{\mathbb{F}}_p$-point x_0 of $\mathscr{A}_g$, there exists a "small" Shimura subvariety $\mathcal{S}_{x_0}$ of positive dimension which passes through x_0. For $\mathscr{A}_g$ such a Shimura subvariety is a Hilbert modular variety, attached to either a totally real number field or a product of a finite number of totally real number fields. This special property of $\mathscr{A}_g$, known as the "Hilbert trick", was used in [1] to prove the density of the Hecke orbit of x_0 in $\mathscr{A}_g$ when x_0 corresponds to an ordinary g-dimensional principally polarized abelian variety.

The "smallness" of a Hilbert modular variety is related to the fact that the $\mathbb{Q}$-rank of the group GL_2 over a totally real number field is 1, and is reflected in the geometric property that for a given leaf $\mathcal{C}$ in a Hilbert modular variety, there are only a *finite* number of possible "linear subvarieties". The last property, combined with irreducibility results of leaves in ramified Hilbert modular varieties [49], implies that the Zariski closure of the prime-to-p Hecke orbit of an $\overline{\mathbb{F}}_p$-point x_0 in $\mathscr{A}_g$ contains many Hilbert modular subvarieties. Our tools allow us to effectively exploit this fact in a number of ways. For instance one can deduce the existence of points with large local stabilizer subgroups in the Zariski closure Z in $\mathcal{C}(x_0)$ of the Hecke orbit x_0. Once we know that Z is appreciably large in a structural way, one can deduce that $Z = \mathcal{C}(x_0)$ in a number of ways.

A proof of Hecke orbit conjecture was sketched in [3]. There have been some advances in the Hecke orbit problem in the years after [3]. The notable ones include (a) a scheme-theoretic definition of central leaves via the notion of sustained p-divisible groups, discussed in §4, and (b) the local rigidity for bi-extension of formal groups, discussed in §6. As already mentioned, it seems very likely that the method for the latter is applicable in the general case, and shows that the Zariski closure of the Hecke orbit of an $\overline{\mathbb{F}}_p$-point x_0 in the leaf $\mathcal{C}(x_0)$ indeed is a "linear subvariety". Despite the above gains on the local structure of Hecke invariant subvarieties, a proof of the Hecke orbit conjecture for $\mathcal{A}_g$ which does not resort to the Hilbert trick, therefor applicable to general PEL modular varieties, remains elusive. Progress in any of the open problems in §9 will bring us closer to this goal.

2. Hecke symmetry on modular varieties of PEL type

2.1. PEL input data. Recall from [24] that a *PEL input data unramified at p* is an 8-tuple $\big(B, \mathscr{O}_B, *, V, (\cdot,\cdot), h, \Lambda_p, K_f^{(p)}\big)$, where

- B is a finite dimensional simple algebra over $\mathbb{Q}$, whose center L is an algebraic number field which is unramified above p,
- $\mathscr{O}_B$ is a $\mathbb{Z}_{(p)}$-order of B whose p-adic completion is $(\mathscr{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-linearly isomorphic to a matrix ring $\mathrm{M}_m(\mathscr{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, where $m = \sqrt{\dim_L(B)}$,
- $*$ is a positive involution on B,
- V is a finite dimensional left B-module,
- $(\cdot,\cdot)$ is a non-degenerate $\mathbb{Q}$-valued alternating form on V such that
$$(bv_1, v_2) = (v_1, b^* v_2)$$
for all $v_1, v_2 \in V$ and all $b \in B$,
- $h : \mathbb{C} \to \mathrm{End}_B(V) \otimes_{\mathbb{Q}} \mathbb{R}$ is a $*$-homomorphism such that the map
$$V_{\mathbb{R}} \times V_{\mathbb{R}} \longrightarrow \mathbb{R} \qquad (v_1, v_2) \mapsto (v_1, h(\sqrt{-1})v_2)$$
is positive definite symmetric bilinear form on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$,
- Λ_p is a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ which is stable under the action of $\mathscr{O}_B$ such that the alternating pairing $(\cdot,\cdot)$ induces a $\mathbb{Z}_p$-valued alternating self-dual pairing
$$(\cdot,\cdot)_p : \Lambda_p \times \Lambda_p \longrightarrow \mathbb{Z}_p$$
on Λ_p, and
- $K_f^{(p)}$ is a compact open subgroup of $G(\mathbb{A}_f^{(p)})$, where
$$\mathbb{A}_f^{(p)} = \prod_{\ell \neq p}{}' \mathbb{Q}_\ell$$
is the ring of all finite prime-to-p adeles over $\mathbb{Q}$, and
$$G = \mathrm{U}(\mathrm{End}_B(V), *')$$
is the unitary group attached to the simple algebra $\mathrm{End}_B(V)$ with involution $*'$ induced by the pairing $(\cdot,\cdot)$.

The left $B_{\mathbb{C}}$-module $V_{\mathbb{C}}$ decomposes into a direct sum $V_{\mathbb{C}} = V_1 \oplus V_2$ of $B_{\mathbb{C}}$-submodules, where

$$V_1 := \{v \in V_{\mathbb{C}} \mid h(z) \cdot v = z \cdot v\}, \quad V_2 := \{v \in V_{\mathbb{C}} \mid h(z) \cdot v = \bar{z} \cdot v\}.$$

The *reflex field* E of a PEL input data as above is by definition the field of definition of the $B_{\mathbb{C}}$-module V_1.

2.2. Modular varieties of PEL type

Given a PEL input data $\big(B, \mathscr{O}_B, *, V, (\cdot,\cdot), h, \Lambda_p, K_f^{(p)}\big)$ unramified at p such that $K_f^{(p)}$ is *sufficiently small*, there is an associated moduli scheme $\mathcal{M} = \mathcal{M}_{K_f^{(p)}}$ over $\mathbb{F}_p$, which is a quasi-projective scheme over $\mathbb{F}_p$, such that for every commutative ring R over $\mathbb{F}_p$, $\mathcal{M}(R)$ is the set of isomorphism classes

$$(A, \iota : \mathscr{O}_B \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}, \lambda : A \to A^t, \bar{\eta})$$

where

- A is an abelian scheme over R up to prime-to-p isogeny,
- ι is a ring homomorphism,
- λ is a polarization of A, and
- $\bar{\eta}$ is a level structure of type $K_f^{(p)}$,

such that the following properties are satisfied.

(i) The homomorphism ι is compatible with the involution $*$ on $\mathscr{O}_B$ and the Rosati involution induced by λ.
(ii) The "determinant condition" in [24, §5 p. 390] is satisfied.

These properties imply that sheaf $\mathrm{R}^1_{\mathrm{et}}\pi_{A/R\,*}\big(\prod_{\ell \neq p} \mathbb{Z}_\ell\big)$ of the first étale homology groups, together with the effect of λ, ι and η on the first homology groups, is "modeled on the given PEL input data". Here $\pi_{A/R}$ denotes the structural morphism $A \to \mathrm{Spec}(R)$ for A.

2.2.1. A sufficient condition for $K_f^{(p)}$ to be "sufficiently small" is the following: there exists a $\mathbb{Z}$-lattice $\Lambda_{\mathbb{Z}}$ in V and a positive integer $n \geq 3$ which is prime to p such that

$$K_f^{(p)} \subseteq \left\{ \gamma_f^{(p)} \in G(\mathbb{A}_f^{(p)}) \,\middle|\, \gamma_\ell \cdot (\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) \subset n \cdot (\Lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) \quad \forall\, \ell | n \right\}.$$

2.2.2. When one varies the compact open subgroup $K_f^{(p)} \subset G(\mathbb{A}_f^{(p)})$ while the other ingredients of the PEL input data remain fixed, one obtains a filtered projective system

$$\widetilde{\mathcal{M}} := \big(\mathcal{M}_{K_f^{(p)}}\big)_{K_f^{(p)}}$$

of modular varieties over $\mathbb{F}_p$.

2.2.3. There is a natural structural morphism from the tower of modular varieties $\widetilde{\mathcal{M}}$ to $\mathrm{Spec}(\mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$, where E is the reflex field of the PEL input data.

2.2.4. The moduli functor associated to a given PEL input data described above makes sense over $\mathbb{Z}_{(p)}$ and is represented by a quasi-projective scheme $\mathcal{M}_{K_f^{(p)},\mathbb{Z}_{(p)}}$ smooth over $\mathscr{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. The moduli space $\mathcal{M}_{K_f^{(p)}}$ is the fiber over $\mathbb{F}_p$ of $\mathcal{M}_{K_f^{(p)},\mathbb{Z}_{(p)}}$. The theory of toroidal compactifications for modular varieties of PEL type, documented in [26], implies the existence of a filtered projective family

$$\left(\mathcal{M}^{\mathrm{tor}}_{K_f^{(p)},\Sigma,\mathbb{Z}_{(p)}}\right)_{K_f^{(p)},\Sigma}\longrightarrow \mathrm{Spec}(\mathscr{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})$$

of proper smooth schemes over $\mathscr{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ indexed by compact open subgroups $K_f^{(p)}\subset G(\mathbb{A}_f^{(p)})$ which are sufficiently small and $K_f^{(p)}$*-admissible cone decompositions* Σ, such that $\mathcal{M}_{K_f^{(p)},\mathbb{Z}_{(p)}}$ is a dense open subscheme of $\mathcal{M}^{\mathrm{tor}}_{K_f^{(p)},\Sigma,\mathbb{Z}_{(p)}}$ for all (sufficiently small) $K_f^{(p)}$.

2.3. Hecke symmetries on the prime-to-p tower of modular varieties

We keep the notation in 2.2. Let $\widetilde{\mathcal{M}}$ be the filtered projective system of moduli schemes over $\mathscr{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. There is a natural $(\mathscr{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)})$-action of the locally compact group $G(\mathbb{A}_f^{(p)})$ operates on the projective system $\widetilde{\mathcal{M}}$. For every sufficiently small open compact subgroup $K_f^{(p)}\subset G(\mathbb{A}_f^{(p)})$, the modular variety $\mathcal{M}_{K_f^{(p)}}$ is the subscheme of $\widetilde{\mathcal{M}}$ fixed under $K_f^{(p)}$, so that $K_f^{(p)}$ is naturally isomorphic to the Galois group of the pro-finite étale Galois cover $\widetilde{\mathcal{M}}/\mathcal{M}_{K_f}$.

When one fixes a (sufficiently small) level subgroup $K_f^{(p)}\subset G(\mathbb{A}_f^{(p)})$, we no longer have a group acting on the modular variety $\mathcal{M}_{K_f^{(p)}}$. What's left of the action of the group $G(\mathbb{A}_f^{(p)})$ on the tower $\widetilde{\mathcal{M}}$ is a family of finite etale algebraic correspondences

$$\mathcal{M}_{K_f^{(p)}}\xleftarrow{\mathrm{pr}}\mathcal{M}_{K_f^{(p)}\cap\gamma^{-1}\cdot K_f^{(p)}\cdot\gamma}\xrightarrow{\gamma\,\cdot}\mathcal{M}_{K_f^{(p)}},\qquad \gamma\in K_f^{(p)}\backslash G(\mathbb{A}_f^{(p)})/K_f^{(p)}$$

on $\mathcal{M}_{K_f^{(p)}}$, indexed by the double coset $K_f^{(p)}\backslash G(\mathbb{A}_f^{(p)})/K_f^{(p)}$. These finite étale algebraic correspondences are the prime-to-p Hecke symmetries on $\mathcal{M}_{K_f^{(p)}}$.

2.4. Remark. (a) Hecke correspondences were introduced by Hecke in [15], [16], [17]. They induce linear operators on elliptic modular forms.

(b) Over $\mathbb{C}$, every ℓ-power Hecke orbit is dense in $\mathcal{M}_{K_f}$ for every prime number ℓ. In contrast, in characteristic $p>0$ there are p-adic invariants which are preserved by all prime-to-p Hecke correspondences. Every such p-adic invariant imposes upper bounds on the Zariski closure of Hecke orbits. For instance the prime-to-p Hecke orbit of any non-ordinary point of $\mathscr{A}_g$ is contained in the zero locus of the Hasse invariant, and the prime-to-p Hecke orbit of any supersingular point of $\mathscr{A}_g$ is finite.

3. Sustained p-divisible groups and central leaves

3.1. Definition. Let $\kappa\supset\mathbb{F}_p$ be a field, and let S be a κ-scheme. Let X_0 be a p-divisible group over κ.

(a1) A p-divisible group over $X \to S$ is *strongly κ-sustained modeled on X_0* if the S-scheme

$$\mathscr{ISOM}_S\big(X_0[p^n]_S, X[p^n]\big) \to S$$

of isomorphisms between the truncated Barsotti-Tate groups $X_0[p^n]$ and $X[p^n]$ is faithfully flat over S for every $n \in \mathbb{N}$. Here $X_0[p^n]_S$ is short for $X_0[p^n] \times_{\mathrm{Spec}(\kappa)} S$; we will use similar abbreviation for base change in the rest of this article if no confusion is likely to arise.

(a2) A p-divisible group $X \to S$ is *strongly κ-sustained* if there exists a p-divisible group X_0 over κ such that $X \to S$ is κ-sustained modeled on X_0. In this case we say that X_0 is a κ-model of X.

(b1) A p-divisible group over $X \to S$ is *κ-sustained* if the $S \times_{\mathrm{Spec}\,\kappa} S$-scheme

$$\mathscr{ISOM}_{S\times_{\mathrm{Spec}\,\kappa} S}\big(\mathrm{pr}_1^* X[p^n], \mathrm{pr}_2^* X[p^n]\big) \to S \times_{\mathrm{Spec}\,\kappa} S$$

is faithfully flat over $S \times_{\mathrm{Spec}\,\kappa} S$ for every $n \in \mathbb{N}$, where $\mathrm{pr}_1, \mathrm{pr}_2 : S \times_{\mathrm{Spec}\,\kappa} S \to S$ are the two projections from $S \times_{\mathrm{Spec}\,\kappa} S$ to S.

(b2) Let $X \to S$ be a κ-sustained p-divisible group over S as in (b1). Let K be a field containing the base field κ. A κ-sustained p-divisible group X_2 over K is said to be a *K/κ-model* of $X \to S$ if the structural morphism

$$\mathscr{ISOM}_{S_K}\big(X_1[p^n] \times_{\mathrm{Spec}(K)} S_K, X[p^n] \times_S S_K\big) \longrightarrow S_K$$

of the above Isom-scheme is faithfully flat, for every positive integer n, where $S_K := S \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(K)$.

See [10] for more information about the notation of sustained p-divisible groups.

3.2. Definition. Let B be finite dimensional simple algebra over $\mathbb{Q}_p$, let $\mathscr{O}_B$ be a maximal order of B, and let $*$ be an involution of B. A polarized $\mathscr{O}_B$-linear p-divisible group over a scheme S is a triple $(X \to S, \iota : \mathscr{O}_B \to \mathrm{End}_S(X), \lambda : X \to X^t)$, where $X \to S$ is a p-divisible group over S, ι is a ring homomorphism, and λ is an isogeny of p-divisible groups compatible with the involution $*$, in the sense that

$$\iota(b)^t \circ \lambda = \lambda \circ \iota(b^*) \quad \forall b \in \mathscr{O}_B.$$

3.3. Definition. Let B be finite dimensional simple algebra over $\mathbb{Q}_p$, let $\mathscr{O}_B$ be a maximal order of B, and let $*$ be an involution of B. Let κ be a field of characteristic $p > 0$, and let S be a scheme over κ. Let $(X \to S, \iota : \mathscr{O}_B \to \mathrm{End}_S(X), \lambda : X \to X^t)$ be a polarized $\mathscr{O}_B$-linear p-divisible group over S.

(a) A polarized $\mathscr{O}_B$-linear p-divisible group

$$(X \to S, \iota : \mathscr{O}_B \to \mathrm{End}_S(X), \lambda : X \to X^t)$$

over S is *strongly κ-sustained modeled on an $\mathscr{O}_B$-linear polarized p-divisible group* $(X_0, \iota_0, \lambda_0)$ over κ if the S-scheme

$$\mathscr{ISOM}_S\big((X_0[p^n]_S, \iota_0, \lambda_0[p^n]_S), (X[p^n], \iota[p^n], \lambda[p^n])\big) \to S$$

is faithfully flat over S for every $n \in \mathbb{N}$.

(b) An $\mathscr{O}_B$-linear polarized p-divisible group $(X \to S, \iota, \lambda)$ is *κ-sustained* if

$$\mathscr{I}\mathscr{S}\mathscr{O}\mathscr{M}_{S\times_{\operatorname{Spec}\kappa} S}\big(\mathrm{pr}_1^*(X[p^n], \iota[p^n], \lambda[p^n]), \mathrm{pr}_2^*(X[p^n], \iota[p^n], \lambda[p^n])\big) \to S \times_{\operatorname{Spec}\kappa} S$$

is faithfully flat for every $n \in \mathbb{N}$. A κ-sustained $\mathscr{O}_B$-linear polarized p-divisible group $(X_2 \to K, \iota_2, \lambda_2)$ over a field K/κ is a K/κ-model of a κ-sustained $\mathscr{O}_B$-linear polarized p-divisible group $(X \to S, \iota, \lambda)$ if the morphism

$$\mathscr{I}\mathscr{S}\mathscr{O}\mathscr{M}_{S_K}\big((X_2[p^n], \iota_2[p^n], \lambda_2[p^n]) \times_{\operatorname{Spec} K} S_K, (X[p^n], \iota[p^n], \lambda[p^n]) \times_S S_K\big) \to S_K$$

is faithfully flat for every $n \in \mathbb{N}$.

3.4. Remark. The notion of sustained p-divisible group is a scheme-theoretic version of the notion of *geometrically fiberwise constant* p-divisible group introduced in [**?**]. Before explaining the relation between the two, we first recall a variant of the notion of geometrically fiberwise p-divisible groups. Let κ be a field of characteristic $p > 0$, let X_0 be a p-divisible group over κ and let $X \to S$ be a p-divisible group over S. We say that X is *geometrically fiberwise constant relative to κ modeled on X_0* if for every point $s \in S$, there exists an algebraically closed field $k \supset \kappa(s)$ and an k-isomorphism between $X_s \times_{\operatorname{Spec}(\kappa(s))} \operatorname{Spec}(k)$ and $X_0 \times_{\operatorname{Spec}(\kappa)} \operatorname{Spec}(k)$.

The promised relation between the two notions is as follows; details are in [10].

(i) If a p-divisible scheme X over a κ-scheme S is strongly sustained modeled on a p-divisible group X_0 over κ, then $X \to S$ is geometrically fiberwise constant relative to κ modeled on X_0.

(ii) Suppose that S is a *reduced* κ-scheme and X_0 is a p-divisible group over κ. If $X \to S$ is a geometrically fiberwise constant p-divisible group relative to κ modeled on X_0, then $X \to S$ is strongly κ-sustained.

3.5. Proposition. *Let S be a scheme over a field $\kappa \supset \mathbb{F}_p$.*

(a) *Let X be a κ-sustained p-divisible group over S. There exists a filtration*

$$X = X^0 \supsetneqq X^1 \supsetneqq \cdots \supsetneqq X^{m-1} \supsetneqq (0)$$

such that

(i) *X^i is a κ-sustained p-divisible group and X^i/X^{i+1} is an isoclinic κ-sustained p-divisible group, for $i = 0, 1, \ldots, m-1$, and*

(ii) $\operatorname{slope}(X^i/X^{i+1}) < \operatorname{slope}(X^{i+1}/X^{i+2})$ *for $i = 0, 1, \ldots, m-2$.*

The above filtration $X = X^0 \supsetneqq X^1 \supsetneqq \cdots \supsetneqq X^{m-1} \supsetneqq (0)$ is uniquely determined by properties (i), (ii) *above. It is called the* slope filtration *of X.*

(b) *Let $\mathscr{O}_B$ be a maximal order in a finite dimensional simple algebra over $\mathbb{Q}_p$. Let (X, ι) be a κ-sustained $\mathscr{O}_B$-linear p-divisible group over S. Let $X = X^0 \supsetneqq X^1 \supsetneqq \cdots \supsetneqq X^{m-1} \supsetneqq (0)$ be the slope filtration of the κ-sustained p-divisible group X.*

(i) *The subgroup X^i of X is stable under the action of $\mathscr{O}_B$ for $i = 1, \ldots, m-1$.*

(ii) *The $\mathscr{O}_B$-linear p-divisible groups X^i and X^i/X^{i+1} is κ-sustained for $i = 0, 1, \ldots, m-1$.*

3.6. Definition. Let $\mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p} := \mathcal{M}_{K_f^{(p)}} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\overline{\mathbb{F}}_p)$ be the modular variety over $\overline{\mathbb{F}}_p$ associated to a be a PEL input data $\big(B, \mathscr{O}_B, *, V, (\cdot,\cdot), h, \Lambda_p, K_f^{(p)}\big)$ unramified at p. Let $x_0 = [(A_0, \iota_0, \lambda_0)] \in \mathcal{M}_{K_f^{(p)}}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$-point of $\mathcal{M}_{K_f^{(p)}}$. The *central leaf* in $\mathcal{M}_{K_f^{(p)}}$ passing through x_0, denoted by $\mathcal{C}(x_0)$, is the maximal element among all locally closed subscheme S $\mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p}$ such that the restriction to S of the universal polarized $\mathscr{O}_B$-linear p-divisible group is $\overline{\mathbb{F}}_p$-sustained modeled on $(A_0[p^\infty], \iota_0[p^\infty], \lambda_0[p^\infty])$.

Remark. In the rest of this chapter we will often omit the adjective "central" when discussing central leaves, because we will not discuss isogeny leaves.

No detailed proof of the statement 3.7 below has been written down, therefore the word "proposition" is set in lower case. The statement is surely correct if $\mathcal{C}(x_0)$ is replaced by $\mathcal{C}(x_0)_{\mathrm{red}}$, with the same underlying topological space but with a structure sheaf. Showing that $\mathcal{C}(x_0)$ is reduced requires more effort.

3.7. proposition. *Notation as in* 3.6. *For every $\overline{\mathbb{F}}_p$-point x_0 of $\mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p}$, the leaf $\mathcal{C}(x_0)$ is a smooth locally closed subscheme of $\mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p}$ stable under all prime-to-p Hecke correspondences on $\mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p}$.*

3.8. The Hecke orbit conjecture. *The prime-to-p Hecke orbit of x_0 is Zariski dense in the leaf $\mathcal{C}(x_0)$, for every $\overline{\mathbb{F}}_p$-point x_0 of $\mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p}$. Equivalently, the only non-empty closed subscheme of the leaf $\mathcal{C}(x_0)$ which is stable under all prime-to-p Hecke correspondences is $\mathcal{C}(x_0)$ itself.*

4. Local structure of leaves

Let $\mathcal{M} = \mathcal{M}_{K_f^{(p)},\overline{\mathbb{F}}_p}$ be a modular variety over $\overline{\mathbb{F}}_p$ attached to a PEL input data unramified at p. The general phenomenon is that the formal completion $\mathcal{C}(x_0)^{/x_0}$ of a leaf $\mathcal{C}(x_0)$ at an $\overline{\mathbb{F}}_p$-point x_0 of $\mathcal{C}(x_0)$ is built up from a family of fibrations, where each fibration is a torsor for a p-divisible formal group. The exact formulation in the general case of leaves in modular varieties of PEL type is a bit complicated. Note that the formal completion $\mathcal{C}^{/x_0}$ is isomorphic to the equi-characteristic p deformation space $\mathcal{D}ef(A_0, \iota_0, \lambda_0)$ of the $\mathscr{O}_B$-linear polarized abelian variety $(A_0, \iota_0, \lambda_0)$, which in turn is isomorphic to the equi-characteristic p deformation space $\mathcal{D}ef(A_0[p^\infty], \iota_0[p^\infty], \lambda_0[p^\infty])$ of the polarized $\mathscr{O}_B$-linear p-divisible group $(A_0[p^\infty], \iota_0[p^\infty], \lambda_0[p^\infty])$ by the Serre-Tate theorem. Moreover the natural map $\mathcal{D}ef(A_0[p^\infty], \iota_0[p^\infty], \lambda_0[p^\infty]) \to \mathcal{D}ef(A_0[p^\infty]$ from the latter deformation space to the equi-characteristic p deformation space of the p-divisible group $A_0[p^\infty]$ is a closed embedding of formal schemes. The readers may consult [10] for the local structure of the leaf in the local deformation space of a p-divisible group over a perfect field of characteristic p.

We will be content with two examples in the case when $\mathcal{M} = \mathscr{A}_g$, and $x_0 = [(A_0, \lambda_0)]$ corresponds to a principally polarized abelian variety A_0 over $\overline{\mathbb{F}}_p$.

4.1. Case when A_0 has exactly two slopes

4.1.1. Suppose that the p-divisible group $A_0[p^\infty]$ of A_0 has two slopes ξ and $1-\xi$, where ξ is a rational number, $0 \le 2\xi < 1$. In other words there $A_0[p^\infty]$ sits in the middle of a short exact sequence

$$0 \to Z_0 \to A_0[p^\infty] \to Y_0 \to 0,$$

where Y_0 and Z_0 are isoclinic p-divisible groups over $\overline{\mathbb{F}_p}$ of height g, $\mathrm{slope}(Y_0) = \xi$, $\mathrm{slope}(Z_0) = 1-\xi$. The principal polarization $\lambda_0[p^\infty]$ on $A_o[p^\infty]$ induces isomorphisms

$$\lambda_{Y_0} : Y_0 \xrightarrow{\sim} Z_0^t, \quad \lambda_{Z_0} = \lambda_{Y_0}^t : Z_0 \xrightarrow{\sim} Y_0^t.$$

4.1.2. Let $\mathrm{M}(Y_0), \mathrm{M}(Z_0)$ be the covariant Dieudonné modules of Y_0 and Z_0 respectively.

(i) We have natural actions of $\mathcal{F}$ and $\mathcal{V}$ on

$$H(Y_0, Z_0)_{\mathbb{Q}} := \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Y_0), \mathrm{M}(Z_0))[1/p]$$

given by

$$(\mathcal{F} \cdot h)(y) = \mathcal{F}(h(\mathcal{V}(y)), \quad (\mathcal{V} \cdot h)(y) = \mathcal{V}(h(\mathcal{V}^{-1}(y))$$

for all $h \in H(Y_0, Z_0)_{\mathbb{Q}}$ and all $y \in \mathrm{M}(Y_0)$. This makes $H(Y_0, Z_0)_{\mathbb{Q}}$ an isocrystal of rank g^2, which is isoclinic of slope $1-2\xi$.

(ii) Note that $W(\overline{\mathbb{F}_p})$-lattice $H_0(Y_0, Z_0) := \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Y_0), \mathrm{M}(Z_0))$ in this isocrystal is stable under the action of $\mathcal{F}$ but not necessarily stable under the action of $\mathcal{V}$.

Denote by $H'(Y_0, Z_0)$ the largest $W(\overline{\mathbb{F}_p})$-submodule of $H_0(Y_0, Z_0)$ which is stable under both $\mathcal{F}$ and $\mathcal{V}$.

(iii) Recall that we have a natural isomorphisms of Dieudonné modules

$$\mathrm{M}(Y_0^t) \xrightarrow{\sim} \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Y_0), W(\overline{\mathbb{F}_p})) =: \mathrm{M}(Y_0)^\vee$$

and similarly an isomorphism

$$\mathrm{M}(Z_0^t) \xrightarrow{\sim} \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Z_0), W(\overline{\mathbb{F}_p})) =: \mathrm{M}(Z_0)^\vee.$$

Define an automorphism $*_0$ on $\mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Y_0), \mathrm{M}(Z_0))$ as the composition of the transposition map

$$\mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Y_0), \mathrm{M}(Z_0)) \longrightarrow \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Z_0)^\vee, \mathrm{M}(Y_0)^\vee),$$

the natural isomorphism

$$\mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Z_0)^\vee, \mathrm{M}(Y_0)^\vee) \xrightarrow{\sim} \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Z_0^t), \mathrm{M}(Y_0^t))$$

and the isomorphism

$$\mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Z_0^t), \mathrm{M}(Y_0^t)) \xrightarrow{\sim} \mathrm{Hom}_{W(\overline{\mathbb{F}_p})}(\mathrm{M}(Y_0), \mathrm{M}(Z_0))$$

induced by λ_{Y_0} and $\lambda_{Z_0}^{-1}$. It is easily verified that $*_0$ is an involution, compatible with $\mathcal{F}$ and $\mathcal{V}$. Therefore $*_0$ induces an involution on the Dieudonné module $H'(Y_0, Z_0)$.

4.1.3. (local structure of leaves in $\mathscr{A}_g$ in the two-slope case)
(a) *The leaf $\mathcal{C}_{\mathcal{D}ef(A_0[p^{infty}])}$ in the deformation space $\mathcal{D}ef(A_0[p^\infty])$ over $\overline{\mathbb{F}_p}$ has a natural structure as a neutral torsor over an isoclinic p-divisible formal group E over $\overline{\mathbb{F}_p}$, with slope $1-2\xi$ and height g^2. The constant p-divisible group*

$$(A_0[p^\infty]) \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathcal{C}_{\mathcal{D}ef(A_0[p^\infty])}$$

over $\mathcal{C}(x_0)^{/x_0}$ corresponds to a section of this torsor.

If $A_0[p^\infty]$ is isomorphic to $Y_0 \times Z_0$, then $\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty])}$ is naturally isomorphic to the p-divisible formal group E.

In the above $\mathcal{C}_{\mathcal{D}ef(A_0[p^{infty}])}$ denotes the maximal closed formal subscheme of the equi-characteristic deformation space $\mathcal{D}ef(A_0[p^\infty]$ over which the universal p-divisible group is strongly $\overline{\mathbb{F}_p}$-sustained modeled on $A_0[p^\infty]$.

(b) *The formal completion $\mathcal{C}(x_0)^{/x_0}$ has a natural structure as a torsor over an isoclinic p-divisible formal group subgroup $P \subset E$ over $\overline{\mathbb{F}_p}$ of height $g(g+1)/2$, with a section which corresponds to the constant principally polarized p-divisible group*

$$(A_0[p^\infty], \lambda_0[p^\infty]) \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathcal{C}(x_0)^{/x_0}$$

over $\mathcal{C}(x_0)^{/x_0}$.

If $A_0[p^\infty]$ is isomorphic to $Y_0 \times Z_0$, then $\mathcal{C}(x_0)^{/x_0}$ is naturally isomorphic to the p-divisible formal group P

(c) *The natural embedding $\mathcal{C}(x_0)^{/x_0} \hookrightarrow \mathcal{C}_{\mathcal{D}ef(A_0[p^\infty])}$ is equivariant with respect to the inclusion $P \hookrightarrow E$.*

(d) *The covariant Dieudonné module of the p-divisible group E is naturally isomorphic to the Dieudonné module $H'(Y_0, Z_0)$ defined in* 4.1.2 (ii). *The covariant Dieudonné module of the p-divisible group P is naturally isomorphic to the submodule of $H'(Y_0, Z_0)$ consisting of all elements fixed by the involution $*_0$.*

4.2. Case when A_0 has three slopes

4.2.1. Suppose that the p-divisible group $A_0[p^\infty]$ has three slopes $\xi, 1/2, 1-\xi$, $0 \le \xi < 1/2$. Moreover we assume for simplicity that $A_0[p^\infty]$ is a product of three isoclinic p-divisible groups over $\overline{\mathbb{F}_p}$:

$$A_0[p^\infty] = Y_1 \times Y_2 \times Y_3,$$

where $\mathrm{slope}(Y_1) = \xi$, $\mathrm{slope}(Y_2) = 1/2$ and $\mathrm{Y}_3 = 1-\xi$. As before the principal polarization λ_0 induces isomorphisms

$$\lambda_{Y_1} : Y_1 \xrightarrow{\sim} Y_3^t, \quad \lambda_{Y_2} = \lambda_{Y_2}^t : Y_2 \xrightarrow{\sim} Y_2^t, \quad \lambda_{Y_3} = \lambda_{Y_1}^t : Y_3 \xrightarrow{\sim} Y_1^t.$$

Let g_1, g_2, g_3 be the height of Y_1, Y_2, Y_3 respective. We have $g_1 = g_3$, $g_2 = 2(g-g_1)$, $\dim(Y_1) = \xi_1 \cdot g_1$, $\dim(Y_2) = g - g_1$, $\dim(Y_3) = (1-\xi_1)\cdot g_1$.

4.2.2. We have natural morphisms

$$\pi : \mathcal{C}_{\mathcal{D}ef(A_0[p^\infty])} \longrightarrow \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times_{\mathrm{Spec}(\overline{\mathbb{F}}_p)} \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$$

over $\overline{\mathbb{F}}_p$.

Recall from 4.1 that $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ and $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ are both isoclinic p-divisible formal groups over $\overline{\mathbb{F}}_p$ of slope $1/2-\xi$ and height $g_1\cdot g_2$. Moreover the covariant Dieudonné module of $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ is $H'(Y_1,Y_2)$, and the covariant Dieudonné module of $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3}$ is $H'(Y_2,Y_3)$.

We have natural isomorphisms

$$\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef((Y_1\times Y_2)^t} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)},$$

where the second isomorphism is induced by the principal polarizations λ_{Y_2} and λ_{Y_3}. The isomorphism $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ is compatible with the natural isomorphism $H'(Y_1,Y_2) \xrightarrow{\sim} H'(Y_2,Y_3)$ coming from linear algebra and the explicit description of the Dieudonné modules $H'(Y_1,Y_2)$ and $H'(Y_2,Y_3)$.

4.2.3. (The structure of $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$).
(a) *There is a natural free action of $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$ on $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$, such that the morphism*

$$\pi : \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \longrightarrow \mathcal{C}_{\mathcal{D}ef((Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$$

is the structural morphism of a $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$-torsor over $\mathcal{C}_{\mathcal{D}ef((Y_1\times Y_2)}\times\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$.
(b) *There is a relative group law*

$$+_1 : \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \times_{\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}} \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \longrightarrow \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$$

such that the natural map

$$\pi : \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \longrightarrow \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$$

is a group homomorphism over $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ when $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ is regarded as the base change from $\overline{\mathbb{F}}_p$ to $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ of the p-divisible formal group $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ over $\overline{\mathbb{F}}_p$. The kernel of this $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$-homomorphism of formal groups over $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ is the base change to $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ of the formal group $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$.
(c) *Similarly there is a relative group law*

$$+_2 : \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \times_{\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}} \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \longrightarrow \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$$

such that π is a $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$-homomorphism from $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$ to (the base change to $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ of) $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$, and the kernel of this homomorphism is (the base change to $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ of) $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$.
(d) *Each of the two relative group laws in* (b) *and* (c) *are compatible with the $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$-torsor structure on $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$, and these two relative group laws are compatible with each other. Thus $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$ acquires a structure as a biextension of the two p-divisible formal groups $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$, $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ by a third p-divisible formal group $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$.*

We refer to [29] for basics of biextensions.

Remark. (a) When $A_0[p^\infty]$ is not a product of its maximal isoclinic subquotients, the morphism

$$\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)} \longrightarrow \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$$

no longer has a biextension structure. Instead it becomes a "bi-coset", meaning that it can be regarded as a "coset" for an extension of (the base change to $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ of) $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ by (the base change to $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ of) $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$, and also as a "coset" of an extension of (the base change to $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ of) $\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ by (the base change to $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$ of) $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$, and these two coset structure (for two different groups) are compatible in a suitable sense. Note that two extensions referred to above do *not* fit together to form a biextension.

(b) The statement 4.2.3 holds for all p-divisible groups which is a product of three isoclinic p-divisible groups with different slopes. The effect of the principal polarization $\lambda_0[p^\infty]$ is that it gives an involution on $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2\times Y_3)}$ which interchanges the two relative group laws.

4.2.4. The principal polarization $\lambda_0[p^\infty]$ on $A_0[p^\infty]$ induces isomorphisms

$$\delta : \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)} \quad \text{and} \quad \delta' : \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}$$

such that $\delta'\circ\delta = \mathrm{id}_{\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)}}$ and $\delta\circ\delta' = \mathrm{id}_{\mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}}$. More precisely, δ is the following composition of isomorphisms

$$\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef((Y_1\times Y_2)^t)} = \mathcal{C}_{\mathcal{D}ef(Y_1^t\times Y_2^t)} \xrightarrow{\alpha_{\lambda_{Y_3},\lambda_{Y_2}}} \mathcal{C}_{\mathcal{D}ef(Y_3\times Y_2)} \xrightarrow{\sim} \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)},$$

where the isomorphism $\alpha_{\lambda_{Y_3},\lambda_{Y_2}}$ is induced by $\lambda_{Y_3} : Y_3 \xrightarrow{\sim} Y_1^t$ and $\lambda_{Y_2} : Y_2 \xrightarrow{\sim} Y_2^t$. The isomorphism δ' is defined similarly.

Let $\Delta \subset \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$ be the graph of δ. Clearly Δ can also be identified with the graph of δ', because δ' is the inverse of δ.

Just as in the two-slope case 4.1, the principal polarizations $\lambda_{Y_1}, \lambda_{Y_3}$ induces an involution $*'$ on the p-divisible formal group $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$ whose Dieudonné module is $H'(Y_1, Y_3)$. The subgroup of all elements of $H'(Y_1, Y_3)$ fixed by $*'$ is an isoclinic p-divisible formal group P' of height $g_1(g_1+1)/2$ and slope $1-2\xi_3$.

4.2.5. (local structure of leaves in $\mathscr{A}_g$ in the three-slope case)
(a) *There exists a unique morphism*

$$\pi' : \mathcal{C}_{\mathcal{D}ef(A_0[p^\infty],\lambda_0[p^\infty])} \to \Delta$$

such that the composition

$$\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty],\lambda_0[p^\infty])} \hookrightarrow \mathcal{C}_{\mathcal{D}ef(A_0[p^\infty])} \xrightarrow{\pi} \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}$$

is equal to the composition

$$\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty],\lambda_0[p^\infty])} \xrightarrow{\pi'} \Delta \hookrightarrow \mathcal{C}_{\mathcal{D}ef(Y_1\times Y_2)} \times \mathcal{C}_{\mathcal{D}ef(Y_2\times Y_3)}.$$

(b) *The formal closed subscheme* $\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty],\lambda_0[p^\infty])}$ *of* $\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty])}$ *is stable under the action of the p-divisible subgroup* P' *of* $\mathcal{C}_{\mathcal{D}ef(Y_1\times Y_3)}$. *The quotient of* $\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty],\lambda_0[p^\infty])}$ *by* P' *is naturally isomorphic to* Δ, *such that* $\mathcal{C}_{\mathcal{D}ef(A_0[p^\infty],\lambda_0[p^\infty])}$ *has a natural structure as a* P'*-torsor over the formal scheme* Δ.

5. Action of the local stabilizer subgroup

5.1. (Local stabilizer principle) *Let* $z \in Z(\overline{\mathbb{F}_p})$ *be a point of a closed subvariety* $Z \subset \mathcal{M}$ *stable under all prime-to-p Hecke correspondences. The formal completion* $\mathcal{Z}^{/z}$ *of* Z *at* z *is stable under the action of an open subgroup of the local stabilizer subgroup* G_z.

To make sense of the above statement, we need to explain what the group G_z is and how it operates on $\mathcal{C}^{/z}$. Let $\big(B, \mathscr{O}_B, *, V, (\cdot,\cdot), h, K_f^{(p)}\big)$ be the PEL input data which defines the modular variety $\mathcal{M}$over $\overline{\mathbb{F}_p}$ as in [24]; see §2 for a quick review. Let $(A_z, \beta_z : B \to \mathrm{End}^0(A_z), \mu_z)$ be the abelian variety with endomorphism by the simple algebra B up to isogeny plus a B-linear polarization parametrized by the point z of the modular variety $\mathcal{M}$. Let $*_z$ be the Rosati involution on $\mathrm{End}^0_B(A_z)$.

Let U_z be the unitary group attached to the semisimple algebra with involution $\big(\mathrm{End}^0_B(A_z)\otimes_{\mathbb{Q}}\mathbb{Q}_p, *_z\big)$ over $\mathbb{Q}_p$. In other words U_z is the algebraic group over $\mathbb{Q}_p$ such that

$$\mathrm{U}_z(R) = \big\{u \in (\mathrm{End}^0_B(A_z)\otimes_{\mathbb{Q}} R)^\times \mid uu^* \in R^\times\big\}.$$

for every commutative $\mathbb{Q}_p$-algebra R. In particular

$$\mathrm{U}_z(\mathbb{Q}_p) = \big\{u \in (\mathrm{End}^0_B(A_z)\otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \mid uu^* \in \mathbb{Q}_p^\times\big\}.$$

The local stabilizer subgroup G_z is the open subgroup of $\mathrm{U}_z(\mathbb{Q}_p)$ defined by

$$G_z := \mathrm{U}_z(\mathbb{Q}_p) \cap \mathrm{Aut}(A_z[p^\infty]).$$

It operates on the formal completing $\mathcal{M}^{/z}$ of $\mathcal{M}$ at z by functoriality. The closed formal subscheme $\mathcal{C}^{/z}$ of $\mathcal{C}^{/z}$ is stable under the natural action of the stabilizer subgroup G_z. The local stabilizer principle is the almost obvious statement that, because the p-adic closure in $\mathrm{U}_z(\mathbb{Q}_p)$ of the subset of all Hecke symmetries having z as a fixed point contains an open neighborhood of $\mathrm{U}_z(\mathbb{Q}_p)$, the formal completion $\mathcal{Z}^{/z}$ must be stable under the action of an open subgroup of G_z.

5.1.1. Remark. (a) The $\mathbb{Q}_p$-group U_z contains the *Frobenius torus* T_z at the $\overline{\mathbb{F}_p}$-rational point z of $\mathcal{M}$. In particular the local stabilizer subgroup G_z is "not too small". This is the reason why we formulated the local stabilizer principle for $\overline{\mathbb{F}_p}$-points.
(b) The local stabilizer principle holds for closed subvarieties Z of the minimal compactification $\overline{\mathcal{M}}$ as well. The precise formulation of the local stabilizer subgroup is omitted here.

5.2. Exploring the action of the local stabilizer subgroups. A wealth of information is encoded in the action of the local stabilizer subgroups. The challenge is to figure out how to effectively mine this source and make the hidden information accessible. The difficulty faced and success achieved so far vary greatly. We comment on four situations below. In situations (b) and (c) below, the local stabilizer subgroup G_z is quite big and the action of G_z is understood, so the local stabilizer principle provides substantial information. For the Hecke orbit problem, the challenge is to show that the Zariski closure of a Heck-invariant subvariety Z of $\mathcal{C}$ contains points as those in (b) and (c).

(a) at points of the leaf $\mathcal{C}$: Local rigidity for subvarieties of leaves

The formal completion at a closed point z of a leaf $\mathcal{C}$ has a "linear structure", in the sense that $\mathcal{C}^{/z}$ is built up from p-divisible formal groups; see 1.2 A1. The action of the local stabilizer subgroup G_z on $\mathcal{C}^{/z}$ is understood. It is expected that the method used in [11] will allow one to show that the formal completion at z of a Hecke-invariant closed subvariety $Z \subset \mathcal{C}$ is a "linear subvariety", in the sense that it is assembled from p-divisible formal subgroups of the building blocks of $\mathcal{C}^{/z}$.

(b) at points the boundary of the modular variety $\mathcal{M}$

If the Zariski closure $\overline{Z}$ of Z in the minimal compactification $\mathcal{M}^*$ of the moduli space $\mathcal{M}$ contains a point z of the boundary $\mathcal{M}^* \smallsetminus \mathcal{M}$, the local stabilizer principle at z yields substantial information. An example is the calculation in [1, §1], where it is shown that if the Zariski closure of a Hecke orbit of an ordinary point of $\mathscr{A}_g$ contains a zero-dimensional cusp, then that Hecke orbit is dense in $\mathscr{A}_g$.

(c) at hypersymmetric points

The notion of hypersymmetric points is defined in [7] and applied in [5], [19], [20] and [9], in combination with the method in [2] to prove irreducibility results in characteristic p.

For an application to the Hecke orbit problem, the difficulty lies in showing that a given Hecke invariant subvariety Z of a leaf $\mathcal{C}$ contains a point z such that the local stabilizer subgroup is substantially bigger than typical $\overline{\mathbb{F}}_p$-point of $\mathcal{C}$. In the case of the Siegel modular variety $\mathscr{A}_g$ one can show that Z contains hypersymmetric points with the help of Hilbert modular subvarieties. Note that the property that every $\mathbb{F}_p$-point is contained in a positive-dimensional Shimura subvariety is special to PEL type C: there exist PEL modular varieties of type A and modular varieties of type D such that the above property does *not* hold.

(d) at supersingular points

It is shown in [1] (as a consequence of [1, §1, Prop. 1]) that the Zariski closure of every Hecke-invariant subvariety of $\mathscr{A}_g$ contains a supersingular point. The argument shows that the same statement holds for PEL-type modular varieties if "supersingular" is replaced by "basic" (in the sense of [25]).

In some sense the local stabilizer subgroup at a supersingular point $z_1 \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ contains a subset "of finite index" in the set of all prime-to-p Hecke symmetries on $\mathscr{A}_g$. So it is not too far-fetched to expect strong rigidity statements for formal subvarieties $\hat{W}$ of $\mathscr{A}_g^{/z_1}$ stable under the action of an open subgroup of the local

stabilizer subgroup G_z. For instance if such a formal subvariety $\hat{W}$ is generically ordinary, one expects that $\hat{W} = \mathcal{A}_g^{/z_1}$. A big obstacle here is that the action of the local stabilizer subgroup G_z on $\mathcal{A}_g^{/z_1}$ is not understood when compared with the action of a local stabilizer group G_{z_2} on the fomral completion $\mathcal{C}(z_2)^{z_2}$ of the central leaf $\mathcal{C}(z_2)$. The formal completion $\mathcal{C}(z_2)$ has a linear structure assembled from p-divisible formal groups, and the action of G_{z_2} on $\mathcal{C}(z_2)$ preserves the linear structure. In contrast we only know that the formal completion $\mathscr{A}^{/z_1}$ is the spectrum of a formal power series ring over $\overline{\mathbb{F}_p}$, and we have little idea about the general pattern of the action of elements of G_z on $\mathcal{A}_g^{/z_1}$, nor do we possess a serviceable asymptotic expansion for the action of G_z on $\mathcal{A}_g^{/z_1}$.

6. Local rigidity for subvarieties of leaves

6.1. As outlined in 5.2 (a), it is expected that every irreducible formal subvariety of the formal completion $\mathcal{C}^{/x_0}$ of a leaf $\mathcal{C}(x_0)$ in a PEL modular variety $\mathcal{M}$ which is stable under the action of the local stabilizer subgroup G_{x_0} is a "linear subvariety" of $\mathcal{C}^{/x_0}$, for any $\overline{\mathbb{F}_p}$-point of $\mathcal{M}$.

We recall that $\mathcal{C}^{/x_0}$ is built up from a finite family of torsors of p-divisible formal groups. A "linear subvariety" $\mathcal{W}$ in $\mathcal{C}^{/x_0}$ is an irreducible formal subscheme $\mathcal{C}^{/x_0}$ which is also built up from a family of torsors of p-divisible formal groups, such that the "building blocks" of $\mathcal{W}$ are p-divisible subgroups of the "building blocks" of $\mathcal{C}^{/x_0}$, and the inclusion map $\mathcal{W} \hookrightarrow \mathcal{C}^{/x_0}$ is compatible with the torsor structures on $\mathcal{W}$ and $\mathcal{C}^{/x_0}$.

In this section we explain two local rigidity results, one for p-divisible formal groups and one for biextensions of p-divisible formal groups. They imply the above linearity statement in the case when $\mathcal{M} = \mathscr{A}_g$, $x_0 = (A_0, \lambda_0)$, and $A_0[p^\infty]$ has at most three slopes. The proofs are in [6] and [11] respectively.

6.2. Definition. Let G be a compact p-adic Lie group and let $\mathrm{Lie}(G)$ be the Lie algebra of G. Let k be a perfect field $k \supset \mathbb{F}_p$.

(a) Let X be a p-divisible group over k. let $\mathrm{M}(X)$ be the covariant Dieudonné module of X, and let $\mathrm{M}(X)_{\mathbb{Q}} := \mathrm{M}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\rho : G \to \mathrm{Aut}(X)$ be an action of G on X. We say that the action of G on X is *strongly non-trivial* if the trivial representation of $\mathrm{Lie}(G)$ does not appear in the Jordan-Hölder series of the representation of the Lie algebra $\mathrm{Lie}(G)$ on $\mathrm{M}(X)_{\mathbb{Q}}$. In other words for every open subgroup G' of G and any two p-divisible subgroups $Y \subsetneqq Z$ of X stable under G', there exists an element $\gamma \in G'$ such that $(\rho(\gamma) - 1)(Z) \not\subseteq Y$.

(b) Let X, Y, Z be p-divisible groups over k. Let $\pi : E \to X \times Y$ be a bi-extension of (X, Y) by Z. An action $\rho : G \to \mathrm{Aut}(E \to X \times Y)$ of G on the bi-extension $\pi : E \to X \times Y$ is *strongly non-trivial* if the induced actions of G on X, Y and Z are all strongly non-trivial as in (a) above.

6.3. Proposition. *Let G be a compact p-adic Lie group and let* $\mathrm{Lie}(G)$ *be the Lie algebra of G. Let k be an algebraically closed field $k \supset \mathbb{F}_p$.*

(a) *Let X be a p-divisible group over k, and let $\rho : G \to \mathrm{Aut}(X)$ be a* strongly non-trivial *action of G on X. Suppose that W is a irreducible closed formal*

subscheme of X stable under the action of G. Then W is a p-divisible formal subgroup of X.

(b) *Let X, Y, Z be p-divisible formal groups over k. Let $\pi : E \to X \times Y$ be a bi-extension of (X, Y) by Z. Let $\rho : G \to \mathrm{Aut}(E \to X \times Y)$ be a* strongly non-trivial *action of G on E compatible with the bi-extension structure. Let W be a irreducible closed formal subscheme of X stable under the action of G. Suppose that every slope of Z is strictly bigger than every slope of X and every slope of Y.*

(i) *There exists a p-divisible subgroup Z_1 of Z and a p-divisible subgroup U of $X \times Y$ such that W is stable under the action of Z_1 and the projection $\pi : E \to X \times Y$ induces an isomorphism $W/Z_1 \xrightarrow{U}$. In other words W is a Z_1-torsor over U and the inclusion map $W \hookrightarrow E$ is equivariant with respect to $Z_1 \hookrightarrow Z$. Note that the p-divisible subgroups $Z_1 \subseteq Z$ and $U \subseteq X \times Y$ in* (i) *are uniquely determined by W.*

(ii) *Suppose that X and Y do not have any slope in common. Then there exists p-divisible subgroups $X_1 \subseteq X$ and $Y_1 \subset Y$, uniquely determined by W, such that $U = X_1 \times Y_1$ and W is a sub-bi-extension of E. In other words W is stable under both relative group laws of the bi-extension E.*

6.4. Remark. The key ingredient of the proof of 6.3 (a) is the following "identity principle" for formal power series over k, proved in [6, §3], which produces identities of the form $f(\underline{y}, \underline{y}) = 0$ in two sets of variables $\underline{x}, \underline{y}$, from a sequence of congruences relations of the form $f(\underline{x}, \underline{x}^{p^n}) \equiv 0 \bmod (\underline{x})^{d_n}$, with $\lim_{n\to\infty} \frac{p^n}{d_n} = 0$.

Let $k \supset \mathbb{F}_p$ be a field. Let $\mathbf{u} = (u_1, \ldots, u_a)$, $\mathbf{v} = (v_1, \ldots, v_b)$ be two tuples of variables. Let $f(\mathbf{u}, \mathbf{v}) \in k[[\mathbf{u}, \mathbf{v}]]$ be a formal power series in the variables $u_1, \ldots, u_a, v_1, \ldots, v_b$ with coefficients in k. Let $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$ be two new sets of variables. Let $(g_1(\mathbf{x}), \ldots, g_a(\mathbf{x}))$ be an a-tuple of power series such that $g_i(\mathbf{x}) \in (\mathbf{x})k[[\mathbf{x}]]$ for $i = 1, \ldots, a$. Let $(h_1(\mathbf{y}), \ldots, h_b(\mathbf{y}))$ be a b-tuple of power series with $h_j(\mathbf{y}) \in (\mathbf{y})k[[\mathbf{y}]]$ for $j = 1, \ldots, b$. Let $q = p^r$ be a power of p for some positive integer r. Let $n_0 \in \mathbb{N}$ be a natural number. Let $(d_n)_{n\in\mathbb{N},\, n\geq n_0}$ be a sequence of natural numbers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$. Suppose we are given power series $\phi_{j,n}(\mathbf{v}) \in k[[\mathbf{v}]]$ for all $j = 1, \ldots, b$ and all $n \geq n_0$ such that

$$R_{j,n}(\mathbf{v}) := \phi_{j,n} - v_j^{q^n} \equiv 0 \pmod{(\mathbf{v})^{d_n}} \quad \forall\, j = 1, \ldots, b,\ \forall n \geq n_0.$$

and

$$f\left(g_1(\mathbf{x}), \ldots, g_a(\mathbf{x}), \phi_{1,n}(h(\mathbf{x})), \ldots, \phi_{b,n}(h(\mathbf{x}))\right) \equiv 0 \pmod{(\mathbf{x})^{d_n}}$$

in $k[[\mathbf{x}]]$, for all $n \geq n_0$. Then

$$f(g_1(\mathbf{x}), \ldots, g_a(\mathbf{x}), h_1(\mathbf{y}), \ldots, h_b(\mathbf{y})) = 0 \ \text{ in } \ k[[\mathbf{x}, \mathbf{y}]].$$

6.5. Remark. The proof of 6.3 (b) uses the notion of *complete restricted perfection* of a complete Noetherian local ring in equi-characteristic p. An example of such a ring is the ring

$$k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\flat}$$

of formal series over k whose underlying abelian group is the set of all formal series $\sum_I b_I \, \underline{t}^I$ with $b_I \in \kappa$ for all I, where I runs through all elements in $\mathbb{N}[1/p]^m$ such that

$$(\flat) \qquad |I|_p \leq \mathrm{Max}\big(C \cdot (|I|_\sigma + d)^E, 1\big) \, .$$

Here the parameters E, C, d are positive real numbers; $\mathbb{N}[1/p]$ is the additive semigroup of all non-negative rational numbers whose denominators divide a power of p. The p-adic absolute value $|I|_p$ and the archimedean absolute value $|I|_\sigma$ of an element $I = (i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m$ are defined by

$$|I|_p = \mathrm{Max}(|i_1|_p, \ldots, |i_m|_p), \quad |I|_\sigma = i_1 + \cdots + i_m \, ,$$

where $|\cdot|_p$ is the p-adic absolute value on $\mathbb{Q}$, normalized by $|p|_p = p^{-1}$. The subset $S_{E,C,d}$ be of $\mathbb{N}[1/p]^m$ consisting of all elements of $\mathbb{N}[1/p]^m$ satisfying the condition $(\flat)$ satisfies the following property: for every real number M, the set of all elements of $S_{E,C,d}$ such that $|I|_\sigma \leq M$ is *finite*. This finiteness property implies that the standard formula for multiplication of two such formal series give a ring structure on $k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}$.

6.6. Remark. We indicate how the complete restricted perfections enters the proof of 6.3 (b), in the case when Z is isoclinic. Furthermore we assume for simplicity that the intersection $Z \cap W$ is a p-divisible subgroup Z_1 of Z. From local rigidity for p-divisible group, we know that the image of W under the projection map $\pi : E \to X \times Y$ is a p-divisible subgroup U of $X \times Y$. We have to show that W is stable under the action of Z_1.

(i) If the bi-extension $E \to X \times Y$ of (X, Y) by Z is trivial, we have a natural retraction morphism $\mathbf{r} : E \to Z$. Then we use the identity principle 6.4 to show that $W \times W$ maps to W under the composition

$$E \times E \xrightarrow{\mathbf{r} \times 1_E} Z \times E \to E,$$

where the second maps $Z \times E \to E$ comes from the Z-torsor structure of E. This is the strategy for the proof of 6.3 (a). However when the bi-extension $E \to X \times Y$ is non-split, we won't be able to produce a natural "retraction" map $\mathbf{r} : W \to Z_1$, even after modifying the bi-extension E by an isogeny. The complete restricted perfection comes to the rescue: there exists a natural retractions $W \dashrightarrow Z_1$ after passing from W to (the formal spectrum of) a suitable complete restricted perfections of the coordinate rings of W.

(ii) The identity principle 6.4 holds when the formal series in the statement belong to rings of the form $k\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle\rangle_{C;\,d}^{E,\,\flat}$. This generalization of 6.4 depends crucially on the finiteness property of the support sets $S_{E,C,d}$. Using the generalized identity principle, one shows that W is stable under the translation action by the image of the "retraction map" $W \dashrightarrow Z_1$. The last statement implies that W is stable under translation by Z_1.

7. Monodromy of Hecke invariant subvarieties

In this section we discuss a general property for prime-to-p monodromy of subvarieties of modular varieties of PEL type in characteristic p which are stable under

all prime-to-p Hecke correspondences. The argument is based largely on group theory. In the case when the Hecke-invariant subvariety in question contains a point with very large local stabilizer subgroups, called *hypersymmetric points*, the same group-theoretic argument leads to information on p-adic monodromy.

We will formulate the results in the case of Siegel modular varieties, then comment on generalization to general PEL modular varieties.

7.1. Proposition. *Let $n \geq 3$ be a positive integer prime to p. Let $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ be the moduli space of g-dimensional principally polarized abelian varieties with symplectic level-n structure over $\overline{\mathbb{F}}_p$. Let Z be a smooth locally closed subscheme of $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ which is stable under the action of all prime-to-p Hecke correspondences. Let ℓ be a prime number such that $\ell \nmid pn$. Let $A \to \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ be the universal abelian scheme over $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$. Assume that Z is not contained in the supersingular locus of $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$.*

(i) *The subscheme Z is irreducible if and only if the prime-to-p Hecke correspondences operates transitively on the set $\pi_0(Z)$ of connected components of Z.*

(ii) *For any connected component Z_0 of Z, the image of the Galois representation attached to ℓ-power torsion points $(A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_0)[\ell^\infty] \to Z_0$ over Z_0 is isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.*

7.2. Remark. (1) Proposition 7.1 was proved in [2]. The proof is essentially group-theoretic.

(2) The "if" part of the statement 7.1 (i) implies that if a subvariety W of $A \to \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ is defined by some numerical invariant fixed under all prime-to-p Hecke correspondences, to show that W is irreducible, it suffices to show that $\pi_0(W)$ consists of a single prime-to-p Hecke orbit. Using this method, the authors proved in [9] that the every non-supersingular Newton polygon stratum of $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$, as well as every non-supersingular central leaf, is irreducible.

(3) The proof of 7.1 depends crucially on the fact that the algebraic group Sp_{2g} is simply connected. Therefore for general PEL type Shimura varieties one needs to pass to the $G^{\mathrm{sc}}_{\mathrm{der}}(\mathbb{A}_f^{(p)})$-tower first and use the Hecke correspondences defined by this tower, before applying the group-theoretic argument. Here G is the reductive group over $\mathbb{Q}$ attached to the given PEL input data, and $G^{\mathrm{sc}}_{\mathrm{der}}$ is the simply connected cover of the derived group of G.

The statement of 7.1 also depends on the fact that Sp_{2g} is almost simple over $\mathbb{Q}$. The way we defined PEL input data has the consequence that the associated reductive group G has the property that G_{der} is almost simple over $\mathbb{Q}$. If the definition of modular varieties of PEL type is generalized to allow the possibility that G_{der} is not necessarily almost $\mathbb{Q}$-simple, the statement of 7.1 needs to be adjusted, to the effect that the $H(\mathbb{A}_f^{(p)})$-Hecke symmetries applied to any $\overline{\mathbb{F}}_p$-point of Z is infinite, for every $\mathbb{Q}$-factor H of $G^{\mathrm{sc}}_{\mathrm{der}}$.

7.3. Definition. (a) An abelian variety A over a field $K \supset \mathbb{F}_p$ is *hypersymmetric* if the natural map

$$\big(\mathrm{End}_{\overline{K}} A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \mathrm{End}_{\overline{K}} A[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$$

is an isomorphism. The last condition is equivalent to the condition that the natural map

$$\big(\mathrm{End}_{\overline{K}} A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow \big(\mathrm{End}_{\overline{K}} A[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

(b) Let B be a finite dimensional simple algebra over $\mathbb{Q}$. A B-linear abelian variety A up to isogeny over a field $K \supset \mathbb{F}_p$ is *hypersymmetric* if the natural map

$$\big(\mathrm{End}^0_{\overline{K},B} A \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow \big(\mathrm{End}^0_{\overline{K},B} A[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

7.4. Remark. (1) The notion of hypersymmetric abelian varieties was introduced in [7].

(2) Clearly an abelian variety which is isogenous to a hypersymmetric abelian variety is itself hypersymmetric. Therefore we formulated 7.3 (b) for abelian varieties up to isogeny.

(3) For a point $z_0 = [(A_0, \lambda_0)] \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ such that A_0 is hypersymmetric, the local stabilizer subgroup G_{z_0} is $\mathrm{U}(\mathrm{End}(A_0[p^\infty]), *_0) \cap \mathrm{Aut}(A_0[p^\infty]$, where $*_0$ is the Rosati involution on $\mathrm{End}(A_0[p^\infty])$ attached to the principal polarization λ_0.

It is easy to show that the prime-to-p Hecke orbit of a hypersymmetric point $z_0 = [(A_0, \lambda_0)] \in \mathscr{A}_g(\overline{\mathbb{F}}_p)$ is dense in the leaf $\mathcal{C}(z_0)$. This is an easy consequence of the local stabilizer principle, the linearization results in 1.2.A and basic representation theory. This example leads to the following observation: if we can show that the Zariski closure in the leaf $\mathcal{C}(x_0)$ of the prime-to-p Hecke orbit of a point $x_0 \in \mathcal{A}_g(\mathbb{F}_p)$ contains a point y_0 with a large stabilizer subgroup G_{y_0}, we have shown that the Zariski closure of this Hecke orbit is not too small.

(4) The notion of hypersymmetric points combined with the argument for the irreducibility result 7.1 (i) leads to an effective way to show that the p-adic monodromy of a Hecke invariant subvariety is large; see [5]. This method is used in [9] to show that the p-adic monodromy of every non-supersingular central leaf in $\mathcal{A}_g$ is maximal.

(5) For modular varieties of PEL type, it can happen that there are central leaves which do not contain any hypersymmetric point. We refer to [52] for a complete solution of the existence problem of hypersymmetric points on modular varieties of PEL type.

8. The Hecke orbit conjecture for $\mathscr{A}_g$

8.1. In this section we outline a proof of the Hecke orbit conjecture for the moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}}_p$. A more detailed sketch of the proof of the Hecke orbit conjecture for $\mathscr{A}_g$ can be found in [3]; see also [4].

The proof uses a special property of $\mathscr{A}_g$, that every $\overline{\mathbb{F}}_p$-point of $\mathscr{A}_g$ is contained in a Hilbert modular subvariety of $\mathscr{A}_g$; see 8.3 It is a consequence of the fact that the endomorphism algebra (A, λ) of every polarized abelian variety over $\overline{\mathbb{F}}_p$ contains a product of totally real fields $F_1 \times \cdots F_r$ fixed by the Rosati involution, with $[F_1 : \mathbb{Q}] + \cdots + [F_r : \mathbb{Q}] = \dim(A)$. The same train of thought, together with the consideration of the action of local stabilizer subgroups, leads to the trick of "splitting at supersingular point"; see 8.2.

8.2. Proposition. *Let $n \geq 3$ be a positive integer prime to p. Let $x_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$ point of $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$. Let $\mathcal{C}(x_0)$ be the central leaf in $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ containing x_0. There exist*

(i) *an $\overline{\mathbb{F}}_p$-point $x_1 = [(A_1, \lambda_1)]$ of the Zariski closure in $\mathcal{C}(x_0)$ of the prime-to-p Hecke orbit of x_0,*
(ii) *totally real number fields $F_1, \ldots, F_r$ with $[F_1 : \mathbb{Q}] + \cdots + [F_r : \mathbb{Q}] = g$ such that $F_i \times_{\mathbb{Q}} \mathbb{Q}_p$ is a field for $i = 1, \ldots, r$, and*
(iii) *a subring of the endomorphism algebra $\mathrm{End}^0(A_1) := \mathrm{End}(A_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ of fixed by the Rosati involution attached to λ_1, which is isomorphic to $F_1 \times \cdots \times F_r$.*

Clearly if we can show that the prime-to-p Hecke orbit of x_1 is dense in $\mathcal{C}(x_1)$, then the prime-to-p Hecke orbit of x_0 is dense in $\mathcal{C}(x_0)$.

8.3. Proposition. *We keep the notation in 8.2. There exist*

(i) *a positive integer $m \geq 3$ prime to p,*
(ii) *a Hecke-equivariant finite morphism*
$$f : \mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p} \times \cdots \mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p} \longrightarrow \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$$
with respect to the embedding $\mathrm{SL}_2(F_1) \times \cdots \mathrm{SL}_2(F_r) \to \mathrm{Sp}_{2g}$ of algebraic groups, where $\mathcal{M}_{F_i,m}$ is the Hilbert modular variety with level-m structure attached to F_i,
(iii) *Hecke-equivariant finite morphisms*
$$h_i : \mathcal{M}_{F_i,m,\overline{\mathbb{F}}_p} \longrightarrow \mathscr{A}_{[F_i:\mathbb{Q}],n,\overline{\mathbb{F}}_p}$$
with respect to the embedding $\mathrm{SL}_2(F_i) \to \mathrm{Sp}_{2[F_i:\mathbb{Q}]}$ for $i = 1, \ldots, r$,
(iii) *an $\overline{\mathbb{F}}_p$-point $(z_1, \ldots, z_r)$ of $\mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p} \times \cdots \mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p}$ with*
$$f_1(z_1, \ldots, z_r) = x_1,$$

such that for any r-tuple of points $y_1, \ldots, y_r$ of $\mathcal{M}_{F_1,m,\overline{\mathbb{F}}_p}, \ldots, \mathcal{M}_{F_r,m,\overline{\mathbb{F}}_p}$ corresponding to $\mathscr{O}_{F_i}$-linear $[F_i : \mathbb{Q}]$-dimensional abelian varieties $B_1, \ldots, B_r$, the abelian variety corresponding to $f_1(y_1, \ldots, y_r)$ is $(F_1 \times \cdots \times F_r)$-linearly isogenous to $B_1 \times \cdots B_r$, and the abelian variety corresponding to $h_i(y_i)$ is isogenous to B_i for $i = 1, \ldots, r$.

The general linearization method implies quickly that the Zariski closure of the prime-to-p Hecke orbit of z_i on the Hilbert modular variety $\mathcal{M}_{F_i,m,\overline{\mathbb{F}}_p}$ contains an open subset of the central leaf $\mathcal{C}_F(z_i)$ in $\mathcal{M}_{F_i,m,\overline{\mathbb{F}}_p}$ passing through z_i for $i = 1, \ldots, r$, because $F_i \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field. The irreducibility result [49] of C.-F. Yu says that $\mathcal{C}_F(z_i)$ is irreducible unless z_i is supersingular, in which case $\mathcal{C}_F(z_i)$

is a finite set and the prime-to-p Hecke correspondences operates transitively on $\mathcal{C}_F(z_i)$. Note that there exist hypersymmetric points in $\mathcal{C}_F(z_i)$ for every i. The statement 8.4 follows.

8.4. Corollary. *Notation as in* 8.3. *There exists an* $\overline{\mathbb{F}}_p$*-point* $x_2 = [(A_2, \lambda_2)] \in \mathcal{C}(x_1)$ *which lies in the Zariski closure the prime-to-p Hecke orbit of* x_1 *such that* A_2 *is isogenous to a product of hypersymmetric abelian varieties.*

8.5. Theorem. *Notation as in* 8.2 *and* 8.3. *The prime-to-p Hecke orbit of every* $\overline{\mathbb{F}}_p$*-point* x_1 *of* $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ *is Zariski dense in the central leaf* $\mathcal{C}(x_1)$ *in* $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$.

As we have mentioned earlier, the linearization method implies that the Zariski closure of the prime-to-p Hecke orbit of the hypersymmetric point $h_i(z_i)$ in the Siegel modular variety $\mathscr{A}_{[F_i:\mathbb{Q}],m,\overline{\mathbb{F}}_p}$ is dense in $\mathcal{C}(h_i(z_i))$. Another application of the linearization method plus easy representation allows us conclude that the Zariski closure of the prime-to-p Hecke orbit of x_1 contains an open subset of $\mathcal{C}(x_1)$. However we know that the leaf $\mathcal{C}(x_1)$ is irreducible unless x_1 is supersingular, so we are done.

9. Open questions

In this section we list several open questions related to the Hecke orbit conjecture. We have not attempted to put these questions in the most general setting possible. Instead we have formulated in relatively simple cases, which we believe still preserves essential aspects of the difficulties.

- Conjectures 9.1 and 9.2 are samples of strong local rigidity predictions in the direction of 5.2 (b). To make progress on them, the first step would be developing a theory of "asymptotic expansion" for the action of elements of the local stabilizer subgroup which are close to 1, on the characteristic p deformation space (or the formal subvariety W in question). Note that in the situation of 4.1 and 4.2 the linear structure of $\mathcal{C}^{/z_0}$ provides such asymptotic expansions, which allows us to apply the identity principle.
- Problem 9.3 is a global rigidity statement with a flavor somewhat different from the Hecke orbit conjecture. Among other things, it asserts that a local geometric condition on a subvariety Z of a PEL modular variety $\mathcal{M}$ over $\overline{\mathbb{F}}_p$, that it is "linear" at one point, implies the existence of many Tate cycles on Z, to the extent that these Tate cycles "cut out" Z in $\mathcal{M}$. The assertion is tempting, but we have no evidence other than our inability to come up with a counter-example.
- Problem 9.4 is a possible approach to 9.3 via p-adic monodromy. Of course both 9.3 and 9.4 can be formulated in the context of "linear subvarieties" of a leaf $\mathcal{C}$ of a PEL modular variety. But we feel that the charm of relative simplicity has its virtue.
- Problem 9.5 is an important and natural question, but we don't have a good idea about the shape of the answer at this point. It bears some connections with 9.2 and also some connection with 9.3.

9.1. Conjecture. *Let $k \supset$ be an algebraically closed field. Let X_0 be a one-dimensional p-divisible formal group of height $h > 1$. Suppose that $\mathcal{W}$ is an irreducible formal subscheme of $\mathcal{D}ef(X_0)$ which is stable under the action of an open subgroup of* $\mathrm{Aut}(X_0$. *Then $\mathcal{W}$ is reduced and formally smooth over k, and $\mathcal{W}$ is the locus in $\mathcal{D}ef(X_0)$ over which the universal p-divisible group has p-rank at most* $\dim(\mathcal{W})$.

9.2. Conjecture. *Let (A_0, λ_0) be a supersingular principally polarized abelian variety over an algebraically closed field $k \supset \mathbb{F}_p$. Suppose that $\mathcal{W}$ is an irreducible formal subscheme of the deformation space $\mathcal{D}ef(A_0, \lambda_0) = \mathcal{D}ef(A_0[p^\infty], \lambda_0[p^\infty])$ which is stable under the action of an open subgroup of* $\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty])$. *If the restriction to the generic point of the universal p-divisible group is ordinary, then $\mathcal{W} = \mathcal{D}ef(A_0, \lambda_0)$.*

9.3. Problem. Let $n \geq 3$ be a positive integer, and let $\mathscr{A}_{g,n}/W(\overline{\mathbb{F}}_p)$ be the moduli space of g-dimensional principally polarized abelian varieties over $W(\overline{\mathbb{F}}_p)$. Let Z be an irreducible closed subscheme of $\mathscr{A}_{g,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{F}}_p)$. Let $z_0 = [(A_0, \lambda_0)] \in Z(\overline{\mathbb{F}}_p)$ be a closed point of Z. Suppose that A_0 is an ordinary abelian variety, and the formal completion $Z^{/z_0}$ of Z is a formal subtorus of the Serre-Tate formal torus $\big(\mathscr{A}_{g,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{F}}_p)\big)^{/z_0}$. Show that there exist a finite extension field L of $W(\overline{\mathbb{F}}_p)[1/p]$ and a reduced closed subscheme $V \subset \mathscr{A}_{g,n} \times_{\mathrm{Spec}(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\mathscr{O}_L)$ such that the following statements hold.

(i) There exists a $\mathbb{Q}_p$-linear embedding $L \hookrightarrow \overline{\mathbb{Q}}_p$ such that geometric generic fiber $V \times_{\mathrm{Spec}(\mathscr{O}_L)} \mathrm{Spec}(\overline{\mathbb{Q}}_p)$ of V is a Shimura subvariety of the moduli space $\mathscr{A}_{g,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{Q}}_p)$.

(ii) The scheme Z is an irreducible component of the closed fiber $V \times_{\mathrm{Spec}(\mathscr{O}_L)} \mathrm{Spec}(\overline{\mathbb{F}}_p)$ of V.

Remark. We do not know whether every irreducible formal subvariety $\mathcal{W}$ of the deformation space of $\mathcal{D}ef(A_0, \lambda_0[p^\infty])$ stable under the action of an open subgroup of $\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty])$ as in 9.2 is "generically linear" in a suitable sense. An affirmative answer will be a substantial progress for conjecture 9.2.

9.4. Problem. Let Z be an irreducible closed subscheme of the Siegel moduli scheme $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p} := \mathscr{A}_{g,n} \times_{Spec(W(\overline{\mathbb{F}}_p))} \mathrm{Spec}(\overline{\mathbb{F}}_p)$ over $\overline{\mathbb{F}}_p$ which is linear at an ordinary point as in 9.3; i.e. there exists a closed point $z_0 = [(A_0, \lambda_0)] \in \mathscr{A}_{g,n}(\overline{\mathbb{F}}_p)$ with A_0 ordinary such that $Z^{/z_0}$ is a formal subtorus of $\mathscr{A}^{/z_0}_{g,n,\overline{\mathbb{F}}_p}$. Let $A \to \mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$ be the universal abelian scheme over $\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}$, and let Z_{ord} be the largest open subset of Z such that $A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}} \to Z_{\mathrm{ord}}$ is ordinary.

(i) Show that the Zariski closure of the image of the Galois representation attached to the maximal étale quotient $A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}}[p^\infty]_{\mathrm{et}}$ of the p-divisible group $A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}}[p^\infty]$ over Z_{ord} is a reductive subgroup G_{Z,naive,z_0} of $\mathrm{GL}_g(\mathrm{V}_p(A_0[p^\infty]_{\mathrm{et}}))$. Here $A_0[p^\infty]_{\mathrm{et}}$ is the maximal étale quotient of the p-divisible group $A_0[p^\infty$, and $\mathrm{V}_p(A_0[p^\infty]_{\mathrm{et}})$ is the p-adic Tate module of $A_0[p^\infty]_{\mathrm{et}}$, non-canonically isomorphic to $\mathbb{Q}_p^g$.

(ii) Let $\mathcal{T}_{Z_{\mathrm{ord}}}$ be the $\mathbb{Q}_p$-linear Tannakian subcategory generated by the isocrystal attached the p-divisible group $A \times_{\mathscr{A}_{g,n,\overline{\mathbb{F}}_p}} Z_{\mathrm{ord}}[p^\infty]$ in the Tannakian category of all overconvergent isocrystals over Z_{ord}.
   (a) Show that the Galois group $\mathrm{Gal}(\mathcal{T}_{Z_{\mathrm{ord}},z_0})$ attached to the Tannakian category $\mathcal{T}_{Z_{\mathrm{ord}}}$ and the fiber functor at z_0 is reductive.
   (b) Show that there exists a parabolic subgroup P of $\mathrm{Gal}(\mathcal{T}_{Z_{\mathrm{ord}},z_0})$ such that the unipotent U radical of P is naturally isomorphic to the cocharacter group of the formal torus $Z^{/z_0}$, and the reductive quotient of P/U is naturally isomorphic to G_{Z,naive,z_0}.

9.5. Problem. Let $\mathcal{C}$ be a leaf in $\mathscr{A}_g$ over $\overline{\mathbb{F}}_p$, let $\overline{\mathcal{C}}$ be the Zariski closure of $\mathcal{C}$ in $\mathscr{A}_g$, and let s_0 be an $\overline{\mathbb{F}}_p$-point of $\overline{\mathcal{C}} \smallsetminus \mathcal{C}$. Analyze the structure of the formal completion $\overline{\mathcal{C}}^{/s_0}$ of $\overline{\mathcal{C}}$ at s_0, and relate the structure of $\overline{\mathcal{C}}^{/s_0}$ to the family of linear structures of $\mathcal{C}^{/z}$ as z varies over points of $\mathcal{C}$.

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