Tate-linear formal varieties and orbital rigidity

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Contents

1 Introduction **1**

2 Localization of nilpotent groups and Mal'cev completion **9**

3 Tate unipotent groups and Lie algebras **18**

4 Tate-linear formal varieties **33**

5 Orbital rigidity: the statement and reduction steps **45**

6 Proof via hypocotyl elongation in tempered perfections **52**

§1. Introduction

(1.1) We begin with an impressionistic description of the first part of the title. Throughout this paper p denotes a prime number, and κ is a field of characteristic p.

A Tate-linear formal variety over κ is a noetherian smooth formal scheme over κ of with an extra structure, called a *Tate-linear structure*. The Tate-linear structure of T is governed by a sheaf $\mathbf{N}$ of nilpotent groups on $\mathrm{Spec}(\kappa)$ with the fpqc topology, and $\mathbf{N}$ determines T. On a formal level, $\mathbf{N}$ looks like a "fundatmental group of T. Thus Tate-linear formal varieties share certain group theoretic features with compact nilmanifolds.

A prominent feature of Tate-linear formal varieties is that every Tate-linear formal variety T over a base field κ of characteristic p is assembled from a finite collection of isoclinic p-divisible groups over κ. More precisely, the Tate-linear structure of T produces

- a finite subset $\mathrm{slope}(T) \subseteq (0,1] \cap \mathbb{Q}$, close the slopes of T,
- a finite family of Tate-linear formal varieties $T_{[a,b]}$ indexed by pairs $a, b \in \mathrm{slope}(T)$ with $a \leq b$,
- a structure of an isoclinic p-divisible group over κ on $T_{[b,b]}$ with slope b, for every $b \in \mathrm{slope}(T)$,

- closed embeddings $j_{[a_1,b],[a_2,b]} : T_{[a_2,b]} \to T_{[a_1,b]}$ for triples $a_1, a_2, b \in \mathrm{slope}(T)$ such that $a_1 \le a_2 \le b$,
- an action of the p-divisible group $T_{[b,b]}$ on $T_{[a,b]}$ compatible with the embedding $j_{[a,b],[b,b]}$ of $T_{[b,b]}$ into $T_{[a,b]}$, for each pair (a,b) in $\mathrm{slope}(T)$ with $a \le b$,
- formally smooth morphisms $\pi_{[a,b_1],[a,b_2]} : T_{[a,b_2]} \to T_{[a,b_1]}$ for triples $a, b_1, b_2 \in \mathrm{slope}(T)$ such that $a \le b_1 \le b_2$.

such that the action of $T_{[b_2,b_2]}$ on $T_{[a,b_2]}$ makes $\pi_{[a,b_1],[a,b_2]}$ a $T_{[b_2,b_2]}$-torsor over $T_{[a,b_1]}$ whenever $(b_1, b_2) \cap \mathrm{slope}(T) = \emptyset$. Thus T is assembled from its isoclinic building blocks $T_{[a,a]}$'s through a family of fibrations with p-divisible formal groups as fibers.

The second part of the title refers to following property of Tate-linear formal varieties.

> *Let T be a Tate-linear formal variety over a perfect field of characteristic p. Let G be a compact p-adic Lie group, and let $\rho : G \to \mathrm{Aut}(T)$ be an action of G on T which respects the Tate-linear structure. Assume that the action ρ of G on T is* strongly nontrivial. *Then every reduced irreducible closed formal subscheme of T stable under the action of G is a Tate-linear formal subvariety.*

It is the main theorem of this article, c.f. 5.1. Two passages in the above statement need to be clarified.

(a) Since the action ρ respects the Tate-linear structure on T, G operates on each isoclinic building block $T_{[a,a]}$ of T. The assumption that ρ is strongly nontrivial means that for every $a \in \mathrm{slope}(T)$, none of the Jordan–Hölder component of the action $d\rho$ of the Lie algebra $\mathrm{Lie}(G)$ of G on the Dieudonné module $\mathbb{D}_*(T_{[aa]}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the trivial representation of $\mathrm{Lie}(G)$.

(b) A Tate-linear formal subvariety is a closed embedding $j : T' \to T$ of a Tate-linear formal variety T' into T which respects the Tate-linear structures on T' and T.

The above assertion appears to be unreasonably strong at first sight, because there isn't any obvious reason why Tate-linear structures are so rigid that a very weak condition on ρ would trigger such a robust response. All we can say is that the orbital rigidity phenomenon is inextricably intertwined with Tate-linear structures. In some sense the class of Tate-linear formal varieties is almost determined by this rigidity property if the examples in 1.2.2 are included; c.f. 1.4 (i)–(iv).

(1.2) What are Tate-linear formal varieties

(1.2.1) Let κ be a field of characteristic p. A *Tate-linear structure* on a noetherian formal scheme T over κ is by definition an isomorphism $\alpha : T \cong \mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ as sheaves on the category $\mathfrak{Sch}_\kappa$ of all κ-schemes with respect to the fpqc topology, where

- $\mathbf{N}$ is a *Tate unipotent group* over κ, and
- $\mathbf{N}_{\mathbb{Q}}$ is the Mal'cev completion of $\mathbf{N}$.

Such a pair (T, α) is called a Tate-linear formal variety over κ, and T is necessarily formally smooth over κ, i.e. it is isomorphic to the formal spectrum of a formal power series ring $\kappa[[u_1, \ldots, u_m]]$ over κ.

We need to explain two critical ingredients in the above definition, Tate unipotent groups and the Mal'cev completion of a torsion free nilpotent group.

(a) A *Tate unipotent group* over κ is a sheaf of groups on the big fpqc site $\mathfrak{Sch}_\kappa$ together with a decreasing filtration $\mathrm{Fil}_{\mathrm{sl}}^{\bullet}\mathbf{N}$ of normal subgroups indexed by $(0, 1]$, with the following properties.

  - There exists a finite subset $\mathrm{slope}(\mathbf{N}) \subseteq (0, 1] \cap \mathbb{Q}$, called the slopes of $\mathbf{N}$, such that
  $$\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} := \mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>s}_{\mathrm{sl}}\mathbf{N}$$
  is non-trivial if and only if $s \in \mathrm{slope}(\mathbf{N})$.
  - $[\mathrm{Fil}^{s_1}_{\mathrm{sl}}\mathbf{N}, \mathrm{Fil}^{s_2}_{\mathrm{sl}}\mathbf{N}]_{\mathrm{grp}} \subseteq \mathrm{Fil}^{s_1+s_2}_{\mathrm{sl}}\mathbf{N}$ for all $s_1, s_2 \in (0, 1]$, where $[\ ,\ ]_{\mathrm{grp}}$ denotes the group commutator $(x, y) \mapsto x^{-1}y^{-1}xy$.
  - For each $s \in \mathrm{slope}(\mathbf{N})$, there exists an isoclinic p-divisible group Y_s over κ and an isomorphism $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} \cong \varprojlim_n Y_s[p^n]$ as fpqc sheaves on $\mathfrak{Sch}_\kappa$. The transition maps $Y_s[p^{n+1}] \to Y_s[p^n]$ in the projective limit above are induced by $[p]_{Y_x}$, multiplication by p on Y_s.

  See definition 3.2.4. Note that the sheaf $\mathbf{N}$ of groups is unipotent of class not exceeding $\mathrm{card}(\mathrm{slope}(\mathbf{N}))$, and is torsion-free and uniquely ℓ-divisible for every prime number $\ell \neq p$.

(b) $\mathbf{N}_{\mathbb{Q}}$ is the Mal'cev completion of the sheaf of torsion-free unipotent groups $\mathbf{N}$. Recall that the Mal'cev completion $N_{\mathbb{Q}}$ of a torsion free unipotent group N is characterized by the following property: $N_{\mathbb{Q}}$ is a uniquely divisible unipotent group containing N such that for every element $x \in N_{\mathbb{Q}}$, there exists a positive integer $n > 0$ such that $x^n \in N$. See §2 for a review and references.

Remark. To illustrate why it is essential to consider sheaves for the fpqc topology on $\mathrm{Spec}(\kappa)$, we only need to examine the example 1.2.2 (i) of Tate-linear formal varieties: a p-divisible formal group X over κ, whose associated Tate unipotent group is the p-adic Tate module $\mathbf{T}_p(X) = \varprojlim_n X[p^n]$ of X, and the Mal'cev completion of $\mathbf{T}_p(X)$ is $\mathbf{T}_p(X)_{\mathbb{Q}} = \mathbf{T}_p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\mathbf{T}_p(X)(S) = (0) = \mathbf{T}_p(X)_{\mathbb{Q}}(S)$ for every noetherian local κ-algebra S. However $\mathbf{T}_p(X)_{\mathbb{Q}}/\mathbf{T}_p(X)$ has many points with values in artinian local κ-algebras R; an element of $(\mathbf{T}_p(X)_{\mathbb{Q}}/\mathbf{T}_p(X))(R)$ corresponds to an element of $(\mathbf{T}_p(X)_{\mathbb{Q}}/\mathbf{T}_p(X))(S)$ in a faithfully flat commutative R-algebra S plus a descent datum for S/R.

It is fair to say that the above definition of Tate-linear formal varieties feels dry and stale. We will try to soften it with some examples and comments.

(1.2.2) Examples of Tate-linear formal varieties

(i) Every p-divisible group X over κ is a Tate-linear formal variety over κ. The Tate unipotent group here is the p-adic Tate module of X, defines as the projective limit $\mathbf{T}_p(X) := \varprojlim_n X[p^n]$ in the category of fpqc sheaves on $\mathfrak{Sch}_\kappa$.

(ii) Let X, Y, Z be p-divisible groups over κ. Every biextension $E \to X \times Y$ of (X, Y) by Z is a Tate-linear formal variety over κ. The Tate unipotent group $\mathbf{N}$ here is a central extension of $\mathbf{X} \times \mathbf{Y}$ by $\mathbf{Z}$, such that $\mathbf{N}$ is split over $\mathbf{X}$ and also over $\mathbf{Y}$, so that $\mathbf{N}$ is a semi-direct product of $\mathbf{Z} \times \mathbf{X}$ with $\mathbf{Y}$ and also a semi-direct product of $\mathbf{Z} \times \mathbf{Y}$ with $\mathbf{X}$. Here $\mathbf{X} = \varprojlim_n X[p^n]$, $\mathbf{Y} = \varprojlim_n Y[p^n]$ and $\mathbf{Z} = \varprojlim_n Z[p^n]$. The Weil pairings $\beta_n : X[p^n] \times Y[p^n] \to Z[p^n]$, $n \geq 1$ of the biextension E determines the group commutator on $\mathbf{N}_{\mathbb{Q}}$, and vice versa. See 4.3 for further explanation, and [21] for the notion of biextensions.

(iii) Let X be a p-divisible group over κ, and let $\mathcal{D}ef\,(X)_{\rm sus}$ be the closed formal subscheme of the equi-characteristic deformation space of X representing all sustained deformations of X. Then $\mathcal{D}ef\,(X)_{\rm sus}$ is a Tate-linear formal variety over κ. The associated Tate unipotent group here is

$$\mathbf{T}_p\big(\mathscr{A}ut^{\rm st}(X)^0\big) := \varprojlim \mathscr{A}ut^{\rm st}(X)^0,$$

the projective limit of the neutral components of the projective system of stabilized Aut group schemes of X. If $s_1 < \cdots < s_r$ are the distinct slopes of X, then $\mathrm{slope}\big(\mathbf{T}_p\big(\mathscr{A}ut^{\rm st}(X)^0\big)\big)$ consists of the $r(r-1)/2$ numbers $s_j - s_i$, $i < j$. Moreover $\mathbf{T}_p\big(\mathscr{A}ut^{\rm st}(X)^0\big)$ is nilpotent of class at most $r-1$. See [7, Ch. 5 §4] for the definition of the projective system $\mathscr{A}ut^{\rm st}(X)^0$, and [7, Ch. 6 §5] for the definition of $\mathcal{D}ef\,(X)_{\rm sus}$.

(iv) Let (Y, λ) be a polarized p-divisible group, and let $\mathcal{D}ef\,(Y, \lambda)_{\rm sus}$ be the closed formal subscheme of the equi-characteristic deformation space of Y representing all sustained deformations of (Y, λ). Then $\mathcal{D}ef\,(Y, \lambda)_{\rm sus}$ is a Tate-linear formal variety over κ attached to the Tate unipotent group

$$\mathbf{T}_p\big(\mathscr{A}ut^{\rm st}(Y, \lambda)^0\big) := \varprojlim \mathscr{A}ut^{\rm st}(Y, \lambda)^0.$$

See [7, Ch. 5 §4] for the definition of $\mathscr{A}ut^{\rm st}(Y, \lambda)^0$, and [7, Ch. 6 §6] for the definition of $\mathcal{D}ef\,(Y, \lambda)_{\rm sus}$.

It follows that for every central leaf $\mathcal{C}$ in a Siegel moduli scheme $\mathscr{A}_{g,d,n}$ which classifies g-dimensional polarized abelian varieties of degree d plus principle level-n structures over an algebraically closed field k of characteristic p, $n \geq 3$ and every k-point x_0 in $\mathcal{C}$, the formal completion $\mathcal{C}^{/x_0}$ is a Tate-linear formal variety over k. We recall that the central leaf $\mathcal{C}(x_0)$ passing through a k-point x_0 is the locally closed smooth subvariety of $\mathcal{C}$ such that $\mathcal{C}(x_0)(k)$ consists of isomorphism classes $[(A,\mu,\zeta)]$ of g-dimensional polarized abelian varieties (A,μ) over k with level-n structure ζ such that the polarized p-divisible group $(A[p^\infty],\mu[p^\infty])$ is isomorphic to (Y,λ); c.f. [22]. See also [8], [6] and [7, Ch. 6] for a scheme-theoretic definition of central leaves via the notion of sustained polarized p-divisible groups.

(1.2.3) Remark. (a) These examples show that for the Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$ attached to a Tate-unipotent group $\mathbf{N}$ over κ, one may regard $\mathbf{N}$ as "the Tate $\mathbb{Z}_p$-module of $\mathrm{TL}(\mathbf{N})$", while the Mal'cev completion $\mathbf{N}_{\mathbb{Q}_p}$ may be thought of as "the Tate $\mathbb{Q}_p$-module" of $\mathrm{TL}(\mathbf{N})$".

(b) The fundamental group Γ of a compact nilmanifold M is a finitely generated nilpotent group. The Mal'cev correspondence says that there exists a nilpotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$ of finite dimension over $\mathbb{Q}$ and an isomorphism between the Mal'cev completion $\Gamma_{\mathbb{Q}}$ of Γ and the $\mathbb{Q}$-points of the nilpotent algebraic group N over $\mathbb{Q}$ with Lie algebra $\mathfrak{n}$. The injective homomorphisms $\Gamma \hookrightarrow \Gamma_{\mathbb{Q}} \cong N(\mathbb{Q}) \hookrightarrow N(\mathbb{R})$ identifies Γ as a co-compact discrete subgroup of the nilpotent Lie group $N(\mathbb{R})$, and M is isomorphic to $N(\mathbb{R})/\Gamma$. This is the analogy between Tate-linear formal varieties and compact nilmanifolds mentioned in 1.2.1.

(c) Many Tate-linear formal varieties, including sustained deformation spaces $\mathcal{D}ef\,(X)_{\rm sus}$ in (ii), are *cascades* whose group-consituents are p-divisible formal groups, in the sense of [20]. It would be interesting to determine which ones among such cascades come from Tate-linear formal varieties.

(1.3) How to prove orbital rigidity of Tate-linear formal varieties

(1.3.1) The orbital rigidity for p-divisible formal groups, first proved for formal tori in [4, §6], and extended to all p-divisible formal groups in [5], relies on the method of *hypocotyl elongation.* This is a result in what Abhyankar called "high school algebra". See [5, Prop. 3.1] for a precise statement, and also 6.7.2 for an equivalent formulation. In a simplified version, this method allows one to deduced a desired *equality* of the form

$$f(g_1(\mathbf{x}),\dots,g_a(\mathbf{x}),h_1(\mathbf{y}),\dots,h_b(\mathbf{y}))=0$$

in a power series ring $\kappa[[x_1,\dots,x_m,y_1,\dots,y_m]]$ over a field κ of characteristic p, where $f(u_1,\dots,u_a,v_1,\dots,v_b)$ is a formal power series over a field κ of characteristic p in two sets of variables $\mathbf{x}$ and $\mathbf{y}$, $g_1(\mathbf{x}),\dots,g_a(\mathbf{x})\in\kappa[[x_1,\dots,x_m]]$, $h_1(\mathbf{y}),\dots,h_b(\mathbf{y})\in\kappa[[y_1,\dots,y_m]]$, from an infinite family of congruences

$$f(g_1(\mathbf{x}),\dots,\mathbf{g}_a(x),h_1(\mathbf{x})^{p^{rn}}),\dots,h_b(\mathbf{x})^{p^{rn}})\equiv 0 \bmod (x_1,\dots,x_m)^{d_n}\quad \forall\, n\geq n_0$$

in a single set of variables $x_1,\dots,x_m$, provided that $\lim_{n\to\infty}\frac{p^{rn}}{d_n}=0$.

In the proof of orbital rigidity of p-divisible formal groups, we are given a p-adic Lie group G acting strongly nontrivially on a p-divisible formal group X, and a reduced irreducible closed formal subscheme W of X stable under G. We may assume that the p-divisible group splits into a product $Y \times Z$, where Z is isoclinic and all slopes of Y are strictly smaller than the slope of Z. We want to show that W is stable under the translation action by $\mathrm{pr}_Z(W)$, where pr_Z is the projection from X to Z. In other words, given any formal function ϕ on X which vanishes on W, we want to show that the pull-back of ϕ to $W \times W$ under the map $(w_1, w_2) \mapsto w_1 + \mathrm{pr}_Z(w_2)$ is the 0-function on $W \times W$. We apply hypocotyl prolongation, where

- $a = \dim(X)$, $b = \dim(Z)$,
- f is the pull-back of ϕ to $X \times X$ under the map $(x_1, x_2) \mapsto x_1 + \mathrm{pr}_Z(x_2)$,
- the functions $(g_1, \ldots, g_a)$ are the coordinates of the inclusion map $W \hookrightarrow X$,
- the functions $(h_1, \ldots h_b)$ are the coordinates of the composition of $W \hookrightarrow Z$ with the endomorphism of Z attached to the action of an element $v \in \mathrm{Lie}(G)$ on Z, after multiplying v by a power of p if necessary, and
- the required infinite family of congruence relations is the result of an easy first order approximation to the "one-parameter subgroup" $\exp_G(p^{tn}v)$, with t/r equal to the slope of Z.

For a given pull-back f as above, we get many equalities as the outputs of hypocotyl elongation, one for each one-dimensional subspace of $\mathrm{Lie}(G)$. The condition that G acts strongly non-trivially implies that the identities from varying v's imply that f vanishes on $W \times W$ for every formal function ϕ on X which vanishes on X, as desired.

(1.3.2) Naturally one tries to use the same method tackle orbital rigidity for Tate-linear formal varieties. But one encounters a serious difficulty, which shows up already in the first nontrivial case of a biextension E of (X, Y) by Z, where X, Y, Z are isoclinic p-divisible groups over κ such that $\mathrm{slope}(X) + \mathrm{slope}(Y) = \mathrm{slope}(Z) = \frac{t}{r}$, $t, r \in \mathbb{N}_{>0}$. Given a "one-parameter subgroup" $\exp_G(p^{tm}v)$ of G, in general there does *not* exist a morphism $h : E \to Z$ such that translation by $[p^{tn}]_Z \circ h$ is a first-order approximation to the action of $\exp_G(p^{tn}v)$ in a large infinitesimal neighborhood U_n of the base point of E. This difficulty led to the uncertainly as to whether orbital rigidity holds only for p-divisible formal groups, or it is a general phenomenon for all Tate-linear formal varieties. In the latter case, there is also the related question on a good notion of Tate-linear formal varieties and their Tate-linear formal subvarieties.

This unfortunate state of affairs lasted many years, until we realized that the compatibility relation between the actions of $\exp_G(p^{tn}v)$ and $\exp_G(p^{t(n+1)}v)$ on E shows the action of the whole one-parameter subgroup $\exp_G(p^{tn}v)$ on E can be approximated by a "generalized formal morphism" from E to Z, whose coordinate functions lie in a non-noetherian complete local ring S of "generalized formal functions" on E, sandwiched between the affine coordinate ring R_E of E and the completion $(R_E^{\mathrm{perf}})^\wedge$ of the perfection of R_E. This ring S

is an example of *tempered perfections* of R_E. The collection of tempered perfections of R_E is a filtered family of subrings of $(R_E^{\mathrm{perf}})^\wedge$. Elements of $(R_E^{\mathrm{perf}})^\wedge$ which lies in some tempered perfection of R_E are called *tempered virtual functions* on the formal scheme E. It turns out that the method of hypocotyl elongation also holds for tempered virtual functions. After this critical upgrade of the main technical tool, the strategy for orbital rigidity of p-divisible formal groups also works for biextensions. See [5] for a first draft of this proof a few years ago, and [7, Ch. 10] for an updated version. We refer to [7, Ch. 10 §7] for more information about tempered perfections, and [7, Ch. 10 §5] for hypocotyl elongation in tempered perfections. See also 6.5 (respectively 6.7) for a review of tempered perfections (respectively hypocotyl elongation in tempered perfections).

(1.4) After the proof of orbital rigidity of biextensions of p-divisible formal groups, there was no doubt that the same method would establish orbital rigidity of other equivariant formal varieties, such as sustained deformation spaces $\mathcal{D}ef\,(X)_{\mathrm{sus}}$ of (certain classes of) p-divisible formal groups X over fields of characteristic p. The question was:

> What would be a *good* notion of Tate-linear formal varieties which includes all examples in 1.2.2 and has the orbital rigidity property?

A candidate was proposed in [6], based on the notion of *terraced Tate unipotent groups* in 3.1. Tao Song proved an orbital rigidity result for sustained deformation spaces of p-divisible groups with at most 4 slopes in his 2022 Penn thesis [29]. In [10] D'Addezio and van Hoften defined a formal schemes over $\overline{\mathbb{F}_p}$ with extra structures, corresponding to Tate-linear formal varieties for which p is strictly bigger than the nilpotency class of the associated Tate unipotent groups, and proved orbital rigidity using the method hypocotyl elongation in tempered perfection.

The definition of Tate-linear formal varieties and subvarieties in this article is more flexible than the one in [6]. In addition, the resulting class of Tate-linear formal varieties satisfy properties (i)–(iv) below.

(i) Sustained deformation spaces of p-divisible groups over κ are Tate-linear formal varieties.

(ii) Orbital rigidity holds for Tate-linear formal varieties.

(iii) Every Tate-linear formal variety T can be embedded, up to isogeny, in a sustained deformation space $\mathcal{D}ef\,(X)_{\mathrm{sus}}$ as a Tate-linear formal subvariety.

(iv) For every Tate-linear formal variety T and every Tate-linear formal subvariety T' of T, there exists an action ρ_{Euler} of an open subgroup of $\mathbb{Z}_p$ on the pair (T, T') which respects the Tate-linear structures.

The statement (iii) follows from 3.2.24, a consequence of an analog of Ado's theorem. The statement (iv) follows from 3.4.6.

(1.5) The rest of this paper organized as follows. In §2 we review the Mal'cev completion of torsion free nilpotent groups, the Mal'cev correspondence between nilpotent uniquely divisible groups and nilpotent Lie $\mathbb{Q}$-algebras, the Baker-Campbell-Hausdorff formula and its inversion. The basics of Tate unipotent groups, their Mal'cev completions, and the associated Tate unipotent Lie algebras are given in §3. The definition and elementary properties of Tate-linear formal varieties are presented in §4. Orbital rigidity of Tate-linear formal varieties are treated in §§5–6. The reduction steps are explained in §5, and the proof of the key theorem 5.2 is given in §6. Here is a more detailed guide.

(i) The scheme of the proof of 5.2 via hypocotyl elongation in tempered perfections is explained in 6.6. In particular, given any element $v \in \mathrm{Lie}(G)$, a tempered virtual morphism $\tilde{\mathfrak{z}}[v] : \mathrm{TL}(\mathbf{N}) \dashrightarrow Z$ constructed in 6.8, where Z is an isoclinic p-divisible group acting on $\mathrm{TL}(\mathbf{N})$ such that $\mathrm{TL}(\mathbf{N})$ is a Z-torsor and $\mathrm{TL}(\mathbf{N})/Z$ is a Tate-linear formal variety all of whose slopes are strictly smaller than the slope of Z. This virtual morphism $\tilde{\mathfrak{z}}[v] : \mathrm{TL}(\mathbf{N}) \dashrightarrow Z$ can be thought of as the first order approximation for the action on $\mathrm{TL}(\mathbf{N})$ of the one-parameter subgroup $\exp_G(p^n v)$ for all sufficiently divisible positive integers n; see 6.8.1 (a). It is fed into the hypocotyl elongation machine to establish the desired equalities.

(ii) The tempered virtual morphism $\tilde{\mathfrak{z}}[v]$ is constructed out of an infinite family of maps $\mathfrak{z}_n[v]\big|_{U_n}$ from infinitesimal neighborhoods U_n of the base points of $\mathrm{TL}(\mathbf{N})$ to Z. They satisfy a compatibility relation 6.2.2. The map is the restriction to U_n of a morphism $\mathfrak{z}_n[v]$ defined in 6.2. A formula for the morphism $\mathfrak{z}_n[v]\big|_{U_n}$ is obtained in 6.3.6 using the Baker-Campbell-Hausdorff formula, which implies the required compatibility property 6.2.2.

(iii) The equalities obtained from the hypocotyl elongation machine indicated in (i) means that in the statement of 5.2, the given reduced irreducible formal subscheme of $\mathrm{TL}(\mathbf{N})$ stable under the action of G is stable under translation by the schematic image of the tempered virtual map $\tilde{\mathfrak{z}}[v]$, for all $v \in \mathrm{Lie}(G)$. Statements 5.2 (a)–(b) follows from this and the assumption that G operates strongly nontrivially on $\mathrm{TL}(\mathbf{N})$. The inseparability assertion 5.2 (c) is deduced from the fact 6.8.1 (b) that $\tilde{\mathfrak{z}}[v]$ respects the Z-torsor structure of $\mathrm{TL}(\mathbf{N})$.

Acknowledgements. We owe an enormous intellectual debts to Mumford's paper [21]. Biextensions of p-divisible groups, introduced in [21], provide an ideal testing ground for determining whether orbital rigidity holds for Tate-linear formal varieties which are not p-divisible formal groups. In addition, the explicit construction of the Weil pairings as structural cocycles of biextensions in [21, §5] kept us from going astray during the conception of tempered virtual formal morphisms $E \dashrightarrow Z$ attached to one-parameter groups of automorphisms of a biextension E of (X, Y) by Z. The first author would also like to thank the support of a Simons Fellowship 561644 and a Simons Foundation collaboration grant 701067 during his research on the two related topics of this article.

§2. Localization of nilpotent groups and Mal'cev completion

In this section we summarize some basic properties of nilpotent groups and their localizations. Proofs can be found in the references cited in the next paragraph.

The Mal'cev completion of finitely generated torsion free nilpotent group was first constructed in [17], which relied heavily on the papers [18] [19] on fundamental groups of nilmanifolds, i.e. homogeneous spaces for connected finite dimensional nilpotent Lie groups. See also Raghunathan [26, Ch. 2] for an account of Mal'cev's original approach. The Mal'cev completion can be regarded as a special case of the theory of localization of (locally) nilpotent groups, and was reworked by Lazard [16], Quillen [25] and others; c.f. [2], [15], [14]. For general background on nilpotent groups, we refer to [11], [2], [30]; see also [9] for a textbook treatment of nilpotent group accessible to advanced undergraduate students.

(2.1) For any group G, denote by

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_i(G) \supseteq \gamma_{i+1}(G) \supseteq \cdots$$

the descending central series (or lower central series) of G, defined inductively by $\gamma_{i+1} = [G, \gamma_i(G)]_{\mathrm{grp}}$ for all $i \geq 1$. Here $[G, \gamma_i(G)]_{\mathrm{grp}}$ is the subgroup of G generated by all commutators $[a, b]_{\mathrm{grp}} := a^{-1}b^{-1}ab$ with $a \in G$ and $b \in \gamma_i(G)$. Let

$$\zeta_1(G) \subseteq \zeta_2(G) \subseteq \cdots \subseteq \zeta_i(G) \subseteq \zeta_{i+1}(G) \subseteq \cdots$$

be the ascending central series of G, defined inductively by $\zeta_1(G) = \mathrm{Z}(G)$, the center of G, and $\zeta_{i+1}(G)/\zeta_i(G) = \mathrm{Z}(G/\zeta_i(G))$ for all $i \geq 1$. We say that G is nilpotent of class at most c, where c is a positive integer, if $\gamma_{c+1}(G) = \{1\}$, or equivalently if $\zeta_c(G) = G$.

(2.2) (Hall–Petresco formula) For every free group $F_m(x_1, \ldots, x_m)$ with free generators $x_1, \ldots, x_m$, define words $\tau_{n,m}(\underline{x}) = \tau_{n,m}(x_1, \ldots, x_m) \in F_m(x_1, \ldots, x_m)$ recursively for $n = 1, 2, \ldots$ by

$$x_1^n \cdots x_m^n = \tau_{1,m}(\underline{x})^{\binom{n}{1}}\, \tau_{2,m}(\underline{x})^{\binom{n}{2}} \cdots \tau_{n,m}(\underline{x})^{\binom{n}{m}}.$$

(2.2.1) Lemma. *The Hall–Petresco word $\tau_{n,m}$ lies in the n-th term of the descending central series of the free group $F_m(x_1, \ldots, x_m)$, i.e.*

$$\tau_{n,m}(x_1, \ldots, x_m) \in \gamma_n(F_m(x_1, \ldots, x_m)) \quad \forall\, n \in \mathbb{N}_{\geq 1}.$$

Moreover for any integer $r \geq 1$, let

$$h_{m,r} : F_{m+r}(y_1, \ldots, y_{m+r}) \to F_m(x_1, \ldots, x_m)$$

be the group homomorphism between free groups such that $h_{m,r}(y_i) = x_i$ for $i = 1, \ldots, m$ and $h_{m,r}(y_{r+j}) = 1$ for $j = 1, \ldots, r$. Then

$$h_{m,r}(\tau_{n,m+r}(y_1, \ldots, y_{m+r})) = \tau_{n,m}(x_1, \ldots, x_m) \quad \forall\, n \in \mathbb{N}_{\geq 1}.$$

An application of the Hall–Petresco formula is the following result due to Blackburn.

(2.2.2) **Proposition.** *For every pair p, c, where p is a prime number and c is a positive integer, define an integer $f(p,c)$ recursively (in c) by $f(p,1)=0$ and*

$$f(p,c) = f(p,c-1) + \max\{i \in \mathbb{N} \mid p^i \le c\} \quad \forall\, c \ge 2.$$

Then for every nilpotent group N of class at most c and every integer $n \ge f(p,c)$, every finite product of p^n-th power of elements of N is the $p^{n-f(p,c)}$-th power of some element of N.

The proof is by induction on the nilpotency class c of N, using the general fact that the subgroup generated by the derived group of N and a single element of N has nilpotency class at most $c-1$.

(2.3) Localization of nilpotent groups

(2.3.1) Let Φ be the set of all prime numbers. Let P be a subset of Φ, and let $P' := \Phi \smallsetminus P$ be the complement of P in Φ. For any non-zero integer n, we say that n is a P-number, written symbolically as $n|P^\infty$, if every prime divisor of n is in P. We say that n is prime to P, or $\gcd(n,P)=1$, if n is a P'-number, i.e. if no prime divisor of n is in P.

(2.3.2) A nilpotent group N is said to be *torsion free* if the map $x \mapsto x^n$ from N to N is injective for every non-zero integer n, or equivalent if the only element of N of finite order is the unity element 1. More generally, for any subset P of the set Φ consisting of all prime numbers, we say that N is *P-torsion free* if the self map $x \mapsto x^n$ of N is injective for every non-zero P-number n. This condition is equivalent to the apparently weaker condition that if $x \in N$ and $x^n = 1$ for some non-zero integer n, then the order of x is prime to P.

A nilpotent group N is said to be *divisible* if the map $x \mapsto x^n$ from N to N is surjective for all non-zero integer n. More generally, N is said to be *P-divisible* for a subset P of Φ if the self map $x \mapsto x^n$ is surjective for every non-zero integer P-number n.

A nilpotent group N which is P-torsion free and P-divisible is also said to be uniquely P-divisible. When $P = \Phi$, we say that N is uniquely divisible.

(2.3.3) **Lemma.** *Let N be a nilpotent group and let $H \le N$ be a subgroup of N. Let $Q \subseteq \Phi$ be a set of prime numbers.*

(a) *The subset*

$$I_Q(H,N) := \{x \in N \mid x^n \in H \text{ for some } Q\text{-number } n\}$$

is a subgroup of N.

(b) *The subgroup $I_Q(H,N)$ of N is Q-*isolated*, i.e. if $y \in N$ and there exists a Q-number m such that $y^m \in I_Q(H,N)$, then $y \in I_Q(H,N)$.*

(c) *If H is a normal subgroup of N, then $I_Q(H,N)$ is also a normal subgroup of N.*

In the special case when $H=\{1\}$ is the trivial subgroup of N, the subgroup $I_{Q,\{1\}}$ consists of all Q-torsion elements of N, and the quotient group $N/I_{Q,\{1\}}$ is Q-torsion free.

(2.3.4) Let $P \subseteq \Phi$ be a set of prime numbers. A group G is said to be *P-local* if the self map $x \mapsto x^n$ of G is bijective for every non-zero integer n with $\gcd(n,P)=1$. Clearly a nilpotent group N is $\emptyset$-local if and only if N is torsion free and divisible.

A group homomorphism $\epsilon_{G,P} : G \to G_P$ is said to be *P-localizing* if G_P is p-local and the map $\epsilon^*_{G,P} : \mathrm{Hom}(G_P,H) \to \mathrm{Hom}(G,H)$ induced by $\epsilon_{G,P}$ is bijective for every P-local group H. This universal property characterizes $\epsilon_{G,P}$ up to unique isomorphism, if such a P-localizing homomorphism $\epsilon_{G,P}$ exists.

The basic theorem on localizations of nilpotent groups is the following.

(2.3.5) Proposition. *For every set $P \subseteq \Phi$ of prime numbers and every nilpotent group N, there exists a P-localizing homomorphism $\epsilon_{N,P} : N \to N_P$, where N_P is a nilpotent p-local group. Moreover N_P is nilpotent of class at most c if N is.*

The assignment $N \rightsquigarrow N_P$ can be regarded as a functor $\mathcal{L}_P$ from the category $\mathfrak{N}$ of nilpotent groups to the category $\mathfrak{N}_P$ of P-local nilpotent groups, which induces functors $\epsilon_{P,c}$ from the category $\mathfrak{N}_c$ of nilpotent groups of class at most c to the category $\mathfrak{N}_{P,c}$ of P-local nilpotent groups of class at most c.

(2.3.6) Proposition. *Let N be a nilpotent group. Let $P \subseteq \Phi$ be a set of prime numbers, and let $P' := \Phi \smallsetminus P$.*

(a) *The kernel $\mathrm{Ker}(\epsilon_{N,P})$ of the P-localizing homomorphism $\epsilon_{N,P} : N \to N_P$ is equal to the subgroup $I_{P'}(\{1\},N)$, consisting of all elements $x \in N$ such that there exists a non-zero integer n prime to P with $x^n = 1$.*

(b) *The localization functor $\mathrm{Loc}_P : \mathfrak{N} \to \mathfrak{N}_P$ is exact. More precisely if $H \trianglelefteq N$ is a normal subgroup of a nilpotent group N, then we have a commutative diagram*

$$\begin{tikzcd}
\{1\} \arrow[r] & H \arrow[r, "j"] \arrow[d, "\epsilon_{H,P}"'] & N \arrow[r, "q"] \arrow[d, "\epsilon_{N,P}"'] & (N/H) \arrow[r] \arrow[d, "\epsilon_{N/H,P}"'] & \{1\} \\
\{1\} \arrow[r] & H_P \arrow[r, "j_P"'] & N_P \arrow[r, "q_P"'] & (N/H)_P \arrow[r] & \{1\}
\end{tikzcd}$$

with exact rows, i.e. j_P identifies H_P as a normal subgroup of N_P, and q_P is an epimorphism.

(c) *Let $\alpha : N \to \bar{N}$ be a homomorphism between nilpotent groups.*

- *The localization $\alpha_P : N_P \to \bar{N}_P$ of α is injective if and only if* $\mathrm{Ker}(\alpha) \subseteq I_{P'}(\{1\}, N)$*, i.e. the order of every element of the kernel of α is finite and prime to P.*
- *The homomorphism $\alpha_P : N_P \to \bar{N}_P$ is surjective if and only if $I_{P'}(\alpha(N), \bar{N}) = \bar{N}$. In other words for every element $y \in \bar{N}$, there exists an element $x \in N$ and an integer n prime to P such that $y_n = \alpha(x)$.*

(2.3.7) By definition the Mal'cev completion $\mathrm{MC}(N)$ of a nilpotent group N is the localization of N with respect the empty subset $\emptyset$ of prime numbers. The universal map $\epsilon_{N,\emptyset} : N \to N_\emptyset = \mathrm{MC}(N)$ identifies $\mathrm{MC}(N)$ is characterized up to a unique isomorphism by the following properties:

(a) $\mathrm{Ker}(\epsilon_{N,\emptyset}) = N_{\mathrm{tor}}$, the subgroup of N consisting of all elements of N of finite order.

(b) $\mathrm{MC}(N)$ is a uniquely divisible nilpotent group.

(c) For every $y \in \mathrm{MC}(N)$, there exists a non-zero integer n and an element $x \in N$ such that $y^n = \epsilon_{N,\emptyset}(x)$.

(2.4) We will review the explicit construction of the Mal'cev completion and the Mal'cev correspondence, following [25, Appendix A3]. The precise statement of the Mal'cev correspondence is given in 2.4.6 and 2.4.7.

(2.4.1) For any group G, denote by $\mathbb{Q}[G]$ the group algebra of G over $\mathbb{Q}$, consisting of all finite formal $\mathbb{Q}$-linear combinations of elements of G. Let I_G be the augmentation ideal of $\mathbb{Q}[G]$, equal to the $\mathbb{Q}$-linear span of all elements of the form $[y] - 1$, $y \in G$. The completed group algebra $\mathbb{Q}[[G]]$ is the formal completion of $\mathbb{Q}[G]$ with respect to the I_G-adic filtration $\left(I_G^n\right)_{n\in\mathbb{N}}$ of $\mathbb{Q}[G]$. Let $I_G^\wedge$ be the closure of I_G in $\mathbb{Q}[[G]]$.

The subset $1 + I_G^\wedge$ of $\mathbb{Q}[[G]]$ is a subgroup of the group $\mathbb{Q}[[G]]^\times$ of invertible elements of $\mathbb{Q}[[G]]$. This group is uniquely divisible, or $\mathbb{Q}$-powered: for every $x \in I_{\mathbb{Q}[[G]]}^\wedge$ and every rational number $a \in \mathbb{Q}$, define $(1+x)^a \in 1 + I_{\mathbb{Q}[[G]]}^\wedge$ by

$$(1+x)^a := 1 + \sum_{m\geq 1} \binom{a}{m} x^m, \quad \binom{a}{m} = \frac{a(a-1)\cdots(a-m+1)}{m!}.$$

So for every element $y \in G$ of finite order, the image of $[y]$ in $\mathbb{Q}[[G]]^\times$ is 1.

(2.4.2) The completed group algebra $\mathbb{Q}[[G]]$ has a natural co-commutative Hopf algebra structure, whose co-multiplication map

$$\Delta_{\mathbb{Q}[[G]]} : \mathbb{Q}[[G]] \longrightarrow \mathbb{Q}[[G]] \hat{\otimes}_{\mathbb{Q}} \mathbb{Q}[[G]]$$

is the continuous $\mathbb{Q}$-algebra homomorphisms such that $\Delta_{\mathbb{Q}[[G]]}([y]) = [y] \otimes [y]$ for all $y \in G$.

(2.4.3) Define a subgroup $\mathrm{Gplk}(\mathbb{Q}[[G]]) \subseteq 1 + I_G^\wedge$ by

$$\mathrm{Gplk}(\mathbb{Q}[[G]]) := \{y \in 1 + I_G^\wedge \mid \Delta_{\mathbb{Q}[[G]]} = y \otimes y\}.$$

Elements of $\mathrm{Gplk}(\mathbb{Q}[[G]])$ are called *group-like elements* in $1 + I_G^\wedge$. Clearly $\mathrm{Gplk}(\mathbb{Q}[[G]])$ is a subgroup of $1 + I_G^\wedge$, and we have a canonical group homomorphism $j_G : G \to \mathrm{Gplk}(\mathbb{Q}[[G]])$.

(2.4.4) Define a Lie $\mathbb{Q}$-subalgebra $\mathrm{Prim}(\mathbb{Q}[[G]])$ of the Lie algebra attached to the associative algebra $\mathbb{Q}[[G]]$ by

$$\mathrm{Prim}(\mathbb{Q}[[G]]) := \{x \in I_G^\wedge \mid \Delta_{\mathbb{Q}[[G]]}(x) = x \otimes 1 + 1 \otimes x.\}$$

Elements of $\mathrm{Prim}(\mathbb{Q}[[G]])$ called *primitive elements* of the co-algebra underlying the Hopf algebra $\mathbb{Q}[[G]]$.

(2.4.5) Denote by $\mathrm{UPrim}(\mathbb{Q}[[G]])$ the universal enveloping algebra of the Lie $\mathbb{Q}$-algebra $\mathrm{Prim}(\mathbb{Q}[[G]])$. Let $\hat{\mathrm{U}}\mathrm{Prim}(\mathbb{Q}[[G]])$ be the completion of $\mathrm{UPrim}(\mathbb{Q}[[G]])$ with respect to the filtration of $\mathrm{UPrim}(\mathbb{Q}[[G]])$ by powers of the augmentation ideal $\mathrm{Prim}(\mathbb{Q}[[G]])\cdot\mathrm{UPrim}(\mathbb{Q}[[G]])$ of $\mathrm{UPrim}(\mathbb{Q}[[G]])$. We have a natural continuous homomorphism

$$h : \hat{\mathrm{U}}\mathrm{Prim}(\mathbb{Q}[[G]]) \longrightarrow \mathbb{Q}[[G]]$$

of $\mathbb{Q}$-algebras.

(2.4.6) Proposition. *Let N be a nilpotent group of class at most c, where c is a positive integer. Let $N_{\mathbb{Q}} := \mathrm{Gplk}(\mathbb{Q}[[N]])$, and let $\mathfrak{n} := \mathrm{Prim}(\mathbb{Q}[[N]])$.*

(a) *The canonical group homomorphism*

$$j_N : N \to N_{\mathbb{Q}}$$

is the localization of N with respect to the empty set $\emptyset$ of primes. In other words the group $N_{\mathbb{Q}}$ is uniquely divisible, the kernel $\mathrm{Ker}(j_N)$ of j_N is the subgroup N_{tor} consisting of all elements of N of finite order, and for every element $x \in N_{\mathbb{Q}}$, there exists a positive integer n and an element $y \in N$ such that $j_N(y) = x^n$. In particular j_N is an isomorphism if and only if N is uniquely divisible.

(b) *The natural map $N/N_{\mathrm{tor}} \to \mathbb{Q}[[N]]/(I_N^\wedge)^{c+1} \cong \mathbb{Q}[N]/I_N^{c+1}$ is injective.*

(c) *The Lie $\mathbb{Q}$-algebra $\mathfrak{n}$ is nilpotent of class at most c.*

(d) *The canonical $\mathbb{Q}$-algebra homomorphism*

$$h : \hat{\mathrm{U}}\mathfrak{n} \longrightarrow \mathbb{Q}[[N]]$$

is an isomorphism of Hopf algebras.

(e) *The exponential and the logarithm series define mutually inverse bijections*

$$\exp : \mathfrak{n} \longrightarrow N_{\mathbb{Q}}, \qquad x \mapsto \exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$$

and

$$\log : N_{\mathbb{Q}} \longrightarrow \mathfrak{n}, \quad y \mapsto \log(y) = \sum_{n\geq 1} (-1)^{n-1} \frac{(y-1)^n}{n}.$$

(f) *The exponential/logarithm pair in* (e) *gives a one-to-one correspondence between the set of all uniquely divisible subgroups of* $\mathrm{Gplk}(\mathbb{Q}[[N]])$ *and the set of all Lie* $\mathbb{Q}$*-subalgebras of* $\mathrm{Prim}(\mathbb{Q}[[N]])$.

Remark. (i) Both infinite sums in (e) converge in the complete Hopf algebra $\mathbb{Q}[[N]]$. Note that for every primitive element x in $I_N^{\wedge} \subseteq \mathbb{Q}[[N]]$, we have

$$\begin{aligned}\Delta(\exp(x)) = \sum_{n\geq 0} \frac{\Delta(x)^n}{n!} &= \exp(x\otimes 1 + 1\otimes x) \\ &= (\exp(x)\otimes 1)\cdot(1\otimes \exp(x)) = \exp(x)\otimes\exp(x)\end{aligned}$$

because $x\otimes 1$ commutes with $1\otimes x$, hence $\exp(x) \in N_{\mathbb{Q}}$. Similarly for every group-like element y in $1 + I_N^{\wedge}$, we have

$$\begin{aligned}\Delta(\log(y)) = \log(\Delta(y)) = \log(y\otimes y) &= \log((y\otimes 1)\cdot(1\otimes y)) \\ &= \log(y\otimes 1) + \log(1\otimes y) = \log(y)\otimes 1 + 1\otimes\log(y),\end{aligned}$$

hence $\log(y)$ is primitive.

(ii) We will call $\mathfrak{n} = \mathrm{Prim}(\mathbb{Q}[[N]])$ in 2.4.6 the Lie algebra of the uniquely divisible nilpotent group $N_{\mathbb{Q}}$.

Let $\mathfrak{n}$ be a nilpotent Lie $\mathbb{Q}$-algebra, let $\mathrm{U}\mathfrak{n}$ be the universal enveloping algebra of $\mathfrak{n}$ over $\mathbb{Q}$. Let $\mathrm{U}\mathfrak{n}^+$ be the augmentation ideal of $\mathrm{U}\mathfrak{n}$, i.e. $\mathrm{U}\mathfrak{n}^+ = \mathfrak{n}\cdot\mathrm{U}\mathfrak{n}$. Let $\hat{\mathrm{U}}\mathfrak{n}$ be the completion of $\mathrm{U}\mathfrak{n}$ with respect to the filtration of $\mathrm{U}\mathfrak{n}$ by powers of $\mathrm{U}\mathfrak{n}^+$, and let $\hat{\mathrm{U}}\mathfrak{n}^+$ be the augmentation ideal of $\hat{\mathrm{U}}\mathfrak{n}$, equal to the closure of $\mathrm{U}\mathfrak{n}^+$ in $\hat{\mathrm{U}}\mathfrak{n}$. The completed universal enveloping algebra $\hat{\mathrm{U}}\mathfrak{n}$ has a natural structure as a co-commutative Hopf algebra, with co-muliplication

$$\Delta_{\hat{\mathrm{U}}\mathfrak{n}} : \hat{\mathrm{U}}\mathfrak{n} \longrightarrow \hat{\mathrm{U}}\mathfrak{n}\,\hat{\otimes}_{\mathbb{Q}}\hat{\mathrm{U}}\mathfrak{n}$$

being the continuous algebra homomorphism such that $\Delta_{\mathrm{U}\mathfrak{n}^+}(x) = x\otimes 1 + 1\otimes x$ for all $x \in \mathfrak{n}$. Denote by

$$\mathrm{Gplk}(\hat{\mathrm{U}}\mathfrak{n}) := \{y \in 1 + \mathrm{U}\mathfrak{n}^+ \mid \Delta_{\hat{\mathrm{U}}\mathfrak{n}}(x) = x\otimes 1 + 1\otimes x\}$$

the subgroup of $1 + \mathrm{U}\mathfrak{n}^+ \leq \hat{\mathrm{U}}\mathfrak{n}^\times$ consisting of all group-like elements in $1 + \mathrm{U}\mathfrak{n}^+$.

(2.4.7) Proposition. *Let $\mathfrak{n}$ be a nilpotent Lie $\mathbb{Q}$-algebra of class at most c, where c is a positive integer as in the previous paragraph.*

(a) *The group* $\mathrm{Gplk}(\hat{\mathrm{U}}\mathfrak{n})$ *is uniquely divisible nilpotent group of class at most c.*

(b) *The canonical homomorphism*

$$\mathfrak{n} \longrightarrow \mathrm{U}\mathfrak{n}/(\mathrm{U}^{+}\mathfrak{n})^{c+1}$$

is injective.

(c) *The canonical continuous homomorphism*

$$\mathbb{Q}[[\mathrm{Gplk}(\hat{\mathrm{U}}\mathfrak{n})]] \longrightarrow \hat{\mathrm{U}}\mathfrak{n}$$

is an isomorphism of complete Hopf algebras.

(2.4.8) Remark. (i) Propositions 2.4.6 and 2.4.7 show that the functor from the category of all uniquely divisible nilpotent groups to and category of all nilpotent Lie $\mathbb{Q}$-algebras, which sends each uniquely divisible nilpotent group N to $\mathrm{Prim}(\mathbb{Q}[[N]])$, is an equivalence, and the functor which sends every nilpotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$ to $\mathrm{Gplk}(\hat{\mathrm{U}}\mathfrak{n})$ is an essential inverse. This is an explicit form of the Mal'cev correspondence.

(ii) Under the Mal'cev correspondence, subgroups correspond to Lie subalgebras, normal subgroups correspond to Lie ideals, and quotient group correspond to quotient Lie algebras. Moreover the Mal'cev correspondence preserves the nilpotency class.

(iii) The statement 2.4.6 (e) says that when a uniquely divisible nilpotent group N corresponds to a nilpotent Lie $\mathbb{Q}$-algebr $\mathfrak{n}$ under the Mal'cev correspondence, we have functorial bijections of sets

$$\log : N \xrightarrow{\sim} \mathfrak{n} \quad \text{and} \quad \exp : \mathfrak{n} \xrightarrow{\sim} N.$$

(2.4.9) (Baker–Campbell–Hausdorff formula) Let $L(x,y)_{\mathbb{Z}}$ be the free Lie $\mathbb{Z}$-algebra with free generators x, y, embedding in the free associative $\mathbb{Z}$-algebra $\mathrm{Ass}(x,y)_{\mathbb{Z}}$ with free generators $\{x,y\}$. Both $L(x,y)_{\mathbb{Z}}$ and $\mathrm{Ass}(x,y)_{\mathbb{Z}}$ are naturally $\mathbb{N}$-graded, such that x,y are homogeneous of degree 1. Denote by $L(x,y)_{\mathbb{Z}}^{n}$ the homogeneous component of $L(x,y)_{\mathbb{Z}}$ of degree n, $n \in \mathbb{N}$. Let $L(x,y)_{\mathbb{Q}} := L(x,y)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $\mathrm{Ass}(x,y)_{\mathbb{Q}} := \mathrm{Ass}(x,y)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\widehat{\mathrm{Ass}}(x,y)_{\mathbb{Q}}$ be the completion of $\mathrm{Ass}(x,y)_{\mathbb{Q}}$ with respect to powers of its augmentation ideal $\mathfrak{m}_{\mathbb{Q}} := \mathfrak{m}_{\mathrm{Ass}(x,y)_{\mathbb{Q}}}$. Let $L(x,y)_{\mathbb{Q}}^{\wedge}$ be the closure of $L(x,y)_{\mathbb{Q}}$ in $\widehat{\mathrm{Ass}}(x,y)_{\mathbb{Q}}$, naturally isomorphic to the completion of $L(x,y)_{\mathbb{Q}}$ with respect to the filtration by $\mathrm{Fil}^{n} L(x,y)_{\mathbb{Q}} := \oplus_{m \geq n} L(x,y)_{\mathbb{Z}}^{m}$. Let

$$\exp : L(x,y)_{\mathbb{Q}}^{\wedge} \longrightarrow \widehat{\mathrm{Ass}}(x,y)_{\mathbb{Q}}$$

be the exponential map given by the standard exponential series.

The Baker–Campbell–Hausdorff formula asserts that there exists a unique element $H(x,y)$ in $L(x,y)^{\wedge}_{\mathbb{Q}}$, which can be written in the form

$$H(x,y) = \sum_{n\geq 1} H_n(x,y) = \sum_{n\geq 1} \sum_{\substack{r+s=n,\\ r,s\geq 0}} H_{r,s}(x,y)$$

such that

$$\exp(x)\cdot\exp(y) = \exp(H(x,y)).$$

where

$$H_{r,s}(x,y) \ \in\ \tfrac{1}{(r+s)!\,(r+s-1)!} L(x,y)^n_{\mathbb{Z}}$$

is an element of $\frac{1}{(r+s)!\,(r+s-1)!}L(x,y)_{\mathbb{Z}}$ of bi-degree (r,s) in (x,y), and

$$H_n(x,y) = \sum_{r+s=n} H_{r,s}(x,y) \ \in\ \tfrac{1}{n!\,(n-1)!} L(x,y)_{\mathbb{Z}}$$

is of total degree n. Moreover $H_{1,0}(x,y) = x$, $H_{0,1}(x,y) = y$ and

$$H_{r,0}(x,y) = 0 = H_{0,s}(x,y) \quad \forall\, r\geq 2,\ \forall\, s\geq 2.$$

Remark. (i) We refer to [28, Part I Ch.II §8] and [3, Ch.II §6] for Dynkin's explicit form of the $H_n(x,y)$'s as an infinite series, from which the estimate of the denominator the coefficients of $H_{r,s}(x,y)$ follows. Better estimates of the denominators are available, but we won't need them.
(ii) The part of $H(x,y)$ linear in y, namely the infinite series $\sum_{r,1} H_{r,1}(x,y)$, is necessarily of the form $h(\mathrm{ad}x)(y)$, where $h(t)$ is a formal power series in $\mathbb{Q}[[t]]$. It is possible to evaluate the formal power series $h(t)$ and write it in a closed form in terms of more familiar functions and their indefinite integrals; we won't need this either.

Define a Lie $\mathbb{Z}$-subalgebra $L(x,y)_!$ of $L(x,y)_{\mathbb{Q}}$ by

$$L(x,y)_! := \sum_{n\geq 0} L(x,y)^n_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/c!].$$

Denote by $L(x,y)^{\wedge}_!$ the completion of $L(x,y)_!$ with respect to the filtration

$$\mathrm{Fil}^n L(x,y)_! = \oplus_{m\geq n} L(x,y)^m \otimes_{\mathbb{Z}} \mathbb{Z}[1/n!].$$

For any $m\geq 1$, the natural map $L(x,y)_!/\mathrm{Fil}^m L(x,y)_! \longrightarrow L(x,y)^{\wedge}_!/\mathrm{Fil}^m L(x,y)^{\wedge}_!$ is an isomorphism, where $\mathrm{Fil}^m L(x,y)^{\wedge}_!$ is the closure of $\mathrm{Fil}^m(x,y)_!$ in $\mathrm{Fil}^m L(x,y)^{\wedge}_!$. Since

$$H_n(x,y) \in L(x,y)^m \otimes_{\mathbb{Z}} \mathbb{Z}[1/n!]$$

for all n, $H(x,y)$ belongs to the image of the canonical injection

$$L(x,y)^{\wedge}_! \to L(x,y)^{\wedge}_{\mathbb{Q}}.$$

This observation is useful for the Lazard correspondence 2.4.11.

(2.4.10) Remark. One way to think about the Mal'cev correspondence is to identify the two sets N and $\mathfrak{n}$ using the mutually inverse bijections log and exp. So we have two structures on this set: as a uniquely divisible nilpotent group and also as a nilpotent Lie $\mathbb{Q}$-algebra. This approach yields the following lowbrow version of the Mal'cev correspondence.

(a) For any nilpotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$, the Baker–Campbell–Hausdorff (BCH) formula defines a binary operation "multiplication" on the set underlying $\mathfrak{n}$, under which $\mathfrak{n}$ acquires the structure of a uniquely divisible nilpotent group. Note that for any two elements $x_1, x_2 \in \mathfrak{n}$, the BCH formula in 2.4.9 for the product $x_1 \cdot x_2$ is a *finite sum*: $z_n(x_1, x_2) = 0$ for all n bigger than the nilpotency class of $\mathfrak{n}$.

(b) Conversely, given any uniquely divisible nilpotent group N, there exists a structure of a nilpotent Lie $\mathbb{Q}$-algebra which gives rise to the group law on N via the BCH formula as in (a). In particular there are two binary operations on the set underlying N, addition and Lie bracket, uniquely determined by the group structure of N, under which N becomes a Lie $\mathbb{Q}$-algebra. Both the addition and the Lie bracket are given by "universal formulas" involving products, inverse, raising to some *rational* power, and passing to the limit. These formulas for $x+y$ and $[x, y]$ are elements of projective limit

$$\varprojlim_n \mathrm{MC}\big(F_2(x,y)/\gamma_n(F_2(x,y))\big),$$

where $\big(\gamma_n(F_2(x,y))\big)_{n\geq 1}$ is the lower central series of $F_2(x, y)$.

These formulas for $x + y$ and $[x, y]$ are just one aspect of the problem on the "inverse Baker–Campbell–Hausdorff formula": characterizing the addition and Lie bracket structure on a uniquely divisible nilpotent group N in terms of the group structure of N. We refer to [16, Ch. II, §2] for further information.

(2.4.11) Proposition. (Lazard correspondence) *Let c be a positive integer. Let $P_{\leq c}$ be the set consisting all prime numbers not exceeding c. Let $P_{>c}$ be the set consisting of all prime numbers strictly bigger than c.*

(a) *Let $\mathfrak{n}_{\mathbb{Z}[1/c!]}$ be a nilpotent Lie $\mathbb{Z}[1/c!]$-algebra of class at most c. The Baker–Campbell–Hausdorff formula gives $\mathfrak{n}_{\mathbb{Z}[1/c!]}$ the structure of a uniquely $P_{\leq c}$-divisible nilpotent group of class at most c on the set underlying $\mathfrak{n}_{\mathbb{Z}[1/c!]}$, such that the product of any two elements $u, v \in \mathfrak{n}_{\mathbb{Z}[1/c!]}$ is given by*

$$u \cdot v := \sum_{1\leq n\leq c} \phi_{(u,v)}(z_n(x, y))$$

where $\phi_{(u,v)} : L(x, y)_!/\mathrm{Fil}^{c+1}L(x, y)_! \longrightarrow \mathfrak{n}_{\mathbb{Z}[1/c!]}$ is the Lie algebra homomorphism such that $\phi_{(u,v)}(x) = u$ and $\phi_{(u,v)}(y) = v$, and $z_n(x, y) \in L(x, y)^n_!$ is the homogeneous component of degree n of the element $z(x, y) \in L(x, y)^\wedge_!$ in 2.4.9.

(b) *Let $\mathfrak{n}_{\mathbb{Z}[1/c!]}$ and $\mathfrak{n}'_{\mathbb{Z}[1/c!]}$ be nilpotent Lie $\mathbb{Z}[1/c!]$-algebras of class at most c, and let $\alpha : \mathfrak{n}_{\mathbb{Z}[1/c!]} \to \mathfrak{n}'_{\mathbb{Z}[1/c!]}$ be a homomorphism of Lie algebras. Then α is also a group* *homomorphism with respect to the group structures on the sets underlying $\mathfrak{n}_{\mathbb{Z}[1/c!]}$ and $\mathfrak{n}'_{\mathbb{Z}[1/c!]}$ via the Baker–Campbell–Hausdorff formula. Moreover the image of α is a Lie ideal of $\mathfrak{n}'_{\mathbb{Z}[1/c!]}$ if and only if it is a normal subgroup for the group structure on $\mathfrak{n}'_{\mathbb{Z}[1/c!]}$.*

(c) *Conversely let $N_{\mathbb{Z}[1/c!]}$ be a uniquely $P_{\leq c}$-divisible nilpotent group of class at most c. There exists a unique nilpotent Lie $\mathbb{Z}[1/c!]$-algebra structure of class at most c on $N_{\mathbb{Z}[1/c!]}$ whose associated nilpotent group law coincides with the group law of $N_{\mathbb{Z}[1/c!]}$. In other words the exact functor from the category of nilpotent Lie $\mathbb{Z}[1/c!]$-algebras of class at most c to the category of uniquely $P_{\leq c}$-divisible nilpotent groups of class at most c described in* (a)–(b) *above is an equivalence.*

§3. Tate unipotent groups and Lie algebras

(3.1) Definition. Let κ be a field of characteristic p. A *terraced Tate unipotent group* over κ is a projective system $N = \big(N_i, \pi_{i,i+1} : N_{i+1} \twoheadrightarrow N_i\big)_{i\geq 1}$ of finite group schemes N_i over κ such that all transition homomorphisms $\pi_{i,i+1}$ are epimorphisms, together with a finite decreasing filtration $\big(\mathrm{Fil}^{\bullet}_{\mathrm{sl}} N_i\big)_{\bullet\in(0,1]}$ indexed by the interval $(0,1]$, called the *slope filtration* of N, which satisfies the following properties.

(i) For every $i \geq 1$ and every $t \in (0,1]$, $\mathrm{Fil}^t_{\mathrm{sl}} N_i$ is a normal subgroup scheme of N_i over κ. Moreover $\mathrm{Fil}^{>0}_{\mathrm{sl}} N_i = N_i$, and $\mathrm{Fil}^{>1}_{\mathrm{sl}} N_i = (0)$ by convention.

(ii) For each $i \geq 1$, the transition map $\pi_{i,i+1} : N_{i+1} \twoheadrightarrow N_i$ respects the slope filtration and induces epimorphisms $\mathrm{Fil}^t_{\mathrm{sl}} N_{i+1} \twoheadrightarrow \mathrm{Fil}^t_{\mathrm{sl}} N_i$ for all $t \in (0,1]$.

(iii) For every $i \geq 1$, we have $[\mathrm{Fil}^t_{\mathrm{sl}} N_i, \mathrm{Fil}^{t'}_{\mathrm{sl}} N_i]_{\mathrm{grp}}] \subseteq \mathrm{Fil}^{t+t'}_{\mathrm{sl}} N_i$ for all $t, t' \in (0,1]$, where $[\ ,\ ]_{\mathrm{grp}}$ denotes the group commutator $(x,y) \mapsto x^{-1}y^{-1}xy$.

(iv) There exists a finite subset $\mathrm{slope}(N)$ of $(0,1] \cap \mathbb{Q}$, such that for every $t \in (0,1]$ and every $i \geq 1$, the quotient group scheme $\mathrm{gr}^t N_i := (\mathrm{Fil}^t_{\mathrm{sl}} N_i)/(\mathrm{Fil}^{>t}_{\mathrm{sl}} N_i)$ is trivial if and only if $s \in \mathrm{slope}(N)$

(v) For every $t \in [0,1]$, there exists a p-divisible group Y_t over κ which is either isoclinic of slope t or trivial, such that the projective system $\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}} N := \big((\mathrm{Fil}^t_{\mathrm{sl}} N_i)/(\mathrm{Fil}^{>t}_{\mathrm{sl}} N_i)\big)_{i\geq 1}$ of commutative finite group schemes over κ is isomorphic to the *projective* system $\big(Y_t[p^i], [p] : Y_t[p^{i+1}] \twoheadrightarrow Y_t[p^i]\big)_{i\geq 1}$ attached to the p-divisible group Y_t. In other words there exists a family of isomorphisms $\alpha_{t,i} : \mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}} N_i \xrightarrow{\sim} Y_t[p^i]$ such that the diagram

$$\begin{array}{ccc}
\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}} N_{i+1} & \xrightarrow[\sim]{\alpha_{t,i+1}} & Y_s[p^{i+1}] \\
{\scriptstyle \mathrm{gr}^t \pi_{i,i+1}}\downarrow & & \downarrow{\scriptstyle [p]_{Y_t[p^{i+1}]}} \\
\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}} N_i & \xrightarrow[\sim]{\alpha_{t,i}} & Y_s[p^i]
\end{array}$$

Note that the p-divisible group Y_t over κ is uniquely determined by $\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}} N$.

(3.1.1) Remark. (a) Conditions 3.1 (iii)–(iv) imply that the distinct members in $\mathrm{Fil}_{\mathrm{sl}}^{\bullet} N_i$ form a finite central series for N_i. Hence N_i is a nilpotent group scheme of class at most $\mathrm{card}\big(\mathrm{slope}(N)\big)$, for all $i \geq 1$.

(b) The definition of terraced Tate unipotent groups can be extended to more general base schemes S. For instance one can replace $\mathrm{Spec}(\kappa)$ by a scheme S in characteristic p, and require that $\mathrm{Fil}_{\mathrm{sl}}^{s} N_i$ is finite locally free over S for every $s \in (0,1]$ and every $i \geq 1$, in addition to conditions (i)–(v) in 3.1. We have refrained from doing so, because the case when the base scheme is a field of characteristic p is sufficient for applications we have in mind.

(c) It is easy to see that the dimension $\dim_\kappa(\mathrm{Lie}(N_i))$ of the tangent space of the finite group scheme N_i is independent of i, and is equal to $\sum_{t \in \mathrm{sl}(N)} \dim_\kappa(Y_t)$. We call this integer the dimension of N, denoted by $\dim(N)$.

(3.1.2) Examples. (a) Clearly a p-divisible group Z over a field κ of characteristic p give rise a terraced Tate unipotent group, with $Z_i = Z[p^i]$ for all $i \in \mathbb{N}$, and the transition maps $Z[p^{i+1}] \to Z[p^i]$ are induced by $[p]_Z$.

It is not difficult to verify that every commutative terraced Tate unipotent group over a field κ arises from a p-divisible group over κ. The proof is left to the readers as an exercise.

(b) The stabilized Aut group schemes of a (polarized) p-divisible group over a field of characteristic p. These examples motivated the general notion of terraced Tate unipotent groups.

(b1) Let X be a p-divisible group over a field κ of characteristic p. The projective system $\mathscr{A}ut^{\mathrm{st}}(X)^0 = \big(\mathscr{A}ut^{\mathrm{st}}(X)^0_n\big)_{n\geq 1}$ of the neutral components of the stabilized Aut group schemes of $X[p^n]$, endowed with the slope filtration induced from the slope filtration of the stabilized End schemes $\mathscr{E}nd^{\mathrm{st}}(X) = \big(\mathscr{E}nd^{\mathrm{st}}(X)_n\big)_{n\geq 1}$, is a terraced Tate unipotent group over κ.

(b2) Similarly let $(Y, \mu : Y \to Y^t)$ be a polarized p-divisible group over κ. The projective system $\mathscr{A}ut^{\mathrm{st}}(Y,\mu)^0 = \big(\mathscr{A}ut^{\mathrm{st}}(Y,\mu)^0_n\big)_{n\geq 1}$ of the neutral components of the stabilized Aut group schemes of $(Y[p^n], \mu[p^n])$, is a terraced Tate unipotent group over κ.

See [7, Ch. 5] for details.

(c) This is a mild generalization of example (b) above. Let E be a p-divisible group over a field κ of characteristic p. Suppose that for each $i \geq 1$, $E_i := E[p^i]$ has a structure as a ring scheme over κ, compatible with the abelian group structure, such that the all epimorphism $\pi_{i,i+1} : E[p^{i+1}] \to E[p^{i+1}]$ is a ring scheme homomorphism over κ. For each i, let E_i^0 be the neutral component of E_i. Then E_i^0 is an ideal of E_i, for each i. Denote by d the number of distinct slopes of $E^0 = \varinjlim E_i^0$. Then the $(d+1)$-st power of the ideal E_i^0 is 0 for all i, and $e \leq d$. Let e be the smallest natural number such that $(E_i^0)^{e+1} = (0)$ for all i.

(c1) Let $N_i = 1 + E_i^0$, which is a subgroup scheme of E_i. Then the projective system $(N_i, \pi_{i,i+1} : E_{i+1,i})_{i\geq 1}$ is a terraced Tate unipotent group over κ, and each N_i is unipotent of class at most e.

(c2) Suppose that in addition, we have involutions τ_i on E_i, which are compatible with the transition maps $\pi_{i,i+1}$. Let N_i' to be the subgroup scheme of $1 + E_i^0$ consisting of functorial points $1+x_i$ of $1+E_i^0$ such that $(1+\tau_i(x_i))\cdot(1+x_i) = 1 = (1+x_i)\cdot(1+\tau_i(x_i))$. Let N_i be the intersection, over all $m \in \mathbb{N}$ of the schematic images of the projection maps $\pi_{i,i+m} : E'_{i+m} \to E'_i$. Then the projective system $(N_i)_{i\geq 1}$ is a terraced Tate unipotent group over κ. The readers are encouraged to verify this assertion directly.

(3.1.3) Remark. In example (c) above, the smallest natural number e such that the $(e+1)$-st power of E_i^0 is 0 may be substantially smaller than the number of slopes of E^0. Consider the case when E is the projective system of stabilized End group schemes $\mathscr{E}nd^{\mathrm{st}}(X)$ of a p-divisible group X over κ. If r is the number of distinct slopes of X, then the number of slopes of E^0 can be as high as $r(r-1)/2$, while $(E_i^0)^r = (0)$. See [7, Ch. 5 §1] for more information about the stabilized End group schemes $\mathscr{E}nd^{\mathrm{st}}(X)$.

(3.2) Definition. Let $N = \left(N_i, \pi_{i,i+1} : N_{i+1} \twoheadrightarrow N_i\right)_{i\geq 1}$ be a terraced Tate unipotent group over a field κ of characteristic p. Denote by $\mathfrak{Sch}_\kappa$ the category whose objects are κ-schemes and morphisms are κ-morphisms between κ-schemes.

(a) Define the Tate module $\mathbf{T}_p(N)$ of N to be the limit

$$\mathbf{T}_p(N) := \varprojlim_i N_i,$$

where N_i is identified with a sheaf of groups for the fpqc topology on the category $\mathfrak{Sch}_\kappa$, and the inverse limit is taken in the category of presheaves on $\mathfrak{Sch}_\kappa$. Since every projective limit of sheaves in the category of presheaves is a sheaf, $\mathbf{T}_p(N)$ is a sheaf of groups for the fpqc topology on the category $\mathfrak{Sch}_\kappa$.

(b) Define the slope filtration $\left(\mathrm{Fil}_{\mathrm{sl}}^\bullet \mathbf{T}_p(N)\right)_{\bullet\in(0,1]}$ of $\mathbf{T}_p(N)$ by

$$\mathrm{Fil}_{\mathrm{sl}}^t \mathbf{T}_p(N) := \varprojlim_i \mathrm{Fil}_{\mathrm{sl}}^t N_i \qquad \forall\, t \in (0,1].$$

Clearly $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}} \mathbf{T}_p(N) \neq 0$ if and only if $s \in \mathrm{slope}(N)$.

(c) Define $\mathrm{MC}(\mathbf{T}_p(N))$ to be the presheaf on $\mathfrak{Sch}_\kappa$ whose value on any κ-scheme S is the Mal'cev completion of $\mathbf{T}_p(N)(S)$.

Similarly, define the slope filtration of the presheaf $\mathrm{MC}(\mathbf{T}_p(N))$ by

$$\mathrm{Fil}_{\mathrm{sl}}^t \mathrm{MC}(\mathbf{T}_p(N)) := \mathrm{MC}(\mathrm{Fil}_{\mathrm{sl}}^t \mathbf{T}_p(N)) \qquad \forall\, t \in (0,1].$$

(d) Define the Tate module $\mathbf{V}_p(N)$ of N, also denoted by $\mathbf{T}_p(N)_{\mathbb{Q}}$, to be the sheafification of the presheaf $\mathrm{MC}(\mathbf{T}_p(N))$ on the category $\mathfrak{Sch}_\kappa$ with respect to the fpqc topology. Similarly for every $t \in (0,1]$, let $\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{V}_p(N)$ be the sheafification of the presheaf $\mathrm{Fil}^{\bullet}_{\mathrm{sl}}\mathbf{T}_p(N)$.

(3.2.1) Remark. If N arises from a p-divisible group Z over κ as in 3.1.2, the resulting fpqc sheaf $\mathbf{T}_p(Z)$ is clearly analogous in spirit to the usual Tate $\mathbb{Z}_\ell$-modules of Z for $\ell \neq p$, but it has not been among the standard tools of algebraic geometers. However the covariant Dieudonnémodule $\mathbb{D}_*(Z)$ is an adequate stand-in for $\mathbf{T}_p(Z)$ in the case when κ is a perfect field.

(3.2.2) Lemma. *Let $N = \big(N_i\big)_{i\geq 1}$ be a terraced Tate unipotent group over a field κ of characteristic p.*

(a) *For every κ-scheme S, $\mathbf{T}_p(N)(S)$ is a torsion free nilpotent group of class at most* $\mathrm{card}(\mathrm{slope}(N))$, *and uniquely ℓ-divisible for every prime number $\ell \neq p$.*

(b) *For every non-zero integer n, the map $x \mapsto x^n$ induces an automorphism on the presheaf of sets underlying the presheaf of nilpotent groups $\mathrm{MC}(\mathbf{T}_p(N))$, and also an automorphism on the sheaf of sets underlying the sheaf of nilpotent groups $\mathbf{V}_p(N)$. In particular for every κ-scheme S, the group $\mathbf{V}_p(N)(S)$ is torsion-free nilpotent and divisible.*

(c) *For every* quasi-compact *κ-scheme S, $\mathbf{V}_p(N)(S)$ is canonically isomorphic to the Mal'cev completion of the nilpotent torsion free group $\mathbf{T}_p(N)(S)$. In other words $\mathbf{V}_p(N)(S) = \mathrm{MC}(\mathbf{T}_p(N)(S))$.*

Proof. The statements (a), (b) are obvious. The statement (c) is proved by induction on $\mathrm{card}(\mathrm{slope}(N))$, using the general statement on sections (and cohomologies) over quasi-compact objects of a filtered colimit of sheaves on a site. See [1] SGA 3 Exp.VI Thm. 5.1, [27, `https://stacks.math.columbia.edu/tag/090G` Lemma 0738].

When $\mathrm{card}(\mathrm{slope}(N)) = 1$, N is the projective system attached to an isoclinic p-divisible group over κ, and $\mathbf{V}_p(N)$ is the limit of the inductive system

$$\mathbf{T}_p(N) \xrightarrow{[p]} \mathbf{T}_p(N) \xrightarrow{[p]} \cdots \xrightarrow{[p]} \mathbf{T}_p(N) \xrightarrow{[p]} \cdots ,$$

which is a very special case of the general statement in the previous paragraph.

Suppose that $\mathrm{card}(\mathrm{slope}(N)) \geq 2$. Let $a = \mathrm{maxslope}(N)$. We have a short exact sequence

$$1 \to \mathrm{Fil}^a_{\mathrm{sl}}N \to N \to N/\mathrm{Fil}^a_{\mathrm{sl}}N \to 1$$

and a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{MC}(\mathbf{T}_p(\mathrm{Fil}^s_{\mathrm{sl}}N)(S)) \arrow[r] \arrow[d, "\alpha"', "\simeq"] & \mathrm{MC}(\mathbf{T}_p(N)(S)) \arrow[r] \arrow[d, "\beta"'] & \mathrm{MC}(\mathbf{T}_p(N/\mathrm{Fil}^s_{\mathrm{sl}}N)(S)) \arrow[r] \arrow[d, "\gamma"', "\simeq"] & 1 \\
1 \arrow[r] & \mathbf{V}_p(\mathrm{Fil}^s_{\mathrm{sl}}N)(S) \arrow[r] & \mathbf{V}_p(N)(S) \arrow[r] & \mathbf{V}_p(N/\mathrm{Fil}^s_{\mathrm{sl}}N)(S) &
\end{tikzcd}$$

where α and γ are bijections. So β is also a bijection. □

(3.2.3) Remark. Let $\phi_i\mathrm{MC}(N)$ be the sub-presheaf of $\mathrm{MC}(N)$ such that

$$\phi_i\mathrm{MC}(N)(S) = \{x \in \mathrm{MC}(N)(S) \mid x^{p^i} \in \mathbf{T}_p(N)(S)\}$$

for every κ-scheme S. It is easily verified that $\phi_i\mathrm{MC}(N)$ is a subsheaf of $\mathbf{V}_p(N)$ for the fpqc topology, and

$$\mathbf{V}_p(N) = \varinjlim_{i\to\infty} \phi_i\mathrm{MC}(N)$$

as an sheaf of sets on $\mathfrak{Sch}_\kappa$ for the fpqc topology. So we can apply the general fact recalled in the proof of 3.2.2 to finish the proof of 3.2.2 (c), without going through the induction argument.

(3.2.4) Definition. Let κ be a field of characteristic p. A *Tate unipotent group* over κ is a sheaf $\mathbf{N}$ on $\mathfrak{Sch}_\kappa$ with respect to the fpqc topology together with a decreasing filtration $\mathrm{Fil}^\bullet_{\mathrm{sl}}\mathbf{N}$ of normal subgroups of $\mathbf{N}$, such that there exists a finite subset $\mathrm{slope}(\mathbf{N}) \subseteq (0,1] \cap \mathbb{Q}$ with the following properties.

- $[\mathrm{Fil}^{s_1}_{\mathrm{sl}}\mathbf{N}, \mathrm{Fil}^{s_2}_{\mathrm{sl}}\mathbf{N}]_{\mathrm{grp}} \subseteq \mathrm{Fil}^{s_1+s_2}_{\mathrm{sl}}\mathbf{N}$ for all $s_1, s_2 \in (0,1]$, where $\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N} = 0$ if $s > 1$ by convention, and $[\ ,\ ]_{\mathrm{grp}}$ denotes the group commutator $(x,y) \mapsto x^{-1}y^{-1}xy$.
- $\mathrm{gr}^s_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} \neq (0)$ if and only if $s \in \mathrm{slope}(\mathbf{N})$.
- For every $t \in \mathrm{slope}(\mathbf{N})$, there exists a non-trivial p-divisible group Y_t over κ such that

  $$\mathrm{gr}^t\mathbf{N} := \mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N} \ \cong\ \varprojlim_n Y_t[p^n],$$

  where $\varprojlim_n Y_t[p^n]$ is the projective limit of the projective system $\big(Y_t[p^n]\big)_{n\geq 1}$ with transition map induced by $[p]_{Y_t}$, and the limit is taken in the category of sheaves on $\mathfrak{Sch}_\kappa$ for the fpqc topology.

(3.2.5) Remark. (i) Clearly the Tate $\mathbb{Z}_p$-module $\mathbf{T}_p(N)$ of terraced Tate unipotent group $N = \big(N_i\big)_{i\geq 1}$ is a Tate unipotent group.

(ii) Conversely a terraced Tate unipotent group $N = \big(N_i\big)_{i\geq 1}$ is determined by its Tate module $\mathbf{T}_p(N)$ and the decreasing family of normal subgroups $\mathbf{U}_i := \mathrm{Ker}(\mathbf{T}_p(N) \twoheadrightarrow N_i)$ of $\mathbf{T}_p(N)$. But unlike the case when N is commutative, we don't know a simple way to recover these normal subgroups $\mathbf{U}_i$'s directly from $\mathbf{T}_p(N)$ and its slope filtration.

(iii) We don't know whether every Tate unipotent group $\mathbf{N}$ is isomorphic to the Tate module $\mathbf{T}_p(N)$ of some terraced Tate unipotent group $N = \big(N_i\big)_{i\geq 1}$.

(3.2.6) Lemma. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ. For every κ-scheme S, $\mathbf{N}$ is a torsion free nilpotent group and is uniquely ℓ-divisible for every prime number $\ell \neq p$.*

Proof. Omitted. □

(3.2.7) Definition. Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p.

(a) Denote by $\mathbf{N}_{\mathbb{Q}}$ the sheafification of the presheaf $\mathrm{MC}(\mathbf{N})$ on $\mathfrak{Sch}_\kappa$ with respect to the fpqc topology.

(b) Define the dimension of $\mathbf{N}$ to be

$$\dim(\mathbf{N}) = \sum_{t\in\mathrm{slope}(\mathbf{N})} \dim(Y_t),$$

where Y_t is a p-divisible group over κ such that $\mathrm{gr}^t\mathbf{N} \cong \varprojlim_n Y_t[p^m]$ as in 3.2.4.

(3.2.8) Lemma. *Let* $\mathbf{N}$ *be a Tate unipotent group over a field* κ *of characteristic* p.

(a) *For every* κ*-scheme* S, $\mathbf{N}_{\mathbb{Q}}(S)$ *is a uniquely divisible.*

(b) *For every quasi-compact* κ*-scheme* S, $\mathbf{N}_{\mathbb{Q}}(S)$ *is the Mal'cev completion of* $\mathbf{N}(S)$.

Proof. Omitted. □

(3.2.9) Definition. Let $\mathbf{N}$ and $\mathbf{N}'$ be Tate unipotent groups over a field κ of characteristic p.

(a) A κ-homomorphism α from $\mathbf{N}$ to $\mathbf{N}'$ is an *isogeny* if α induces an isomorphism from $\mathbf{N}_{\mathbb{Q}}$ to $\mathbf{N}'_{\mathbb{Q}}$.

(b) A κ-homomorphism *up to isogeny* from $\mathbf{N}$ to $\mathbf{N}'$ is a homomorphism from $\mathbf{N}_{\mathbb{Q}}$ to $\mathbf{N}'_{\mathbb{Q}}$.

(c) A *quasi-isogeny* over κ from $\mathbf{N}$ to $\mathbf{N}'$ is a group isomorphism over κ from $\mathbf{N}_{\mathbb{Q}}$ to $\mathbf{N}_{\mathbb{Q}}$. If there exists a quasi-isogeny from $\mathbf{N}$ to $\mathbf{N}'$, we say that $\mathbf{N}$ and $\mathbf{N}'$ are isogenous.

Remark. The Mal'cev correspondence gives the following equivalent definitions of 3.2.9 (a)–(c), where $\mathrm{Lie}\,\mathbf{N}_{\mathbb{Q}}$ and $Lie\mathbf{N}'_{\mathbb{Q}}$ are the fpqc sheaves of Lie algebras on $\mathfrak{Sch}_\kappa$ attached to $\mathbf{N}_{\mathbb{Q}}$ and $\mathbf{N}'_{\mathbb{Q}}$ as in 3.2.14.

(a′) A κ-homomorphism α from $\mathbf{N}$ to $\mathbf{N}'$ is an *isogeny* if α induces an isomorphism from $\mathrm{Lie}\,\mathbf{N}_{\mathbb{Q}}$ to $\mathrm{Lie}\,\mathbf{N}'_{\mathbb{Q}}$.

(b′) A κ-homomorphism *up to isogeny* from $\mathbf{N}$ to $\mathbf{N}'$ is a homomorphism of fpqc sheaves of Lie $\mathbb{Q}_p$-algebras from $\mathrm{Lie}\,\mathbf{N}_{\mathbb{Q}}$ to $\mathrm{Lie}\,\mathbf{N}'_{\mathbb{Q}}$.

(c′) A *quasi-isogeny* over κ from $\mathbf{N}$ to $\mathbf{N}'$ is a Lie $\mathbb{Q}_p$-algebra isomorphism from $\mathrm{Lie}\,\mathbf{N}_{\mathbb{Q}}$ to $\mathrm{Lie}\,\mathbf{N}'_{\mathbb{Q}}$.

(3.2.10) Definition. Let $N = \big(N_i, \mathrm{Fil}^{\bullet}_{\mathrm{sl}} N_i, \pi_{i,i+1} : N_{i+1} \to N_i\big)_{i\geq 1}$ be a terraced Tate unipotent group over a field κ of characteristic p. A *terraced Tate unipotent subgroup* N' is a family of subgroup schemes $\big(N'_i \subseteq N_i\big)_{i\geq 1}$ such that

(a) The restriction to N'_i of every transition morphism $\pi_{i,i+1}|_{N_{i+1}}$ factors through N'_i and defines an epimorphism $\pi'_{i,i+1} : N'_{i+1} \twoheadrightarrow N'_i$.

(b) The projective family of finite group schemes N_i, together with the filtration

$$\mathrm{Fil}^{\bullet}_{\mathrm{sl}} N'_i := N'_i \cap \mathrm{Fil}^{\bullet}_{\mathrm{sl}} N_i \quad \forall \bullet \in (0, 1],$$

is a terraced Tate unipotent group $\big(N'_i, \mathrm{Fil}^{\bullet}_{\mathrm{sl}} N'_i, \pi'_{i,i+1}\big)_{i\geq 1}$ over κ.

(3.2.11) Lemma. *Let κ be a field of characteristic p. Let N be a terraced Tate unipotent group over κ, and let N' be a terraced Tate unipotent subgroup over κ. The Tate $\mathbb{Z}_p$-module $\mathbf{T}_p(N')$ of N' is co-torsion free in $\mathbf{T}_p(N)$, i.e.*

$$\mathbf{T}_p(N') = \mathbf{T}_p(N) \cap \mathbf{V}_p(N').$$

The proofs of lemma 3.2.11 and lemma 3.2.12 below are left to the readers.

(3.2.12) Lemma. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let n be a positive integer.*

(a) *The subgroup $\mathbf{N}^{p^n}$ of $\mathbf{N}$ generated by all p^n-th powers of local sections of $\mathbf{N}$ is a Tate unipotent subgroup of $\mathbf{N}$ over κ.*

(b) *The subgroup $\mathbf{N}^{p^n}$ of $\mathbf{N}_{\mathbb{Q}}$ generated by all local sections of $\mathbf{N}_{\mathbb{Q}}$ whose p^n-th power are in $\mathbf{N}$ is a Tate unipotent group over κ containing $\mathbf{N}$.*

(3.2.13) Definition. Let κ be a field of characteristic p.

(a) A *Tate unipotent Lie $\mathbb{Z}_p$-algebra* $\mathfrak{N}$ over κ is a sheaf of Lie $\mathbb{Z}_p$-algebras for the fpqc topology on $\mathfrak{Sch}_\kappa$ such that there exist a p-divisible group Y over κ and family of isomorphisms

$$\alpha_n : \mathfrak{N}/p^n\mathfrak{N} \xrightarrow{\sim} Y[p^n], \quad n \geq 1,$$

which are compatible in the sense that the diagram

$$\begin{tikzcd} \mathfrak{N}/p^{n+1}\mathfrak{N} \arrow[r, "\alpha_{n+1}", "\sim"'] \arrow[d, "\pi_n"] & Y[p^{n+1}] \arrow[d, "{[p]}"'] \\ \mathfrak{N}/p^n\mathfrak{N} \arrow[r, "\alpha_n", "\sim"'] & Y[p^n] \end{tikzcd}$$

commutes for every $n \geq 1$, where π_n is the natural epimorphism from $\mathfrak{N}/p^{n+1}\mathfrak{N}$ to $\mathfrak{N}/p^n\mathfrak{N}$.

(b) A *Tate unipotent Lie $\mathbb{Q}_p$-algebra* over κ is a sheaf of Lie $\mathbb{Q}_p$-algebras for the fpqc topology on $\mathfrak{Sch}_\kappa$ isomorphic to $\mathfrak{N} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for some Tate unipotent Lie $\mathbb{Z}_p$-algebra $\mathfrak{N}$ over κ.

(3.2.14) Definition. Let $\mathbf{N}$ be a Tate unipotent group over κ, and let $\mathbf{N}_{\mathbb{Q}}$ be the Mal'cev completion of $\mathbf{N}$.

(a) Denote by $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ the sheaf of Lie $\mathbb{Q}$-algebras on $\mathfrak{Sch}_{\kappa}$ for the fpqc topology attached to $\mathbf{N}_{\mathbb{Q}}$ under the Mal'cev correspondence. The $\mathbb{Q}$-module structure on $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ extends uniquely to a $\mathbb{Q}_p$-module structure. This sheaf of Lie $\mathbb{Q}_p$-algebras $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ is called the Lie $\mathbb{Q}_p$-algebra of $\mathbf{N}$ and $\mathbf{N}_{\mathbb{Q}}$.

(b) Suppose that $\mathbf{N}$ is nilpotent of class at most $p-1$. Under the Lazard correspondence, $\mathbf{N}$ corresponds to a sheaf $\mathfrak{Lie}\mathbf{N}$ of Lie $\mathbb{Z}_p$-algebras on $\mathfrak{Sch}_{\kappa}$ for the fpqc topology. We call it the Lie $\mathbb{Z}_p$-algebra of $\mathbf{N}$

Lemma 3.2.15 below follows directly from definition 3.2.14.

(3.2.15) Lemma. *Let $\mathbf{N}$ be a Tate unipotent group over κ.*

(a) *The Lie $\mathbb{Q}_p$-algebra $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ is a Tate unipotent Lie $\mathbb{Q}_p$-algebra over κ.*

(b) *If $\mathbf{N}$ is nilpotent of class at most $p-1$, then the sheaf $\mathfrak{Lie}\mathbf{N}$ corresponding to $\mathbf{N}$ under the Lazard correspondence is a Tate unipotent Lie $\mathbb{Z}_p$-algebra over κ, which is nilpotent of class at most c.*

(3.2.16) Lemma. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ be the Lie algebra of the Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{N}$, and $\log_{\mathbf{N}_{\mathbb{Q}}} : \mathbf{N}_{\mathbb{Q}} \to \mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ be the associated logarithm map. The Lie $\mathbb{Z}_p$-subalgebra generated by $\log_{\mathbf{N}_{\mathbb{Q}}}(\mathbf{N})$ is a Tate unipotent Lie $\mathbb{Z}_p$-algebra over κ.*

Proof. Omitted. □

(3.2.17) Lemma. *Let $\mathfrak{N}$ be a Tate unipotent $\mathbb{Z}_p$-Lie algebra over $\mathbb{Q}$.*

(a) *There exists a Tate unipotent group $\mathbf{N}$ over κ, unique up to isogeny, such that $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ is isomorphic to $\mathfrak{N} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

(b) *If $\mathfrak{N}$ is nilpotent of class at most $p-1$, then there exists a Tate unipotent Lie group $\mathbf{N}$, unique up to isomorphism, such that $\mathfrak{Lie}\mathbf{N}$ is isomorphic to $\mathfrak{N}$.*

Proof. The statement (b) is immediate from the Lazard correspondence. The sufficiency part of statement (a) follows from the Mal'cev correspondence. It remains to prove the existence part (a).

Let c be the nilpotency class of $\mathfrak{N}$. Let $\mathbf{N}_{\mathbb{Q}_p}$ be the fpqc sheaf of nilpotent groups on $\mathfrak{Sch}_{\kappa}$ attached to $\mathfrak{N}$ under the Mal'cev correspondence. The BCH formula recalled in 2.4.9 implies that for any positive integer m satisfying the condition that

$$2\,\mathrm{ord}_p(n!) \leq m(n-1) \quad \forall\, n \in \mathbb{N} \text{ with } 2 \leq n \leq c, \tag{3.2.17.1}$$

the subsheaf $\exp(p^m\mathfrak{N})$ is a subgroup of $\mathbf{N}_{\mathbb{Q}_p}$. Moreover this subgroup of $\mathbf{N}_{\mathbb{Q}_p}$ is a Tate $\mathbb{Z}_p$-module of a Tate unipotent group. □

(3.2.18) Remark. From the standard estimate $\mathrm{ord}_p(n!) < \frac{n}{p-1}$, one sees that the condition (3.2.17.1) on m is satisfied if $m \geq \frac{4}{p-1}$ and $p \geq 3$, or if $m \geq 2$.

(3.2.19) Lemma. *Let N be a torsion free nilpotent group of class at most c, where c is a positive integer. Let $N_{\mathbb{Q}}$ be the Mal'cev completion of N. Let $\mathfrak{n}_{\mathbb{Q}}$ be the Lie $\mathbb{Q}$-algebra attached to $N_{\mathbb{Q}}$ under the Mal'cev correspondence, and let $\log_{N_{\mathbb{Q}}} : N_{\mathbb{Q}} \xrightarrow{\sim} \mathfrak{n}_{\mathbb{Q}}$ be the logarithm map.*

There exists a positive integer $n_0 = n_0(c)$, depending only on c, such that the subset $\log_{N_{\mathbb{Q}}}(N^n)$ of $\mathfrak{n}$ is a Lie $\mathbb{Z}_{(p)}$-subalgebra of $\mathfrak{n}$ for all positive integers n which are multiples of n_0. Here N^n denotes the subgroup of N generated by the subset $\{z^n | z \in N\}$ of all n-th powers in N.

Proof. Let $F = F(x,y)$ be the free group in variables x, y, let $\big(\gamma_i(F(x,y))\big)_{i\geq 1}$ be the descending central series of F_2. Consider the torsion free nilpotent group $F/\gamma_{c+1}(F)$ of class c and its Mal'cev completion $(F/\gamma_{c+1}(F))_{\mathbb{Q}}$. It suffice to show that when $N = F/\gamma_{c+1}(F)$, we have

$$\log_{N_{\mathbb{Q}}}(x^n) + \log_{N_{\mathbb{Q}}}(y^n),\ [\log_{N_{\mathbb{Q}}}(x^n), \log_{N_{\mathbb{Q}}}(y^n)] \in \log_{N_{\mathbb{Q}}}(N^n) \quad \forall n \text{ s.t. } n_0(c)|n,$$

where $n_0(c)$ is a constant depending only on c.

The last statement can be proved by induction on c, a standard method in the theory of nilpotent groups. We will use the method of *typical sequences* in [16, Ch. II, §§1–2], which is more illuminating. $F_{\mathbb{Q}}$ be the Mal'cev completion of $F = F(x,y)$, Let $F_{\mathbb{Q}}^{\wedge}$ be the completion of $F_{\mathbb{Q}}$ with respect to its lower central series. According to [16, Ch. II, Thm. 1.5], there exist uniquely determined elements $z_i, w_i \in \gamma_i(F_{\mathbb{Q}}^{\wedge})$, $i \in \mathbb{N}_{\geq 1}$, such that

$$\exp\big(\log(x^t) + \log(y^t)\big) = z_1^t z_2^{t^2} \cdots z_i^{t^i} \cdots, \quad \exp\big([\log(x^t), \log(y^t)]\big) = w_1^t w_2^{t^2} \cdots w_i^{t^i} \cdots.$$

It is clear from the BCH formula that $z_1 = xy$ and $w_1 = 1$. Denote by $\bar{z}_i$ (respectively $\bar{w}_i$) the image of z_i (respectively w_i) in $F_{\mathbb{Q}}^{\wedge}/\gamma_{c+1}(F_{\mathbb{Q}}^{\wedge}) \cong F_{\mathbb{Q}}/\gamma_{c+1}(F_{\mathbb{Q}})$. Pick positive integers $a_2, \ldots, a_c$, such that

$$\bar{z}_i^{a_i}, \bar{w}_i^{b_i} \in F^{\wedge}/\gamma_{c+1}(F^{\wedge}) \cong F/\gamma_{c+1}F \quad \text{for } i = 2, \ldots, c.$$

Let $n_0 = \mathrm{lcm}(a_2, \ldots, a_c)$. Then $\bar{z}_i^{t^i}, \bar{w}_i^{t^i} \in (F/\gamma_{c+1}(F))^n$ for all $i = 1, \ldots, c$. $\square$

Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p, which is nilpotent of class at most c, where c is a positive integer. Let $\mathfrak{N}_{\mathbb{Q}_p} := \mathfrak{Lie}\mathbf{N}_{\mathbb{Q}_p}$ be the Lie $\mathbb{Q}_p$-algebra of the Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{N}$. Let $\log : \mathbf{N}_{\mathbb{Q}} \xrightarrow{\sim} \mathfrak{N}_{\mathbb{Q}_p}$ be the logarithm map from $\mathbf{N}_{\mathbb{Q}}$ to $\mathfrak{N}_{\mathbb{Q}_p}$. For each positive integer n, denote by $\mathbf{N}^{p^n}$ the fpqc sheaf on $\mathfrak{Sch}_{\kappa}$ which is the smallest sheaf of subgroups of $\mathbf{N}$ generated by all p^n-th powers of local sections of $\mathbf{N}$.

(3.2.20) Corollary. *Notation as in the preceding paragraph. There exists a positive integer $n_1 = n_1(c)$, depending only on c, such that for every integer $n \geq n_1$, the subsheaf $\log(\mathbf{N}^{p^n})$ of $\mathfrak{N}_{\mathbb{Q}_p}$ is stable under addition and Lie bracket. Consequently $\log(\mathbf{N}^{p^n})$ is a Tate unipotent Lie $\mathbb{Z}_p$-subalgebra of $\mathfrak{N}_{\mathbb{Q}_p}$. Moreover $\log(\mathbf{N}^{p^n}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathfrak{N}_{\mathbb{Q}_p}$.*

PROOF. This assertion holds with $n_1(c) = \mathrm{ord}_p(n_0(c))$, where $n_0(c)$ is as in 3.2.19. $\square$

(3.2.21) LEMMA. *Let $\mathbf{N}_1, \mathbf{N}_2$ be Tate unipotent groups over a field κ of characteristic p, and let $\alpha : \mathbf{N}_1 \to \mathbf{N}_2$ be an isogeny. There exists a positive integer n such that $\mathbf{N}_2^{p^n} \subseteq \alpha(\mathbf{N}_1)$.*

(3.2.22) LEMMA. *Let κ be a field of characteristic p. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be Tate unipotent groups over κ. Let α be a quasi-isogeny from $\mathbf{N}_1$ to $\mathbf{N}_2$.*

(a) *There exists a Tate unipotent group $\mathbf{N}_3$ over κ and an isogeny $\beta : \mathbf{N}_3 \to \mathbf{N}_1$ such that $\alpha \circ \beta$ is an isogeny from $\mathbf{N}_3$ to $\mathbf{N}_1$.*

(b) *There exists a Tate unipotent group $\mathbf{N}_4$ over κ and an isogeny $\gamma : \mathbf{N}_2 \to \mathbf{N}_4$ such that $\gamma \circ \alpha$ is an isogeny from $\mathbf{N}_1$ to $\mathbf{N}_4$.*

The proofs of lemma 3.2.21 and 3.2.22 are left to the readers.

(3.2.23) LEMMA. *Let $\mathfrak{N}_{\mathbb{Q}}$ be a Tate unipotent Lie $\mathbb{Q}_p$-algebra over a field κ of characteristic p. There exists a p-divisible group X over κ and an embedding of the fpqc sheaf $\mathfrak{N}_{\mathbb{Q}}$ of Lie $\mathbb{Q}_p$-algebras to the fpqc sheaf of Lie $\mathbb{Q}_p$-algebras underlying the sheaf of nilpotent associative $\mathbb{Q}_p$-algebras (without unity) $\mathbf{V}_p(\mathscr{E}nd^{\mathrm{st}}(X)^0)$ on $\mathfrak{Sch}_\kappa$.*

PROOF. Let $U(\mathfrak{N}_{\mathbb{Q}})$ be the universal enveloping algebra of $\mathfrak{N}_{\mathbb{Q}}$ over $\mathbb{Q}_p$, an fpqc sheaf of associative algebras on $\mathfrak{Sch}_\kappa$ with a canonical injection $\jmath : \mathfrak{N}_{\mathbb{Q}} \to U(\mathfrak{N}_{\mathbb{Q}})$ and an increasing filtration $\mathrm{Fil}_{\deg}$, where

$$\mathrm{Fil}_{\deg\leq n} U(\mathfrak{N}_{\mathbb{Q}}) = \sum_{0\leq m\leq n} \big(\otimes^m \mathfrak{N}_{\mathbb{Q}} \to U(\mathfrak{N}_{\mathbb{Q}})\big).$$

Each $\mathrm{Fil}_{\deg\leq n}$ has a natural decreasing slope filtration indexed by $[0,\infty)$. These slope filtrations are compatible with the inclusions $\mathrm{Fil}_{\deg\leq n} \hookrightarrow \mathrm{Fil}_{\deg\leq n}$. Together they define a slope filtration $\mathrm{Fil}_{\mathrm{slope}}$ on $U(\mathfrak{N}_{\mathbb{Q}})$.

Consider the quotient $U(\mathfrak{N}_{\mathbb{Q}})/\mathrm{Fil}^{>1}_{\mathrm{slope}} U(\mathfrak{N}_{\mathbb{Q}})$, whose slopes are contained in the interval $[0,1]$. There exists a p-divisible group X over κ, unique up to isogeny, such that $(\varprojlim_n X[p^n]) \otimes_{\mathbb{Z}} \mathbb{Q} \cong U(\mathfrak{N}_{\mathbb{Q}})/\mathrm{Fil}^{>1}_{\mathrm{slope}} U(\mathfrak{N}_{\mathbb{Q}})$. Since $\mathrm{Fil}^{>1}_{\mathrm{slope}} U(\mathfrak{N}_{\mathbb{Q}})$ is an ideal of $U(\mathfrak{N}_{\mathbb{Q}})$, the quotient $U(\mathfrak{N}_{\mathbb{Q}})/\mathrm{Fil}^{>1}_{\mathrm{slope}} U(\mathfrak{N}_{\mathbb{Q}})$ is a sheaf of algebras over $\mathbb{Q}_p$. The composition

$$\mathfrak{N}_{\mathbb{Q}} \xrightarrow{\jmath} U(\mathfrak{N}_{\mathbb{Q}}) \longrightarrow U(\mathfrak{N}_{\mathbb{Q}})/\mathrm{Fil}^{>1}_{\mathrm{slope}} U(\mathfrak{N}_{\mathbb{Q}})$$

is an injection because every slope of $\mathfrak{N}_{\mathbb{Q}}$ is contained in $(0,1]$. So the restriction to $\mathfrak{N}_{\mathbb{Q}}$ of the regular representation of $U(\mathfrak{N}_{\mathbb{Q}})/\mathrm{Fil}^{>1}_{\mathrm{slope}} U(\mathfrak{N}_{\mathbb{Q}})$ is also an injection. $\square$

REMARK. Lemma 3.2.23 is an analog of Ado's theorem for Tate unipotent Lie $\mathbb{Q}_p$-algebras. Corollary 3.2.24 rephrases it in terms of Tate unipotent groups.

(3.2.24) COROLLARY. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. There exists a p-divisible group X over κ and an embedding $\mathbf{N}_{\mathbb{Q}} \hookrightarrow \mathbf{V}_p\big(\mathscr{A}ut^{\mathrm{st}}(X)^0\big)$.*

PROOF. Immediate from 3.2.23 and the Mal'cev correspondence. $\square$

(3.3) Dieudonné theory of Tate unipotent Lie algebras over perfect fields

In this subsection κ is a *perfect* field of characteristic p.

(3.3.1) Definition. Let $\mathfrak{N}$ be a Tate unipotent Lie $\mathbb{Z}_p$-algebra over κ. The covariant Dieudonné module $\mathbb{D}_*(\mathfrak{N})$ of $\mathfrak{N}$ is the projective limit of the classical covariant Dieudonné modules of the commutative group schemes representing $\mathfrak{N}/p^n\mathfrak{N}$:

$$\mathbb{D}_*(\mathfrak{N}) := \varprojlim_n \mathbb{D}_*(\mathfrak{N}/p^n\mathfrak{N}).$$

In other words $\mathbb{D}_*(\mathfrak{N})$ is the Dieudonné module of the p-divisible group Y over κ such that $\mathfrak{N} \cong \varprojlim_n Y[p^n]$ in the category of fpqc sheaves on $\mathfrak{Sch}_\kappa$. The Lie bracket on $\mathfrak{N}$ induces a $\Lambda(\kappa)$-bilinear pairing

$$[\ ,\ ] = [\ ,\ ]_{\mathbb{D}_*(\mathfrak{N})} : \mathbb{D}_*(\mathfrak{N}) \times \mathbb{D}_*(\mathfrak{N}) \longrightarrow \mathbb{D}_*(\mathfrak{N})$$

which satisfies the Jacobi identity.

In the above we have followed the notation scheme in [7], denoting the ring of p-adic Witt vectors with entries in κ by $\Lambda(\kappa)$. The Dieudonné modules $\mathbb{D}_*(\mathfrak{N}/p^n\mathfrak{N})$ are left modules over the Dieudonné ring R_κ, which contains $\Lambda(\kappa)$ and elements $\mathtt{F}$ and $\mathtt{V}$. We refer to [24] for an exposition of covariant Dieudonné theory.

(3.3.2) Lemma. *Let $\mathfrak{N}$ be a Tate unipotent Lie $\mathbb{Z}_p$-algebra over κ, and let $\mathbb{D}_*(\mathfrak{N})$ be its Dieudonné module. The following identities*

$$\begin{aligned}
[\mathtt{V}x, \mathtt{V}y]_{\mathbb{D}_*(\mathfrak{N})} &= \mathtt{V}([x,y]_{\mathbb{D}_*(\mathfrak{N})}) \\
[\mathtt{F}x, y]_{\mathbb{D}_*(\mathfrak{N})} &= \mathtt{F}([x,\mathtt{V}y]_{\mathbb{D}_*(\mathfrak{N})}) \\
[x, \mathtt{F}y]_{\mathbb{D}_*(\mathfrak{N})} &= \mathtt{F}([\mathtt{V}x,y]_{\mathbb{D}_*(\mathfrak{N})})
\end{aligned}$$

hold for all $x, y \in \mathbb{D}_*(\mathfrak{N})$.

(3.3.3) Definition. A *connected Dieudonné Lie algebra* over κ is a left module L over the Dieudonné ring R_κ together with a $\Lambda(\kappa)$-bilinear pairing

$$[\ ,\ ] : L \times L \to L$$

such that the following conditions hold.

- L is a free $\Lambda(\kappa)$-module of finite rank.
- All slopes of L are strictly positive.
- The bilinear pairing $[\ ,\ ]$ on L satisfies the Jacobi identity and the identities in 3.3.2.

All connected Dieudonné Lie algebras over κ for (the objects of) an additive category. A morphism from an object L_1 to an object L_2 consists of all R_κ-module homomorphisms $h : L_1 \to L_2$ such that

$$h\big([x,y]_{\mathbb{D}_*(\mathfrak{N}_1)}\big) = [h(x),h(y)]_{\mathbb{D}_*(\mathfrak{N}_2)}, \quad h(\mathtt{F}x) = \mathtt{F}(h(x)), \quad h(\mathtt{V}x) = \mathtt{V}(h(x))$$

for all $x, y \in L_1$.

Lemmas 3.3.4 and 3.3.5 below follow from multilinear Dieudonné theory; see [12, Ch. 2] and [13, 1.2].

(3.3.4) Lemma. *The functor $\mathfrak{N} \rightsquigarrow \mathbb{D}_*(\mathfrak{N})$ from the category of Tate unipotent Lie $\mathbb{Z}_p$-algebras over κ to the category of connected Dieudonné Lie algebras over κ is fully faithful. Explicitly, this means the following.*

Let $\mathfrak{N}_1, \mathfrak{N}_2$ be Tate unipotent Lie $\mathbb{Z}_p$-algebras over κ, and let $\mathbb{D}_*(\mathfrak{N}_i)$ be the Dieudonné module of $\mathfrak{N}_i$, $i = 1, 2$. Denote by $\mathrm{Hom}_\kappa^{\text{Tate unip}}(\mathfrak{N}_1, \mathfrak{N}_2)$ the set consisting of all homomorphisms of Tate unipotent groups over κ from $\mathfrak{N}_1$ to $\mathfrak{N}_2$. Let $\mathrm{Hom}_{R_\kappa}^{\mathrm{Lie}}(\mathbb{D}_*(\mathfrak{N}_1), \mathbb{D}_*(\mathfrak{N}_2))$ be the set consisting of all $\Lambda(\kappa)$-linear homomorphisms $h : \mathbb{D}_*(\mathfrak{N}_1) \to \mathbb{D}_*(\mathfrak{N}_2)$ such that

$$h\big([x,y]_{\mathbb{D}_*(\mathfrak{N}_1)}\big) = [h(x),h(y)]_{\mathbb{D}_*(\mathfrak{N}_2)}, \quad h(\mathtt{F}x) = \mathtt{F}(h(x)), \quad h(\mathtt{V}x) = \mathtt{V}(h(x))$$

for all $x, y \in \mathbb{D}_*(\mathfrak{N}_1)$. The map which send each element $\alpha \in \mathrm{Hom}_\kappa^{\text{Tate unip}}(\mathfrak{N}_1, \mathfrak{N}_2)$ to the element $\mathbb{D}_*(\alpha) \in \mathrm{Hom}_{R_\kappa}^{\mathrm{Lie}}(\mathbb{D}_*(\mathfrak{N}_1), \mathbb{D}_*(\mathfrak{N}_2))$ induced by α is a bijection from $\mathrm{Hom}_\kappa^{\text{Tate unip}}(\mathfrak{N}_1, \mathfrak{N}_2)$ to $\mathrm{Hom}_{R_\kappa}^{\mathrm{Lie}}(\mathbb{D}_*(\mathfrak{N}_1), \mathbb{D}_*(\mathfrak{N}_2))$.

(3.3.5) Lemma. *The covariant Dieudonné functor $\mathbb{D}_*$ induces an equivalence from the category of Tate unipotent Lie $\mathbb{Z}_p$-algebras over κ to the category of connected Dieudonné Lie algebras over κ.*

(3.3.6) Definition. A connected Dieudonné Lie algebra over κ up to isogeny is a left $(R_\kappa \otimes_{\mathbb{Z}} \mathbb{Q})$-module $L_{\mathbb{Q}}$ together with a $\Lambda(\kappa)_{\mathbb{Q}}$-linear Lie bracket $[\ ,\ ]_{L_{\mathbb{Q}}} : L_{\mathbb{Q}} \times L_{\mathbb{Q}} \to L_{\mathbb{Q}}$, such that there exists a connected Dieudonné Lie algebra L over κ, such that $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ as a left module over $R \otimes_{\mathbb{Z}} \mathbb{Q}$, and the Lie bracket $[\ ,\ ]_{L_{\mathbb{Q}}}$ is induced from the Lie bracket $[\ ,\ ]_L : L \times L \to L$ for L.

(3.3.7) Corollary. *The covariant Dieudonné functor $\mathbb{D}_*$ induces an equivalence from the category of Tate unipotent Lie $\mathbb{Q}_p$-algebras over κ to the category of connected Dieudonné Lie algebras over κ up to isogeny.*

(3.4) The group of automorphisms of a Tate unipotent group

(3.4.1) Definition. Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p, and let $\mathbf{N}_{\mathbb{Q}}$ be its Mal'cev completion.

(a) Denote by $\mathrm{Aut}(\mathbf{N})$ the compact p-adic Lie group consisting of all automorphisms of the fpqc sheaf $\mathbf{N}$ of nilpotent groups on $\mathfrak{Sch}_\kappa$.

(b) Denote by $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ the locally compact p-adic Lie group consisting of all automorphisms of the fpqc sheaf $\mathbf{N}_{\mathbb{Q}}$ on $\mathfrak{Sch}_{\kappa}$.

(3.4.2) Lemma. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p, and let $\mathbf{N}_{\mathbb{Q}}$ be the Mal'cev completion of $\mathbf{N}$. Let $\mathfrak{N}_{\mathbb{Q}} = \mathrm{Lie}\,\mathbf{N}_{\mathbb{Q}}$ be the Tate unipotent Lie $\mathbb{Q}_p$-algebra attached to $\mathbf{N}_{\mathbb{Q}}$ via the Mal'cev correspondence.*

(a) *There is a natural embedding $\mathrm{Aut}(\mathbf{N}) \hookrightarrow \mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$, which identifies $\mathrm{Aut}(\mathbf{N})$ as a compact subgroup of the locally compact p-adic Lie group $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$.*

(b) *The p-adic Lie group $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ is naturally isomorphic to the group $\mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}})$ of automorphisms of the Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{N}_{\mathbb{Q}}$.*

(c) *Suppose that the field κ is perfect. Then $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ is naturally isomorphic to the group of all automorphisms α of the left $(R_\kappa \otimes_{\mathbb{Z}} \mathbb{Q})$-module $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$ such that*

$$[\alpha(x), \alpha(y)] = \alpha([x, y]) \quad \forall x, y \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}}).$$

Here $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$ is the connected Dieudonné Lie algebra up to isogeny attached to the Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{N}_{\mathbb{Q}}$.

Proof. The statement (a) follows from the theory of localization of nilpotent groups. The statement (b) follows from the Mal'cev correspondence. The statement (c) follows from (a) and the Dieudonné theory of Tate unipotent Lie algebras. □

Remark. Note that the description of $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ in (c) shows that $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ is the group of $\mathbb{Q}$-points of a linear algebraic group over $\mathbb{Q}_p$.

(3.4.3) Lemma. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let $\mathrm{slope}(N)$ be the set of slopes of $\mathbf{N}$. For each $t \in \mathrm{slope}(\mathbf{N})$, let $\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N} := \mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N}$.*

(a) *The canonical homomorphism*

$$\mathrm{Aut}(\mathbf{N}) \xrightarrow{\gamma} \prod_{t\in\mathrm{slope}(\mathbf{N})} \mathrm{Aut}(\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N})$$

between compact p-adic groups is a closed embedding.

(b) *The canonical homomorphism*

$$\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}) \xrightarrow{\gamma_{\mathbb{Q}}} \prod_{t\in\mathrm{slope}(\mathbf{N}_{\mathbb{Q}})} \mathrm{Aut}(\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N}_{\mathbb{Q}})$$

between locally compact p-adic groups is a closed embedding.

Proof. Omitted. □

(3.4.4) Lemma. *Notation as in* 3.4.2.

(a) *The embedding* $\mathrm{Aut}(\mathbf{N}) \hookrightarrow \mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ *of p-adic Lie groups induces an isomorphism*

$$\mathrm{Lie}(\mathrm{Aut}(\mathbf{N})) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}))$$

of their Lie algebras.

(b) *The Lie algebra* $\mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}))$ *of the p-adic Lie group* $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ *is naturally isomorphic to the Lie* $\mathbb{Q}_p$*-algebra* $\mathrm{Der}(\mathfrak{N}_{\mathbb{Q}})$ *consisting of all derivations of the fpqc sheaf of Lie algebras* $\mathfrak{N}_{\mathbb{Q}}$ *on* $\mathfrak{Sch}_{\kappa}$.

The proofs of 3.4.4 and 3.4.5 are left as an exercise.

(3.4.5) Lemma. *Let* $\mathfrak{N}_{\mathbb{Q}}$ *be a Tate unipotent Lie* $\mathbb{Q}_p$*-algebra. For each* $t \in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}})$*, let* $\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}} := \mathrm{Fil}^t_{\mathrm{sl}}\mathfrak{N}_{\mathbb{Q}}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathfrak{N}_{\mathbb{Q}}$*. Let* $\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}} := \oplus_{t\in\mathrm{slope}(\mathfrak{N}_{\mathbb{Q}})}\mathrm{gr}^t_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}$*, with induced Lie bracket*

$$[\ ,\ ]_{\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}} : (\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}) \times (\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}) \to \mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}.$$

(a) *The canonical homomorphisms*

$$\mathrm{End}_{\mathrm{Lie}}(\mathfrak{N}_{\mathbb{Q}}) \longrightarrow \mathrm{End}_{\mathrm{Lie}}\big(\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}\big)$$

and

$$\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{N}_{\mathbb{Q}}) \longrightarrow \mathrm{Aut}_{\mathrm{Lie}}\big(\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}\big)$$

are injections.

(b) *Suppose that* κ *is a perfect field, and let* $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$ *be the Dieudonné Lie algebra over* κ *up to isogeny associated to* $\mathfrak{N}_{\mathbb{Q}}$.

  (b1) *The Tate unipotent Lie* $\mathbb{Q}_p$*-algebras* $\mathfrak{N}_{\mathbb{Q}}$ *and* $\mathrm{gr}^\bullet_{\mathrm{Fil}_{\mathrm{sl}}}\mathfrak{N}_{\mathbb{Q}}$ *are isomorphic.*

  (b2) *The natural map from* $\mathrm{End}_{\mathrm{Lie}}(\mathfrak{N}_{\mathbb{Q}})$ *(respectively* $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{N}_{\mathbb{Q}})$*) to the set of all endomorphisms (respectively automorphisms)* α *of the left* $(R_\kappa \otimes_{\mathbb{Z}} \mathbb{Q})$*-module* $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$ *such that* $\alpha([x,y]) = [\alpha(x), \alpha(y)]$ *for all* $x, y \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$ *is a bijection.*

  (b3) *The Lie algebra* $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}})$ *of* $\mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}})$ *is naturally isomorphic to the set* $\mathrm{Der}_{\mathrm{Lie}}\big(\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})\big)$ *of all derivations of the connected Dieudonné Lie algebra up to isogeny* $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$*, consisting of all endomorphisms* D *of the left* $(R_\kappa \otimes_{\mathbb{Z}} \mathbb{Q})$*-module* $\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})$ *such that*

$$D([x,y]_{\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})}) = [Dx, y]_{\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})} + [x, Dy]_{\mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}})} \quad \forall\, x, y \in \mathbb{D}_*(\mathfrak{N}_{\mathbb{Q}}).$$

(3.4.6) Euler's flow on Tate unipotent Lie algebras.

Let κ be a field of characteristic p. Let $\mathfrak{N}_{\mathbb{Q}}$ be a Tate unipotent Lie $\mathbb{Q}_p$-algebra over p. *Assume that slope filtration of* $\mathfrak{N}_{\mathbb{Q}}$ *splits*, i.e. there exists $\mathbb{Q}_p$-submodules $\mathfrak{N}_{\mathbb{Q},s}$ of $\mathfrak{N}_{\mathbb{Q}}$ attached to isoclinic p-divisible groups X_s over κ of slope of slope s, where s ranges over $\mathrm{slope}(\mathfrak{N}_{\mathbb{Q}})$, such that

$$\mathfrak{N}_{\mathbb{Q}} = \oplus_{s\in\mathrm{slope}(\mathfrak{N}_{\mathbb{Q}})}\mathfrak{N}_{\mathbb{Q},s}.$$

Then under the Lie bracket of $\mathfrak{N}_{\mathbb{Q}}$, we have

$$[\mathfrak{N}_{\mathbb{Q},s}, \mathfrak{N}_{\mathbb{Q},s'}] \subseteq \mathfrak{N}_{\mathbb{Q},s+s'} \quad \forall\, s, s' \in (0,1],$$

with the convention that $\mathfrak{N}_{\mathbb{Q},s} = (0)$ if $s \notin \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}})$. Note that assumption on $\mathfrak{N}_{\mathbb{Q}}$ holds automatically if the base field κ is perfect.

Let b_0 be the least common multiple of denominators of slopes of $\mathfrak{N}_{\mathbb{Q}}$. Define additive endomorphisms ϕ^{mb_0} of $\mathfrak{N}_{\mathbb{Q}}$ by

$$\phi^{mb_0}\big|_{\mathfrak{N}_{\mathbb{Q},s}} = p^{mb_0 s}\cdot \mathrm{id}_{\mathfrak{N}_{\mathbb{Q},s}} \quad \forall\, s \in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}}).$$

for all $m \in \mathbb{N}$. Then ϕ^{mb_0} is an automorphism of the Tate unipotent Lie algebra $\mathfrak{N}_{\mathbb{Q}}$, and is a linear analog of the mb_0-th iterate of the relative Frobenius map for $\mathfrak{N}_{\mathbb{Q}}$.

Define a derivation D_{Euler} of $\mathfrak{N}_{\mathbb{Q}}$ by

$$D_{\mathrm{Euler}}\big|_{\mathfrak{N}_{\mathbb{Q},s}} = s\cdot \mathrm{id}_{\mathfrak{N}_{\mathbb{Q},s}} \quad \forall\, s \in \mathrm{slope}(\mathfrak{N}_{\mathbb{Q}}).$$

The family of automorphisms ϕ^{mb_0} indexed by $\mathbb{N}$ can be thought of as a discrete version of the flow associated with the derivation D_{Euler}.

(3.4.7) Definition. Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let $\mathfrak{N}_{\mathbb{Q}}$ be the Tate unipotent Lie $\mathbb{Q}_p$-algebra attached to the Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{N}$. Let $\bar{\kappa}$ be an algebraic closure of κ. Let $(\mathfrak{N}_{\mathbb{Q}})_{\bar{\kappa}}$ be the base chage of $\mathfrak{N}_{\mathbb{Q}}$ from $\mathrm{Spec}(\kappa)$ to $\mathrm{Spec}(\bar{\kappa})$. Let G be a p-adic Lie group, and let $\rho : G \to \mathrm{Aut}(\mathbf{N})$ be a continuous group homomorphism. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G, and let $d\rho : \mathfrak{g} \to \mathrm{Lie}(\mathrm{Aut}(\mathbf{N}))$ be the homomorphism of Lie algebras induced by ρ.

We say that the action ρ of G on $\mathbf{N}$ is *strongly nontrivial* if the action of $\mathfrak{g}$ on $\mathbb{D}_*((\mathfrak{N}_{\mathbb{Q}})_{\bar{\kappa}})$ induced by $d\rho$ does not contain the trivial representation of $\mathfrak{g}$ as a subquotient.

Remark. In 3.4.7, the condition that the action ρ on $\mathbf{N}$ is strongly nontrivial is equivalent to the condition that the action of G on the base change $(\mathrm{gr}^t\mathbf{N})_{\bar{\kappa}}$ of $\mathrm{gr}^t\mathbf{N}$ is strongly nontrivial for every slope t of $\mathbf{N}$.

§4. Tate-linear formal varieties

(4.1) Definition. Let κ be a field of characteristic p. Let $\mathfrak{Art}_\kappa$ be the category of augmented artinian local κ-schemes; it is the opposite category of the category whose objects are $(R, j : \kappa \to R, \epsilon^* : R \to \kappa)$, where R is a commutative local ring with 1, j and ϵ^* are unital ring homomorphisms, and $\epsilon^* \circ j = \mathrm{id}_\kappa$. Let $\mathfrak{Art}_{\kappa,\mathrm{fppf}}$ be the site on the category $\mathfrak{Art}_\kappa$ with fppf topology.

Let $\mathbf{N}$ be a Tate unipotent group over κ. Let $\mathbf{N}_{\mathbb{Q}}$ be the Mal'cev completion of $\mathbf{N}$, which is a sheaf on $\mathfrak{Sch}_\kappa$ of uniquely divisible nilpotent groups for the fpqc topology. Consider the sheaf $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ of left $\mathbf{N}$-coset on the site $\mathfrak{Sch}_{\kappa,\mathrm{fpqc}}$. Its restriction to $\mathfrak{Art}_{\kappa,\mathrm{fppf}}$ is a sheaf on $\mathfrak{Art}_{\kappa,\mathrm{fppf}}$, which we denote by $\mathrm{TL}(\mathbf{N})$:

$$\mathrm{TL}(\mathbf{N}) := \big(\mathbf{N}_{\mathbb{Q}}/\mathbf{N}\big)\big|_{\mathfrak{Art}_{\kappa,\mathrm{fppf}}}.$$

Proposition 4.2 below says that the fppf sheaf $\mathrm{TL}(\mathbf{N})$ is represented by a connected smooth formal κ-scheme topologically of finite type over κ. Abusing the notation, we denote this smooth formal κ-scheme again by $\mathrm{TL}(\mathbf{N})$.

(4.1.1) Remark. The sheaves $\mathbf{N}$ and $\mathbf{N}_{\mathbb{Q}}$ do not have many sections over noetherian κ-schemes. In fact $\mathbf{N}(\mathrm{Spec}(R)) = 0 = \mathbf{N}_{\mathbb{Q}}(\mathrm{Spec}(R))$ for any commutative noetherian local κ-algebra R. In contrast the quotient sheaf $\mathbf{N}_{\mathbb{Q}}/\mathbf{N}$ has many points over spectra of artinian κ-algebras, as shown in 4.2 below. The proof of 4.2 depends on 4.2.1, an analog of Hilbert's theorem 90 for p-divisible formal groups.

(4.1.2) Lemma. *Let κ be a field of characteristic p. Let $\mathbf{N}$ be a Tate unipotent group over κ, and let $\mathbf{Z}$ be a Tate unipotent subgroup of $\mathbf{N}$ contained in the center of $\mathbf{N}$ such that the quotient $\mathbf{N}/\mathbf{Z}$ is also a Tate unipotent group over κ. Then the action of the p-divisible group $\mathrm{TL}(\mathbf{Z}) = \mathbf{Z}_{\mathbb{Q}}/\mathbf{Z}$ on $\mathrm{TL}(\mathbf{N})$ induced by the translation action of $\mathbf{Z}$ on $\mathbf{N}$ makes $\mathrm{TL}(\mathbf{N})$ a $\mathrm{TL}(\mathbf{Z})$-torsor over $\mathrm{TL}(\mathbf{N}/\mathbf{Z})$.*

The proof is easy and omitted.

(4.2) Proposition. *We use the notation in* 4.1.

(a) *For any $s, t \in (0, 1]$ with $t \geq s$, the fppf sheaf $\mathrm{TL}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N})$ on $\mathfrak{Art}_{\kappa,\mathrm{fppf}}$, naturally isomorphic to $\Big(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}_{\mathbb{Q}}/\big(\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N}_{\mathbb{Q}} \cdot \mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}\big)\Big)\Big|_{\mathfrak{Art}_{\kappa,\mathrm{fppf}}}$, is represented by a connected augmented smooth formal κ-scheme whose dimension is $\dim(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N})$.*

(b) *For any $s, t_1, t_2 \in (0, 1]$ with $t_1 \geq t_2 \geq s$, the natural map*

$$\pi_{s,t_1,t_2} : \mathrm{TL}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t_1}_{\mathrm{sl}}\mathbf{N}) \to \mathrm{TL}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t_2}_{\mathrm{sl}}\mathbf{N})$$

is represented by a smooth formal κ-morphism between smooth formal κ-schemes. In particular π_{s,t_1,t_2} induces a surjection

$$\mathrm{TL}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t_1}_{\mathrm{sl}}\mathbf{N})(R) \twoheadrightarrow \mathrm{TL}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t_2}_{\mathrm{sl}}\mathbf{N})(R)$$

for every artinian local κ-algebra R.

Proof. The assertion that the fpqc sheaves

$$\mathrm{TL}\big(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N}\big) \quad \text{and} \quad \Big(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}_{\mathbb{Q}}/\big(\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N}_{\mathbb{Q}}\big)\cdot \mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}\big)\Big)\Big|_{\mathfrak{Art}_{\kappa,\mathrm{fppf}}}$$

are naturally isomorphic follows from the exactness of localization functors on the category of nilpotent groups.

The rest of the statement (a) is proved by induction on $\mathrm{card}(\mathrm{slope}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N}))$. The case when $\mathrm{card}(\mathrm{slope}(\mathrm{Fil}^s_{\mathrm{sl}}\mathbf{N}/\mathrm{Fil}^{>t}_{\mathrm{sl}}\mathbf{N})) = 1$ is obvious. To simplify the notation, we may and do assume that $s = \min(\mathrm{slope}(\mathbf{N}))$, and $t = \max(\mathrm{slope}(\mathbf{N}))$. Consider the short exact sequence

$$1 \longrightarrow \mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N} \longrightarrow \mathbf{N} \longrightarrow \mathbf{N}/\mathrm{Fil}^t\mathbf{N} \longrightarrow 1,$$

with $\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}$ contained in the center of $\mathbf{N}$, and the associated short exact sequence

$$1 \longrightarrow \mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}_{\mathbb{Q}} \longrightarrow \mathbf{N}_{\mathbb{Q}} \longrightarrow (\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})_{\mathbb{Q}} \longrightarrow 1.$$

Since $\mathrm{TL}(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})$ is naturally isomorphic to $\mathbf{N}_{\mathbb{Q}}/(\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}_{\mathbb{Q}}\cdot\mathbf{N})$, the natural map

$$\pi : \mathrm{TL}(\mathbf{N}) \longrightarrow \mathrm{TL}(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})$$

has a natural structure as a Y_t-torsor, where Y_t is the p-divisible group over κ such that $\varprojlim_n Y_t[p^n] \cong \mathrm{gr}_{\mathrm{Fil}_{\mathrm{sl}}}\mathbf{N}$. To show that $\mathrm{TL}(\mathbf{N})$ is represented by an augmented formal scheme of finite type over κ, it suffices to show that for every augmented artinian local scheme S over κ and every κ-morphism $z : S \to \mathrm{TL}(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})$, the fpqc sheaf

$$\mathcal{L}_z := (\mathbf{N}_{\mathbb{Q}}/\mathbf{N}) \times_{((\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})_{\mathbb{Q}}/(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}),\, z)} S$$

over S has a section over S, and is representable by a formal scheme over S. Lemma 4.2.1 below shows that $\mathcal{L}_z$ has a section over S. That $\mathcal{L}_z$ is representable by a trivial Y_t-torsor follows from fpqc descent. We have proved the statement (a).

The statement (b) follows from (a), because π_{s,t_1,t_2} is a formal morphism between smooth formal κ-schemes topologically of finite type over κ, which induces a surjection on tangent spaces. $\square$

(4.2.1) Lemma. *Let κ be a field of characteristic p. Let X be a p-divisible formal group over κ, i.e. all slopes of X are strictly positive. Let $(R, \mathfrak{m})$ be a commutative artinian local κ-algebra. For every $i \geq 1$, we have*

$$\mathrm{H}^i_{\mathrm{fpqc}}(\mathrm{Spec}(R), X) = \mathrm{H}^i_{\mathrm{fppf}}(\mathrm{Spec}(R), X) = \mathrm{H}^i_{\mathrm{et}}(\mathrm{Spec}(R), X) = \mathrm{H}^i_{\mathrm{zar}}(\mathrm{Spec}(R), X) = 0.$$

In particular, every X-torsor over $\mathrm{Spec}(R)$ *for the fpqc topology admits a section over* $\mathrm{Spec}(R)$.

Proof. This statement is "well-known" for $\mathrm{H}^i_{\mathrm{et}}(\mathrm{Spec}(R), X)$ and $\mathrm{H}^i_{\mathrm{zar}}(\mathrm{Spec}(R), X)$. We give a proof that $\mathrm{H}^i_{\mathrm{fpqc}}(\mathrm{Spec}(R), X) = 0$. The rest can be proved by the same argument.

Let n_0 be the smallest natural number such that $\mathfrak{m}^{n_0+1} = 0$. For each $j = 0, 1, \ldots, n_0$, we have a natural closed embedding $\iota_j : \mathrm{Spec}(R/\mathfrak{m}^{j+1}) \hookrightarrow \mathrm{Spec}(R)$. The sheaf $X = \varinjlim_n X[p^n]$ on the category of all κ-schemes with respect to the fpqc topology has a decreasing filtration

$$\mathrm{Fil}^j X = \mathrm{Ker}\big(X \longrightarrow \iota_{j*}\iota_j^* X\big), \quad j = 0, 1, \ldots, n_0$$

with $\mathrm{Fil}^{n_0} = 0$. Moreover we have natural isomorphisms

$$\mathrm{Fil}^j X/\mathrm{Fil}^{j+1} X \cong \mathcal{O}_{\mathrm{Spec}(R)} \otimes_R (\mathfrak{m}^j/\mathfrak{m}^{j+1}), \quad j = 0, 1, \ldots, n_0.$$

Since the cohomology group $\mathrm{H}^i_{\mathrm{fpqc}}\big(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)} \otimes_R (\mathfrak{m}^j/\mathfrak{m}^{j+1})\big)$ with coefficient sheaf associated to the coherent $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\mathcal{O}_{\mathrm{Spec}(R)} \otimes_R (\mathfrak{m}^j/\mathfrak{m}^{j+1})$ is zero for every $i \geq 1$, we conclude that $\mathrm{H}^i_{\mathrm{fpqc}}(\mathrm{Spec}(R), X) = 0$ for all $i \geq 1$. □

(4.2.2) Definition. Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Denote by $\mathcal{D}ef_{\mathcal{T}or}(\mathbf{N})$ the deformation functor of the trivial *right* $\mathbf{N}$-torsor on the category $\mathfrak{Art}_\kappa$ of augmented artinian local κ-schemes.

(4.2.3) Proposition. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. The natural map*

$$\delta_{\mathbf{N}} : \mathrm{TL}(\mathbf{N}) \longrightarrow \mathcal{D}ef_{\mathcal{T}or}(\mathbf{N})$$

is an isomorphism of formal schemes over κ. In other words for every augmented artinian local κ-scheme S, $\delta_{\mathbf{N}}$ induces a bijection $\mathrm{TL}(\mathbf{N})(S) \xrightarrow{\sim} \mathcal{D}ef_{\mathcal{T}or}(\mathbf{N})(S)$*..*

Proof. Suppose first that $\mathrm{card}(\mathrm{slope}(\mathbf{N})) = 1$, i.e. $\mathbf{N}$ is isomorphic to $\varprojlim_n X[p^n]$ for an isoclinic p-divisible formal group X over κ. Consider the exact sequence

$$\mathrm{H}^0(S, \mathbf{N}_{\mathbb{Q}}) \longrightarrow \mathrm{TL}(\mathbf{N})(S) \xrightarrow{\delta_{\mathbf{N}}} \mathrm{H}^1_{\mathrm{fpqc}}(S, \mathbf{N}) \longrightarrow \mathrm{H}^1_{\mathrm{fpqc}}(S, \mathrm{TL}(\mathbf{N}))$$

attached to the short exact sequence

$$0 \longrightarrow \mathbf{N} \longrightarrow \mathbf{N}_{\mathbb{Q}} \longrightarrow \mathrm{TL}(\mathbf{N}) \longrightarrow 0$$

of fpqc sheaves of abelian groups on $\mathrm{Spec}(\kappa)$. We know from 4.2.1 that

$$\mathrm{H}^1_{\mathrm{fpqc}}(S, \mathrm{TL}(\mathbf{N})) \cong \mathrm{H}^1_{\mathrm{fpqc}}(S, X) = (0).$$

We also know that

$$\mathrm{H}^0(S, \mathbf{N}) \cong \mathrm{Hom}_S(\mathbb{Q}_p/\mathbb{Z}_p, X) = 0$$

because $\mathrm{Hom}_\kappa(\mathbb{Q}_p/\mathbb{Z}_p, X) = 0$. We have shown that 4.2.3 holds when $\mathrm{card}(\mathrm{slope}(\mathbf{N})) = 1$.

The general case is proved by an easy induction on card(slope($\mathbf{N}$)). Let $t = \max(\mathrm{slope}(\mathbf{N}))$. Consider the short exact sequence $1 \to \mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N} \to \mathbf{N} \to \mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N} \to 1$ and the commutative diagram

$$\begin{tikzcd}
\mathrm{TL}(\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})(S) \arrow[r] \arrow[d, "\delta_{\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N},S}"] & \mathrm{TL}(\mathbf{N})(S) \arrow[r] \arrow[d, "\delta_{\mathbf{N},S}"] & \mathrm{TL}(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})(S) \arrow[d, "\delta_{\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N},S}"] \\
\mathcal{D}ef_{\mathcal{T}or}\left(\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}\right)(S) \arrow[r] & \mathcal{D}ef_{\mathcal{T}or}\left(\mathbf{N}\right)(S) \arrow[r] & \mathcal{D}ef_{\mathcal{T}or}\left(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}\right)(S).
\end{tikzcd}$$

For the top row of the above diagram, 4.2 (b) implies that the inverse image of every element of $\mathrm{TL}(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})(S)$ under the map $\mathrm{TL}(\mathbf{N})(S) \to \mathrm{TL}(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})(S)$ is a torsor for the commutative group $\mathrm{TL}(\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N})$. For the bottom row, the inverse image of every element of $\mathcal{D}ef_{\mathcal{T}or}\left(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}\right)(S)$ under the map $\mathcal{D}ef_{\mathcal{T}or}\left(\mathbf{N}\right)(S) \to \mathcal{D}ef_{\mathcal{T}or}\left(\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}\right)(S)$ is either empty or is a torsor for the commutative group $\mathcal{D}ef_{\mathcal{T}or}\left(\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}\right)(S) \cong Y_t(S)$, where Y_t is a p-divisible group over κ such that $\varprojlim_n Y_t[p^n] \cong \mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N}$. We have seen that the vertical arrow $\delta_{\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N},S}$ is bijective, while the vertical arrow $\delta_{\mathbf{N}/\mathrm{Fil}^t_{\mathrm{sl}}\mathbf{N},S}$ is bijective by induction. It follows that $\delta_{\mathbf{N},S}$ is a bijection. □

(4.2.4) Definition. Let κ be a field of characteristic p.

(i) The *Tate-linear formal variety* attached to a Tate unipotent group $\mathbf{N}$ over κ is the smooth formal scheme T over κ which represents the (restriction to $\mathfrak{Art}_\kappa$ of the) fpqc sheaf $\mathrm{TL}(\mathbf{N})$ on $\mathfrak{Sch}_\kappa$. We will abuse notation and use $\mathrm{TL}(\mathbf{N})$ to denote the formal scheme representing it, if no confusion is likely.

(ii) A Tate-linear formal variety over κ is a pair $\left(T, \zeta : T \xrightarrow{\sim} \mathrm{TL}(\mathbf{N})\right)$, where T is a formal scheme over κ, $\mathbf{N}$ is a Tate unipotent group over κ, and ζ is an isomorphism of formal schemes over κ. The isomorphism ζ is said to be a Tate-linear structure on the formal scheme T.

Remark. A Tate-linear formal variety T in 4.2.4 is a homogeneous space under $\mathbf{N}$ in the category of fpqc sheaves on $\mathfrak{Sch}_\kappa$. From this perspective, Tate-linear formal varieties are analogous to compact nilmanifolds.

(4.3) Biextensions of p-divisible formal groups as Tate-linear formal varieties.

Let κ be a field of characteristic p. We will show that every biextension of p-divisible formal groups over κ is a Tate-linear formal variety attached to a Tate unipotent group with nilpotency class at most 2. The construction also shows that every Tate-linear formal variety attached to Tate unipotent groups of nilpotency class at most 2 is isogenous to a biextension of p-divisible formal groups, if the base field κ is perfect.

(4.3.1) We set up notation for the rest of 4.3. Let $\mathbf{N}$ be a Tate unipotent group over κ, which contains *commutative* Tate unipotent subgroups $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, such that $\mathbf{N}$ is a central extension of $\mathbf{X}\times\mathbf{Y}$ by $\mathbf{Z}$ which is split over $\mathbf{X}$ and also over $\mathbf{Y}$. In other words the following conditions are satisfied.

(i) $\mathbf{Z}$ is contained in the center of $\mathbf{N}$, and $\mathbf{N}/\mathbf{Z}$ is commutative.

(ii) The maps $\mathbf{Z}\times\mathbf{X}\to\mathbf{Z}\cdot\mathbf{X}$ and $\mathbf{Z}\times\mathbf{Y}\to\mathbf{Z}\cdot\mathbf{Y}$ given by the group law of $\mathbf{N}$ are isomorphisms. Moreover the subgroups $\mathbf{Z}\cdot\mathbf{X}$ and $\mathbf{Z}\cdot\mathbf{Y}$ of $\mathbf{N}$ are both normal.

(iii) $\mathbf{N}$ is a semi-direct product of $\mathbf{Z}\times\mathbf{X}$ with $\mathbf{Y}$, and also a semi-direct product of $\mathbf{Z}\times\mathbf{X}$ with $\mathbf{Y}$

Passing to the Mal'cev completions, we see that the Mal'cev completion $\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{N}$ is a central extension of $\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}}$ by $\mathbf{Z}_{\mathbb{Q}}$ which is split over $\mathbf{X}_{\mathbb{Q}}$ and also over $\mathbf{Y}_{\mathbb{Q}}$. Let $q:\mathbf{N}\twoheadrightarrow\mathbf{X}\times\mathbf{Y}$ and $q_{\mathbb{Q}}:\mathbf{N}_{\mathbb{Q}}\twoheadrightarrow\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}}$ be the quotient maps of the central extension $\mathbf{N}$ of $\mathbf{X}\times\mathbf{Y}$ by $\mathbf{Z}$ (respectively $\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}}$ by $\mathbf{Z}_{\mathbb{Q}}$). The group commutator

$$[\ ,\ ]_{\mathrm{grp},\mathbf{N}_{\mathbb{Q}}}:\mathbf{N}_{\mathbb{Q}}\times\mathbf{N}_{\mathbb{Q}}\longrightarrow\mathbf{N}_{\mathbb{Q}},\quad [\mathbf{n}_1,\mathbf{n}_2]_{\mathrm{grp}}=\mathbf{n}_1^{-1}\mathbf{n}_2^{-1}\mathbf{n}_1\mathbf{n}_2\ \ \forall\,\mathbf{n}_1,\mathbf{n}_2\in\mathbf{N}_{\mathbb{Q}}$$

on $\mathbf{N}_{\mathbb{Q}}$ induces a skew-symmetric bilinear pairing

$$\langle\ ,\ \rangle_{\mathbf{N}_{\mathbb{Q}}}:(\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}})\times(\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}})\longrightarrow\mathbf{Z}_{\mathbb{Q}}$$

such that the diagram

$$\begin{tikzcd}
\mathbf{N}_{\mathbb{Q}}\times\mathbf{N}_{\mathbb{Q}} \arrow[r, "{[\ ,\ ]_{\mathrm{grp},\mathbf{N}_{\mathbb{Q}}}}"] \arrow[d, "q_{\mathbb{Q}}\times q_{\mathbb{Q}}"] & \mathbf{N}_{\mathbb{Q}} \\
(\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}})\times(\mathbf{X}_{\mathbb{Q}}\times\mathbf{Y}_{\mathbb{Q}}) \arrow[r, "\langle\ ,\ \rangle_{\mathbf{N}_{\mathbb{Q}}}"] & \mathbf{Z}_{\mathbb{Q}} \arrow[u, hook]
\end{tikzcd}$$

commutes. Let $X:=\mathrm{TL}(\mathbf{X})=\mathbf{X}_{\mathbb{Q}}/\mathbf{X}$, $Y:=\mathrm{TL}(\mathbf{Y})=\mathbf{Y}_{\mathbb{Q}}/\mathbf{Y}$ and $Z:=\mathrm{TL}(\mathbf{Z})=\mathbf{Z}_{\mathbb{Q}}/\mathbf{Z}$ be the p-divisible groups corresponding to $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ respectively. Group laws on $\mathbf{X},\mathbf{Y},\mathbf{Z}$ and X,Y,Z will be written *additively*, to conform with the usual notation for p-divisible groups and biextensions.

Let $\pi:\mathrm{TL}(\mathbf{N})\to\mathrm{TL}(\mathbf{X}\times\mathbf{Y})$ be the morphism attached to the quotient maps q and $q_{\mathbb{Q}}$. Let $E:=\mathrm{TL}(\mathbf{N})$, so that $\pi:E\to X\times Y$ has a natural structure as a Z-torsor.

(4.3.2) We use the notation in 4.3.1. The Z-torsor structure on E over $X\times Y$ admits a natural enhancement to a biextension of (X,Y) by Z, with the relative group laws $+_1: E\times_Y E\to E$ and $+_2:E\times_X E\to E$ defined as follows.[1]

[1] Here $+_1$ is "addition along the first set of variables", while $+_2$ is "addition along the second set of variables". This is the convention in [21, pp. 320–321], where Mumford explains the construction of the Weil pairings of biextensions of p-divisible groups. Unfortunately this convention is *opposite* to the convention in [21, pp. 310–311], where $+_1$ and $+_2$ denote the group laws relative to the first and second factor of the base respectively.

(1a) Given a functorial point y of Y, pick a functorial point $\mathbf{y}_0$ on $\mathbf{Y}_{\mathbb{Q}}$ lifting y. The fiber $(\mathrm{pr}_2 \circ \pi)^{-1}(y)$ of E over y consists of all elements of the form $[\mathbf{z} \cdot \mathbf{x} \cdot \mathbf{y}_0]$, where $\mathbf{z}$ is a functorial point of $\mathbf{Z}_{\mathbb{Q}}$, $\mathbf{x}$ is a functorial point of $\mathbf{Z}_{\mathbb{Q}}$, and $[\mathbf{z} \cdot \mathbf{x} \cdot \mathbf{y}_0]$ is the image of $\mathbf{z} \cdot \mathbf{x} \cdot \mathbf{y}_0$ in $\mathrm{TL}(\mathbf{N}) = E$. One checks that two functorial points $[\mathbf{z}_1 \cdot \mathbf{x}_1 \cdot \mathbf{y}_0]$ and $[\mathbf{z}_2 \cdot \mathbf{x}_2 \cdot \mathbf{y}_0]$ of E_y over the same κ-scheme are equal if and only

$$\mathbf{x}_1 - \mathbf{x}_2 \in \mathbf{X} \quad \text{and} \quad \langle \mathbf{x_1} - \mathbf{x}_2, \mathbf{y}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} + \mathbf{z}_1 - \mathbf{z}_2 \in \mathbf{Z}.$$

Here $\mathbf{z}_i \cdot \mathbf{x}_i \cdot \mathbf{y}_0$ refers to product according to the group law on $\mathbf{N}_{\mathbb{Q}}$, $i = 1, 2$, and $\langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{y}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} + \mathbf{z}_1 - \mathbf{z}_2$ is a shorthand version of $\langle \mathbf{x}_1 -_{\mathbf{X}_{\mathbb{Q}}} \mathbf{x}_2, \mathbf{y}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} +_{\mathbf{Z}_{\mathbb{Q}}} \mathbf{z}_1 -_{\mathbf{Z}_{\mathbb{Q}}} \mathbf{z}_2$.

For any two functorial points $[\mathbf{z}_1 \cdot \mathbf{x}_1 \cdot \mathbf{y}_0]$ and $[\mathbf{z}_2 \cdot \mathbf{x}_2 \cdot \mathbf{y}_0]$ of E_y over the same κ-scheme, define their sum under $+_1$ by

$$[\mathbf{z}_1 \cdot \mathbf{x}_1 \cdot \mathbf{y}_0] +_1 [\mathbf{z}_2 \cdot \mathbf{x}_2 \cdot \mathbf{y}_0] := [(\mathbf{z}_1 + \mathbf{z}_2) \cdot (\mathbf{x}_1 + \mathbf{x}_2) \cdot \mathbf{y}_0].$$

It is not difficult to verify that this gives a well-defined morphism $+_1 : E \times_Y E$ to E.

(1b) It follows directly from the definition in (1a) that for each functorial point y of Y, the sheaf of commutative groups $(\mathrm{pr}_2 \circ \pi)^{-1}(y)$ under the group law $+_1$ is the push-out of the top row by the vertical arrow ξ_y in the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{X} \arrow[r] \arrow[d, "\xi_y"'] & \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{X}_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbf{X}_{\mathbb{Q}}/\mathbf{X} = X \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & Z \arrow[r] & (\mathrm{pr}_2 \circ \pi)^{-1}(y) \arrow[r] & X \arrow[r] & 1\,.
\end{tikzcd}$$

with exact rows. Here $\xi_y : \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{X} \to Z$ is the group homomorphism given by

$$\xi_y(\mathbf{z} \cdot \mathbf{x}) = \langle \mathbf{x}, \mathbf{y}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} \bmod \mathbf{Z} \qquad \forall\, \mathbf{z} \in \mathbf{Z}_{\mathbb{Q}},\ \forall\, \mathbf{x} \in \mathbf{X},$$

where $\mathbf{y}_0$ is a representative of y in $\mathbf{X}_{\mathbb{Q}}$. Clearly ξ_y depends only on $y \in \mathbf{Y}_{\mathbb{Q}}/\mathbf{Y}$, and not on the representative $\mathbf{y}_0$ of y.

(2a) Similarly, given any functorial point x of X, pick a representative $\mathbf{x}_0$ of x in $\mathbf{X}_{\mathbb{Q}}$. Points of the fiber $(\mathrm{pr}_1 \circ \pi)^{-1}(x)$ of $E \to X \times Y$ over y consists of all points of the form $[\mathbf{z} \cdot \mathbf{y} \cdot \mathbf{x}_0]$, where $\mathbf{z}$ (respectively $\mathbf{y}$) runs through all points of $\mathbf{Z}_{\mathbb{Q}}$ (respectively $\mathbf{Y}_{\mathbb{Q}}$). Two points $[\mathbf{z}_1 \cdot \mathbf{y}_1 \cdot \mathbf{x}_0]$ and $[\mathbf{z}_2 \cdot \mathbf{y}_2 \cdot \mathbf{x}_0]$ are equal if and only if

$$\mathbf{y}_1 - \mathbf{y}_2 \in \mathbf{Y} \quad \text{and} \quad \mathbf{z}_1 - \mathbf{z}_2 + \langle \mathbf{y}_1 - \mathbf{y}_2, \mathbf{x}_0 \rangle_{\mathbf{N}_{\mathbb{Q}}} \in \mathbf{Z}.$$

The sum of two points $[\mathbf{z}_1 \cdot \mathbf{y}_1 \cdot \mathbf{x}_0]$ and $[\mathbf{z}_2 \cdot \mathbf{y}_2 \cdot \mathbf{x}_0]$ of $(\mathrm{pr}_1 \circ \pi)^{-1}(x)$ under $+_2$ is defined as

$$[\mathbf{z}_1 \cdot \mathbf{y}_2 \cdot \mathbf{x}_0] +_2 [\mathbf{z}_1 \cdot \mathbf{y}_2 \cdot \mathbf{x}_0] := [(\mathbf{z}_1 + \mathbf{z}_2) \cdot (\mathbf{y}_1 + \mathbf{y}_2) \cdot \mathbf{x}_0].$$

(2b) The sheaf of commutative groups $(\mathrm{pr}_1 \circ \pi)^{-1}(x)$ in (2a) is an extension of Y by Z, which sits in the push-out diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{Y} \arrow[r] \arrow[d, "\eta_x"'] & \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{Y}_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathbf{Y}_{\mathbb{Q}}/\mathbf{Y} = Y \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & Z \arrow[r] & (\mathrm{pr}_1 \circ \pi)^{-1}(x) \arrow[r] & Y \arrow[r] & 1\,,
\end{tikzcd}$$

where $\eta_x : \mathbf{Z}_{\mathbb{Q}} \cdot \mathbf{Y} \to Z$ is given by

$$\eta_x(\mathbf{z} \cdot \mathbf{y}) = \langle \mathbf{y}, \mathbf{x}_0 \rangle \bmod \mathbf{Z} = -\langle \mathbf{x}_0, \mathbf{y} \rangle \bmod \mathbf{Z} \quad \forall\, \mathbf{z} \in \mathbf{Z}_{\mathbb{Q}},\ \forall\, \mathbf{y} \in \mathbf{Y}.$$

Logically, one still needs to check that the two relative group laws $+_1, +_2$ are compatible, to complete the proof that $(\pi : E \to X \times Y, +_1, +_2)$ is a biextension of (X, Y) by Z. The require compatibility condition is this: for any functorial points x_1, x_2 of X, y_1, y_2 of Y, and points u_{ij} of E above (x_i, y_j),

$$(u_{1,1} +_1 u_{2,1}) +_2 (u_{1,2} +_1 u_{2,2}) = (u_{1,1} +_2 u_{1,2}) +_1 (u_{2,1} +_2 u_{2,2})$$

holds. The verification is straight forward and left to the readers.

(4.3.3) To identify the biextension $\pi : E \to X \times Y$ of (X, Y) by Z constructed above, we need to compute its Weil pairings

$$\beta_n : X[p^n] \times Y[p^n] \longrightarrow Z[p^n], \quad n \geq 1,$$

since every biextension can be reconstructed from its Weil pairings. There is of course a choice of sign in the definition of the Weil pairings of a biextension. We will follow the convention in [21]. We recall the recipe in [21, pp. 320–321].

(a) For each positive integer n, construct a canonical splitting

$$\phi_n : X[p^n] \times Y[p^{2n}] \longrightarrow (1_X \times [p^n]_Y)^* E|_{(X[p^n] \times Y[p^{2n}])}$$

of the restriction to $X[p^n] \times Y[p^{2n}]$ of $(1_X \times [p^n]_Y)^* E$, a biextension of $(X[p^n], Y[p^{2n}])$ by Z, as follows.

- Given any pair (x_n, y_{2n}) of functorial points of $X[p^n] \times Y[p^{2n}]$, after passing to a suitable finite faithfully flat cover of the base scheme of (x_n, y_{2n}), one chooses a functorial point $u_n \in E_{(x_n, y_{2n})}$ such that $[p^n]_{+1}(u_n)$ is equal to $\epsilon_{+_1}(y_{2n})$, where $\epsilon_{+_1} : Y \to E$ is the 0-section of the relative group law $+_1$. Clearly u_n is unique up to translations by elements of $Z[p^n]$.
- Define $\phi_n(x_n, y_{2n})$ by
$$\phi_n(x_n, y_{2n}) = +_2(u_n).$$

Clearly $\phi_n(x_n, y_{2n})$ is a well-defined point of E above $(x_n, p^n y_{2n})$. One easily verifies that the map

$$\phi_n : X[p^n] \times Y[p^{2n}] \to (1_X \times [p^n]_Y)^* E|_{(X[p^n]\times Y[p^{2n}])}$$

is bi-additive.

(b) For each $b_n \in Y[p^n]$, there exists a unique element $\beta_n(x_n, y_n, b_n) \in Z$ such that

$$\beta_n(x_n, y_n, b_n) * \phi_n(x_n, y_n + b_n) = \phi_n(x_n, y_n),$$

where $\beta_n(x_n, y_n, b_n) * \phi_n(x_n, y_n + b_n)$ is the translation of $\phi_n(x_n, y_n + b_n)$ by the point $\beta_n(x_n, y_n, b_n)$ for the Z-torsor structure of E. It turns out the $\beta_n(x_n, y_n, b_n)$ is independent of y_n, and descends to a bilinear map

$$\beta_n : X[p^n] \times Y[p^n] \longrightarrow Z[p^n].$$

Remark. The splittings ϕ_n above are constructed through normalizations using the group law $+_1$. Reversing the roles of $+_1$ and $+_2$, one can construct canonical splittings ψ_n of $([p^n]_X \times 1_Y)^* E|_{(X[p^{2n}]\times Y[p^n])}$, and define bilinear pairings $\gamma_n : X[p^n] \times Y[p^n] \to Z[p^n]$ by $\gamma_n(a_n, y_n) * \psi_n(x_{2n} + a_n, y_n) = \psi_n(x_{2n}, y_n)$ for all $x_{2n} \in X[p^{2n}]$, all $a_n \in X[p^n]$ and all $y_n \in Y[p^n]$. It turns out that $\gamma_n = -\beta_n$ on $X[p^n] \times Y[p^n]$.

(4.3.4) Going back to the biextension $\pi : E = \mathrm{TL}(\mathbf{N}) \to X \times Y$ attached to the Tate unipotent group $\mathbf{N}$, we want to compute its Weil pairings $\beta_n : X[p^n] \times Y[p^n] \to Z[p^n]$. Of course one suspect that these pairings are likely related to the skew symmetric pairing $\langle\ ,\ \rangle_{\mathbf{N}_\mathbb{Q}} : \mathbf{X}_\mathbb{Q} \times \mathbf{Y}_\mathbb{Q} \to \mathbf{Z}_\mathbb{Q}$, but there is an obvious question about the sign.

First we compute the canonical splittings. Given any functorial points (x_n, y_{2n}) of E above $X[p^n] \times Y[p^{2n}]$, choose liftings $\mathbf{x}_n$ of x_n and $\mathbf{y}_{2n}$ of y_{2n} in $p^{-n}\mathbf{X}$ and $p^{-2n}\mathbf{Y}$ respectively, after passing to a suitable fpqc cover of the base scheme of (x_n, y_{2n}). Passing to a higher fpqc cover if necessary, one chooses a functorial point $\mathbf{z}_n$ of $\mathbf{Z}_\mathbb{Q}$ such that $[\mathbf{z}_n \cdot \mathbf{x}_n \cdot \mathbf{y}_{2n}]$ is normalized with respect to $+_1$, in the sense that $[p^n]_{+_1}([\mathbf{z}_n \cdot \mathbf{x}_n \cdot \mathbf{y}_{2n}]) = [p^n\mathbf{z}_n \cdot p^n\mathbf{x}_n \cdot \mathbf{y}_{2n}]$ is the 0-element of the group $\big((\mathrm{pr}_2 \circ \pi)^{-1}(y), +_1\big)$. This means that

$$\langle p^n\mathbf{x}_n, \mathbf{y}_{2n}\rangle_{\mathbf{N}_\mathbb{Q}} +_{\mathbf{Z}_\mathbb{Q}} p^n\mathbf{z}_n \in \mathbf{Z}.$$

By definition,

$$\begin{aligned}
\phi_n(x_n, y_{2n}) &= [p^n]_{+_2}([\mathbf{z}_n \cdot \mathbf{x}_n \cdot \mathbf{y}_{2n}]) = [p^n]_{+_2}\big([\,(\mathbf{z}_n + \langle \mathbf{x}_n, \mathbf{y}_{2n}\rangle_{\mathbf{N}_\mathbb{Q}}) \cdot \mathbf{y}_{2n} \cdot \mathbf{x}_n\,]\big) \\
&= \big[\big(p^n\mathbf{z}_n + \langle p^n\mathbf{x}_n, \mathbf{y}_{2n}\rangle_{\mathbf{N}_\mathbb{Q}}\big) \cdot p^n\mathbf{y}_{2n} \cdot \mathbf{x}_n\big] \\
&= \big[\,(-\langle p^n\mathbf{x}_n, \mathbf{y}_{2n}\rangle) \cdot \mathbf{x}_n \cdot p^n\mathbf{y}_{2n}\,\big]\,.
\end{aligned}$$

Notice that the last expression, $\big[\,(-\langle p^n\mathbf{x}_n, \mathbf{y}_{2n}\rangle) \cdot \mathbf{x}_n \cdot p^n\mathbf{y}_{2n}\,\big]$, indeed depends only on (x_n, y_n). If follows immediately that for any functorial point (x_n, b_n) of $X[p^n] \times Y[p^n]$, we have

$$\beta_n(x_n, b_n) = p^n\langle \mathbf{x}_n, \mathbf{b}_n\rangle_{\mathbf{N}_\mathbb{Q}}$$

for all liftings $(\mathbf{x}_n, \mathbf{b}_n)$ in $(p^{-n}\mathbf{X}) \times (p^{-n}\mathbf{Y})$ of (x_n, b_n). We state this result in 4.3.5 below.

(4.3.5) Proposition. *Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $\mathbf{X} = \varprojlim_n X[p^n]$, $\mathbf{Y} = \varprojlim_n Y[p^n]$, $\mathbf{Z} = \varprojlim_n Z[p^n]$ be the commutative Tate unipotent groups attached to X, Y, Z respectively. Let $\mathbf{N}$ be a Tate unipotent group over κ which is a central extension of $\mathbf{X} \times \mathbf{Y}$ of $\mathbf{Y}$ split over $\mathbf{X}$ and $\mathbf{Y}$ as specified in* 4.3.1. *Let $E \to X \times Y$ be the biextension of (X, Y) by Z attached to $E = \mathrm{TL}(\mathbf{N})$. Then the Weil pairing*

$$\beta_n : X[p^n] \times Y[p^n] \longrightarrow Z[p^n], \quad n \geq 1$$

of the biextension E is given by

$$\beta_n(x_n, y_n) = p^n \langle \mathbf{x}_n, \mathbf{y}_n \rangle_{\mathbf{N}_{\mathbb{Q}}}$$

for all functorial points (x_n, y_n) of $X[p^n] \times Y[p^n] \cong (p^{-n}\mathbf{X}/\mathbf{X}) \times (p^{-n}\mathbf{Y}/\mathbf{Y})$ and any lifting $(\mathbf{x}_n, \mathbf{y}_n)$ of (x_n, y_n) in $(p^{-n}\mathbf{X}) \times (p^{-n}\mathbf{Y})$, with values in an fpqc cover of the base scheme of (x_n, y_n). Here $\langle\ ,\ \rangle_{\mathbf{N}_{\mathbb{Q}}} : \mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}} \to \mathbf{Z}_{\mathbb{Q}}$ is the $\mathbb{Q}_p$-bilinear map

$$(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}, \mathbf{y}]_{\mathrm{grp}, \mathbf{N}_{\mathbb{Q}}} \quad \mathbf{x} \in \mathbf{X}_{\mathbb{Q}},\ \mathbf{y} \in \mathbf{Y}_{\mathbb{Q}}$$

from $\mathbf{X}_{\mathbb{Q}} \times \mathbf{Y}_{\mathbb{Q}}$ to $\mathbf{Z}_{\mathbb{Q}}$.

(4.3.6) Corollary. *Let X, Y, Z be p-divisible groups over a field κ of characteristic p. Let $\mathbf{X} = \varprojlim_n X[p^n]$, $\mathbf{Y} = \varprojlim_n Y[p^n]$, $\mathbf{Z} = \varprojlim_n Z[p^n]$ be the commutative Tate unipotent groups attached to X, Y, Z respectively. Given any biextension $\pi : E \to X \times Y$, there exists a Tate unipotent group $\mathbf{N}$ over κ which is a central extension $\mathbf{N}$ of $\mathbf{X} \times \mathbf{Y}$ by $\mathbf{Z}$ which is split over $\mathbf{X}$ and $\mathbf{Y}$, such that the biextension $\mathrm{TL}(\mathbf{N}) \to X \times Y$ of (x, Y) by Z is isomorphic to $\pi : E \to X \times Y$.*

Proof. Let $\big(\beta_n : X[p^n] \times Y[p^n] \to Z[p^n]\big)_{n \geq 1}$ be the family of Weil pairings of the biextension E. Because the β_n's satisfy the compatibility condition

$$\beta_n(px_{n+1}, py_{n+1}) = p\beta_{n+1}(x_{n+1}, y_{n+1}) \quad \forall\, x_{n+1} \in X[p^{n+1}],\ \forall\, y_{n+1} \in Y[p^{n+1}],$$

There exists a unique $\mathbb{Z}_p$-bilinear pairing $\tilde{\beta} : \mathbf{X} \times \mathbf{Y} \longrightarrow \mathbf{Z}$ such that for any point (x_n, y_n) of $X[p^n] \times Y[p^n]$

$$\beta_n(x_n, y_n) = p^{-n}\, \tilde{\beta}(x, y) \bmod \mathbf{Z},$$

where $\mathbf{x}$ and $\mathbf{y}$ are points of $\mathbf{X}$ and $\mathbf{Y}$ such that $x_n = p^{-n}\mathbf{x} \bmod \mathbf{X}$ and $y_n = p^{-n}\mathbf{y} \bmod \mathbf{X}$. It suffices to show that there exists a central extension $\mathbf{N}$ which is split over $\mathbf{X}$ and $\mathbf{Y}$, such that

$$[\mathbf{x}, \mathbf{y}]_{\mathrm{grp}} = \tilde{\beta}(\mathbf{x}, \mathbf{y})$$

for all functorial points $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$.

Define a fpqc sheaf of nilpotent algebras $\mathbf{A}$ *without unity* on $\mathfrak{Sch}_\kappa$ whose underlying additive group is $\mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$, with multiplication defined as follows.

- $\mathbf{Z} \cdot \mathbf{X} = \mathbf{X} \cdot \mathbf{Z} = \mathbf{Z} \cdot \mathbf{Y} = \mathbf{Y} \cdot \mathbf{Z} = \mathbf{Z} \cdot \mathbf{Z} = \mathbf{Y} \cdot \mathbf{Y} = \mathbf{X} \cdot \mathbf{X} = \mathbf{Y} \cdot \mathbf{X} = 0.$

- $\mathbf{x} \cdot \mathbf{y} = \tilde{\beta}(\mathbf{x}, \mathbf{y})$ for all functorial points (x, y) of $\mathbf{X} \times \mathbf{Y}$.

Let $\mathbf{N}$ be the subgroup $1 + \mathbf{A}$ of units in the sheaf of algebras $\mathbb{Z}_p \oplus \mathbf{A}$, and let $\eta : \mathbf{A} \to \mathbf{N}$ be the map $\eta(\mathbf{a}) = 1 + \mathbf{a}$ for all functorial points $\mathbf{a}$ of $\mathbf{A}$. Define commutative subgroups $\mathbf{X}', \mathbf{Y}', \mathbf{Z}'$ of $\mathbf{N}$ by $\mathbf{X}' = 1 + \mathbf{X}$, $\mathbf{Y}' = 1 + \mathbf{Y}$, and $\mathbf{Z}' = 1 + \mathbf{Z}$. Clearly η induces canonical group isomorphisms $\mathbf{X} \cong \mathbf{X}'$, $\mathbf{Y} \cong \mathbf{Y}'$ and $\mathbf{Z} \cong \mathbf{Z}'$. It is not difficult to see that $\mathbf{N}$ is a Tate unipotent group. Clearly $\mathbf{N}$ is a central extension of $\mathbf{X}' \times \mathbf{Y}'$ by $\mathbf{Z}'$ which is split over $\mathbf{X}'$ and $\mathbf{Y}'$. Moreover the group commutators of elements of $\mathbf{X}'$ with elements of $\mathbf{Y}'$ are given by

$$[\eta(\mathbf{x}), \eta(\mathbf{y})]_{\mathrm{grp}} = \eta(\tilde{\beta})$$

for all functorial points $(\mathbf{x}, \mathbf{y})$ of $\mathbf{X} \times \mathbf{Y}$. $\square$

(4.3.7) Definition. Let $\mathbf{N}$ and $\mathbf{N}'$ be Tate unipotent groups over κ.

(a) A formal morphism $f : \mathrm{TL}(\mathbf{N}) \to \mathrm{TL}(\mathbf{N}')$ over κ is *Tate-linear*, or a TL-morphism, if there exists a homomorphism $\alpha : \mathbf{N} \to \mathbf{N}'$ such that $f = \mathrm{TL}(\alpha)$.

(b) A TL-morphism $\mathrm{TL}(\mathbf{N}) \to \mathrm{TL}(\mathbf{N}')$ over κ induced by a homomorphism $\alpha : \mathbf{N} \to \mathbf{N}'$ of Tate unipotent groups over κ is an *isogeny* if α is an isogeny.

(c) The Tate-linear formal varieties $\mathrm{TL}(\mathbf{N})$ and $\mathrm{TL}(\mathbf{N}')$ are *isogenous* if $\mathbf{N}$ and $\mathbf{N}'$ are isogenous.

(4.3.8) Definition. Let $\mathrm{TL}(\mathbf{N})$ be a Tate-linear formal variety attached to a Tate unipotent group $\mathbf{N}$ over a field κ of characteristic p. A Tate-linear formal subvariety, or a TL-subvariety, of $\mathrm{TL}(\mathbf{N})$ is a closed formal subscheme which is the image of a closed TL-embedding $\mathrm{TL}(\mathbf{N}') \hookrightarrow \mathrm{TL}(\mathbf{N})$, where $\mathbf{N}'$ is a co-torsion free Tate unipotent subgroup of $\mathbf{N}$.

(4.3.9) Corollary. *Let $\mathbf{N}$ be a central extension of $\mathbf{X} \times \mathbf{Y}$ by $\mathbf{Z}$ as in* 4.3.1*. Let $X = \mathrm{TL}(\mathbf{X})$, $Y = \mathrm{TL}(\mathbf{Y})$ and $Z = \mathrm{TL}(\mathbf{Z})$. Let $E = \mathrm{TL}(\mathbf{N})$, endowed with the natural biextension structure of (X, Y) by Z as in* 4.3.2*. An closed formal subscheme T of E is a Tate-linear formal subvariety of E in the sense of* [7, §10.3] *if and only if T is a Tate-linear formal subvariety in the sense of* 4.3.8*.*

Note that in [7, §10.3], the notion of Tate-linear formal subvarieties of a biextension p-divisible formal groups is defined directly in terms of the biextension structure of$E$. The proof of 4.3.9 is an easy exercise using proposition 4.3.5.

(4.3.10) Corollary. *Let $\mathbf{N}$ be a Tate unipotent group over a* perfect *field κ of characteristic p. Suppose that there exists Tate unipotent subgroup $\mathbf{Z}$ of the center of $\mathbf{N}$ such that $\mathbf{N}/\mathbf{Z}$ is a commutative Tate unipotent subgroup. There exist p-divisible formal groups X, Y over κ, a biextension E of (X, Y) by Z, a Tate-linear formal subvariety W of E, and an isogeny $W \to \mathrm{TL}(\mathbf{N})$ of Tate-linear formal varieties.*

PROOF. We prove the following equivalent statement.

> *Let $\mathfrak{N}$ be a Tate unipotent Lie $\mathbb{Q}_p$-algebra over κ. Suppose that there Tate unipotent Lie $\mathbb{Q}_p$-subalgebra $\mathfrak{Z}$ of the center of $\mathfrak{N}$ such that the quotient $\mathfrak{N}/\mathfrak{Z}$ is commutative. There exist commutative Tate unipotent Lie $\mathbb{Q}_p$-algebras $\mathfrak{X}, \mathfrak{Y}$ over κ, a central extension $\mathfrak{E}$ of $\mathfrak{X} \times \mathfrak{Y}$ split over $\mathfrak{X}$ and $\mathfrak{Y}$, and a Tate unipotent Lie $\mathbb{Q}_p$-subalgebra $\mathfrak{W}$ of $\mathfrak{E}$, and an isomorphism $\mathfrak{N} \xrightarrow{\sim} \mathfrak{W}$.*

Since κ is perfect, there exists a commutative Tate unipotent Lie $\mathbb{Q}_p$-algebra $\mathfrak{U}$ over κ and an isomorphism $\mathfrak{N} \cong \mathfrak{Z} \oplus \mathfrak{U}$ compatible with the $\mathbb{Q}_p$-module structures. Of course. The Lie bracket $[\ ,\ ]_{\mathfrak{N}}$ on $\mathfrak{N}$ induces a skew symmetric $\mathbb{Q}_p$-linear pairing $\langle\ ,\ \rangle : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{Z}$.

Define a skew symmetric pairing

$$[\ ,\ ] : (\mathfrak{U} \oplus \mathfrak{U}) \times (\mathfrak{U} \oplus \mathfrak{U}) \longrightarrow \mathfrak{Z}$$

by

$$[u_1 + v_1, u_2 + v_2] := \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle$$

for all functorial points (u_1, v_1), (u_2, v_2) of $\mathfrak{U} \oplus \mathfrak{U}$ over the same base scheme. This skew symmetric pairing $[\ ,\ ]$ defines a Tate unipotent Lie $\mathbb{Q}_p$-algebra structure on $\mathfrak{E} := \mathfrak{Z} \oplus \mathfrak{U} \oplus \mathfrak{V}$

Define commutative Tate unipotent Lie $\mathbb{Q}_p$-subgroups $\mathfrak{X}$ (respectively $\mathfrak{Y}$) of $\mathfrak{U} \oplus \mathfrak{U}$ to be the images of the diagonal homomorphism $u \mapsto (u, u)$ (respectively the anti-diagonal homomorphism $u \mapsto (u, -u)$) from $\mathfrak{U}$ to $\mathfrak{U} \oplus \mathfrak{U}$. Clearly $\mathfrak{U} \oplus \mathfrak{U} = \mathfrak{X} \oplus \mathfrak{Y}$. Moreover the central extension $\mathfrak{E}$ of $\mathfrak{U} \oplus \mathfrak{U}$ by $\mathfrak{Z}$ splits over $\mathfrak{X}$ and $\mathfrak{Y}$. □

(4.4) Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. We give a crude congruence estimate for element in a "one-parameter subgroups" $\exp(p^n B)$ in the automorphism group $\mathrm{Aut}(\mathbf{N})$ of $\mathbf{N}$, for n sufficiently large, where B is an element of the Lie algebra of $\mathrm{Aut}(\mathbf{N})$

(4.4.1) DEFINITION. For each $n \in \mathbb{N}$, let

$$\mathrm{Fr}^n_{\mathrm{TL}(\mathbf{N})/\kappa} : \mathrm{TL}(\mathbf{N}) \longrightarrow \mathrm{TL}(\mathbf{N})^{(p^n)}$$

be the n-th iterate of the relative Frobenius morphism for $\mathrm{TL}(\mathbf{N})$. Define $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^n]$ to be

$$\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^n] := \left(\mathrm{Fr}^n_{\mathrm{TL}(\mathbf{N})/\kappa}\right)^{-1}(*_{\mathrm{TL}(\mathbf{N})^{(p^n)}}),$$

where $*_{\mathrm{TL}(\mathbf{N})^{(p^n)}}$ is the base closed point of the formal scheme $\mathrm{TL}(\mathbf{N})^{(p^n)}$.

The following proposition 4.4.2 gives "trivial estimates" for the action of one-parameter families of automorphisms of a Tate-linear formal variety.

(4.4.2) PROPOSITION. *Let $\mathbf{N}$ be a Tate unipotent group over a field κ of characteristic p. Let $\mathfrak{N}_{\mathbb{Q}_p} = \mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ be the Tate unipotent $\mathbb{Q}_p$-Lie algebra associated to the Mal'cev completion*

$\mathbf{N}_{\mathbb{Q}}$ *of* $\mathbf{N}$. *Let* $s \in (0,1]$ *such that* $\max(\mathrm{slope}(\mathbf{N})) \le s \le 1$. *Let* U *be a finitely generated* $\mathbb{Z}_p$*-submodule of the Lie algebra* $\mathrm{Der}(\mathfrak{N}_{\mathbb{Q}})$ *of the* p*-adic Lie group* $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}) = \mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}_p})$. *There exist constants* $c_0, n_0 \in \mathbb{N}$ *such that for every* $n \ge n_0$ *and every* $B \in U$, *the exponential* $\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^n B)$ *of* $p^n B$ *is an element of* $\mathrm{Aut}(\mathbf{N})$ *which operates trivially on the infinitesimal neighborhood* $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/s\rfloor - c_0}]$ *of the base point of* $\mathrm{TL}(\mathbf{N})$.

PROOF. The basic idea here is that the statement 4.4.2 follows from the following two estimates.

(a) Since the all slopes of $\mathfrak{N}_{\mathbb{Q}_p} = \mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ are $\le s$, the effect of Fr^m should "divide $[p^{ms-c_1}]$" for some positive constant c_1.

(b) The effect of $\exp(p^n B) - 1$ should be "divisible by p^n" for n large.

PRELIMINARY REDUCTIONS

By 3.2.20, there exists a constant m_1 such that $\log(\mathbf{N}^{p^m})$ is a Tate unipotent Lie $\mathbb{Z}_p$-subalgebra of $\mathfrak{N}_{\mathbb{Q}_p}$ such that $\log(\mathbf{N}^{p^m}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathfrak{N}_{\mathbb{Q}_p}$, for all $m \ge m_1$. So we may and do assume that $\log(\mathbf{N})$ is a Tate unipotent Lie $\mathbb{Z}_p$-subalgebra $\mathfrak{N}_{\mathbb{Z}_p}$ of $\mathfrak{N}_{\mathbb{Q}_p}$, $\mathfrak{N}_{\mathbb{Q}_p} = \mathfrak{N}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\exp(\mathfrak{N}_{\mathbb{Z}_p}) = \mathbf{N}$. The functoriality of the Frobenius morphisms and the exponential/logarithm pairs for $\mathfrak{N}_{\mathbb{Q}}$ and $\mathbf{N}_{\mathbb{Q}}$ tells us that for each $r \in \mathbb{N}$, $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^r]$ is the image in $\mathrm{TL}(\mathbf{N})$ the image of

$$\left(\mathrm{Fr}^r_{\mathfrak{N}_{\mathbb{Q}}}\right)^{-1}\left(\mathfrak{N}_{\mathbb{Z}_p}^{(p^r)}\right)$$

under the exponential map $\exp_{\mathbf{N}_{\mathbb{Q}}} : \mathfrak{N}_{\mathbb{Q}_p} \longrightarrow \mathbf{N}_{\mathbb{Q}}$.

where $\mathrm{Fr}^r_{\mathfrak{N}_{\mathbb{Q}}} : \mathfrak{N}_{\mathbb{Q}_p} \to \mathfrak{N}_{\mathbb{Q}_p}^{(p^r)}$ is the r-th power of the relative Frobenious of $\mathfrak{N}_{\mathbb{Q}}$.

Let $G = \mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}) = \mathrm{Aut}(\mathfrak{N}_{\mathbb{Q}})$. We may and do assume that $\exp_G(B) \in G$ for all $B \in U$.

APPLY THE SLOPE ESTIMATE for the p-divisible group $\mathfrak{N}_{\mathbb{Q}_p}/\mathfrak{N}_{\mathbb{Z}_p}$

Since $s \ge \max(\mathrm{slope}(\mathbf{N}))$ there exists constants $r_1, d_1 \in \mathbb{N}$ such that

$$\left(\mathrm{Fr}^r_{\mathfrak{N}_{\mathbb{Q}}}\right)^{-1}\left(\mathfrak{N}_{\mathbb{Z}_p}^{(p^r)}\right) \subseteq p^{-(rs+d_1)}\mathfrak{N}_{\mathbb{Z}_p} \quad \forall\, r \ge r_1.$$

The formula

$$\exp_G(p^n B) = \mathrm{id} + \sum_{j\ge 1} \tfrac{p^{nj}B^j}{j!}$$

plus the standard estimate for $\mathrm{ord}_p(j!)$ shows that the restriction to the image in $\mathrm{TL}(\mathbf{N})$ of

$$\left(\mathrm{Fr}^{\lfloor n/s\rfloor - \lceil d_1/s\rceil}_{\mathfrak{N}_{\mathbb{Q}}}\right)^{-1}\left(\mathfrak{N}_{\mathbb{Z}_p}^{(p^{\lfloor n/s\rfloor - \lceil d_1/s\rceil})}\right)$$

of the action of $\exp_G(p^n B)$ is the identity map, for all $n \ge s(r_1 + d_1) + 2$. □

(4.5) REMARK. There are many questions one may ask related to the notion of Tate-linear formal varieties. We mentioned one in 1.2.3 (c). Another line of inquiry is to develop a theory of families of Tate-linear formal varieties, which include (a) below as examples, and make progress on questions (b)–(d).

(a) Let $\mathcal{C}$ be a central leaf on a moduli space of abelian varieties of PEL type over $\overline{\mathbb{F}_p}$. Let $(\mathcal{C}\times\mathcal{C})^{\Delta\mathcal{C}}$ be the formal completion of $\mathcal{C}\times\mathcal{C}$ along the diagonal $\Delta\mathcal{C}$. Then the projection map $\mathrm{pr}_1 : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ makes $(\mathcal{C}\times\mathcal{C})^{\Delta\mathcal{C}}$ a family of Tate-linear formal varieties over $\mathcal{C}$.

(b) Let $\mathcal{W}$ be a Newton polygon stratum in Siegel modular variety over $\overline{\mathbb{F}_p}$. For each $\overline{\mathbb{F}_p}$-point x_0 of $\mathcal{W}$, there is a flat formal morphism $\pi_{x_0} : \mathcal{W}^{/x_0}\to\mathrm{Spf}(R_{x_0})$, where R_{x_0} is completion local $\overline{\mathbb{F}_p}$-domain, such that $\mathcal{W}^{/x_0}$ is a family of Tate-linear formal varieties over $\mathrm{Spf}(R_{x_0})$, and the closed fiber $\pi_{x_0}^{-1}(x_0)$ is $\mathcal{C}(x_0)^{/x_0}$, the formal completion at x_0 of the central leaf $\mathcal{C}(x_0)$ passing through x_0.

(c) In the case when the base field is a finite field, develop a theory of quasi-canonical liftings of Tate-linear formal varieties over mixed-characteristic complete discrete valuation rings with finite residue fields.

(d) Let $\mathcal{C}$ be a central leaf in a Siegel modular variety $\mathscr{A}_{g,d,n,\overline{\mathbb{F}_p}}$ over $\overline{\mathbb{F}_p}$. Develop a good notion of Tate-linear subvarieties of $\mathcal{C}$ and their quasi-canonical liftings (if the latter exist).

§5. Orbital rigidity: the statement and reduction steps

Theorem 5.1 below is the main rigidity result of this article.

(5.1) Theorem. *Let κ be a perfect field of characteristic p, and let $\mathbf{N}$ be a Tate unipotent group over κ. Let G be a p-adic Lie group acting strongly nontrivially on $\mathbf{N}$. Let W be a reduced irreducible closed formal subscheme of $\mathrm{TL}(\mathbf{N})$. If W is stable under the strongly nontrivial action of G on $\mathrm{TL}(\mathbf{N})$, then W is a Tate-linear formal subvariety of $\mathrm{TL}(\mathbf{N})$. In other words, there exists a unique co-torsion-free Tate unipotent subgroup $\mathbf{N}'$ of $\mathbf{N}$ such that $W=\mathrm{TL}(\mathbf{N}')$.*

(5.1.1) Remark. (a) The statement 5.1 depends only on the isogeny class $\mathbf{N}$, because the map $\mathrm{TL}(\alpha):\mathrm{TL}(\mathbf{U})\to\mathrm{TL}(\mathbf{N})$ induced by an isogeny $\alpha:\mathbf{U}\to\mathbf{N}$ is purely inseparable.

(b) An easy descent argument shows that it suffices to prove 5.1 when the base field κ is algebraically closed.

(5.1.2) Corollary. *Let κ be a perfect field of characteristic p, and let $\mathbf{N}_1,\mathbf{N}_2$ be Tate unipotent groups over κ. Let G be a p-adic Lie group acting strongly nontrivially on Tate-linear formal varieties $\mathrm{TL}(\mathbf{N}_1)$ and $\mathrm{TL}(\mathbf{N}_2)$ attached to $\mathbf{N}_1$ and $\mathbf{N}_2$ respectively. Then every G-equivariant morphism of formal schemes $f:\mathrm{TL}(\mathbf{N}_1)\to\mathrm{TL}(\mathbf{N}_2)$ over κ is Tate-linear, i.e. there exists a unique homomorphism $h:\mathbf{N}_1\to\mathbf{N}_2$ over κ of Tate unipotent groups such that $f=\mathrm{TL}(h)$.*

Proof. Apply 5.1 to the graph of f. □

(5.1.3) Remark. Theorem 5.1 has been proved in two cases:

- When $\mathbf{N}$ is commutative, i.e. $\mathbf{N}$ corresponds to a p-divisible group over κ. This is [5, Thm. 4.3].
- When $\mathbf{N}$ is a central extension of a product $\mathbf{X}\times\mathbf{Y}$ of two commutative Tate unipotent groups by a commutative Tate unipotent group Z which is split over $\mathbf{X}$ and also over $\mathbf{Y}$ as in 4.3.1. This is the main result [7, §10.6] in view of 4.3.9.

Therefore 4.3.10 tells us that 5.1 holds whenever $\mathbf{N}$ is nilpotent of class at most 2, i.e. if there exists a Tate unipotent subgroup $\mathbf{Z}$ contained in the center of $\mathbf{N}$ such that $\mathbf{N}/\mathbf{Z}$ is a commutative Tate unipotent group.

(5.1.4) Remark. Theorem 5.1 will be proved by induction on card(slope($\mathbf{N}$)). In 5.4 we show that 5.1 follows formally from theorems 5.2 and 5.3 below. We will prove 5.3 later in this section, by reducing it to the case of biextensions. Theorem 5.2 will be proved in §6 using the method of hypocotyl elongation in tempered perfections.

Lemma 5.1.5 says that orbital rigidity of $\mathbf{N}$ is a property depending only on the isogeny class of $\mathbf{N}$, or equivalently the Tate unipotent $\mathbb{Q}_p$-Lie algebra $\mathfrak{Lie}\mathbf{N}_{\mathbb{Q}}$ of $\mathbf{N}$. In the rest of this section, this method of "modification by a suitable isogeny" will be used frequently, without formally invoking 5.1.5.

(5.1.5) Lemma. *Let $\mathbf{N}$ and $\mathbf{N}'$ be isogenous Tate unipotent groups over a perfect field κ of characteristic p. The statement 5.1 holds for $\mathbf{N}$ if and only if it holds for $\mathbf{N}'$.*

Proof. Let $\alpha : \mathbf{N}'' \to \mathbf{N}$ and $\alpha' : \mathbf{N}'' \to \mathbf{N}'$ be isogenies of Tate unipotent groups over There exists an open subgroup G_1 of G such that the action of G on $\mathbf{N}$ lifts to $\mathbf{N}''$ and descends to $\mathbf{N}'$, so that α and α' are both G_1-equivariant. The G_1-equivariant maps $\mathrm{TL}(\alpha) : \mathrm{TL}(\mathbf{N}'') \to \mathrm{TL}(\mathbf{N})$ and $\mathrm{TL}(\alpha') : \mathrm{TL}(\mathbf{N}'') \to \mathrm{TL}(\mathbf{N}')$ are both purely inseparable. It follows that a reduced formal subscheme W of $\mathrm{TL}(\mathbf{N})$ is G_1-equivariant (respectively a Tate-linear formal subvariety) if and only if $(\mathrm{TL}(\alpha)^{-1}(W))_{\mathrm{red}}$ is. The same holds for $\mathrm{TL}(\alpha')$. ∎

We will use the following notation in theorem 5.2.

- Let κ be a perfect field of characteristic p, and let $\mathbf{N}$ be a Tate unipotent group over κ.
- Let G be a p-adic Lie group acting strongly nontrivially on the Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$ attached to $\mathbf{N}$.
- Let W be a reduced irreducible closed formal subscheme of $\mathbf{N}$ which is stable under the action of G.
- Let $\lambda_1 = \max(\mathrm{slope}(\mathbf{N}))$, let $\mathbf{Z} = \mathrm{Fil}_{\mathrm{sl}}^{\lambda_1}\mathbf{N}$. Let $Z = \mathrm{TL}(\mathbf{Z})$ be the p-divisible group attached to $\mathbf{Z}$, which operates naturally on $\mathrm{TL}(\mathbf{N})$.

- Let $\mathbf{N}_2 := \mathbf{N}/\mathbf{Z}$.
- Let $\pi : \mathrm{TL}(\mathbf{N}) \longrightarrow \mathrm{TL}(\mathbf{N}_2)$ be the map induced by the quotient map $\mathbf{N} \twoheadrightarrow \mathbf{N}_2$.

Clearly $\mathrm{TL}(\mathbf{N})$ is a torsor over $\mathrm{TL}(\mathbf{N}_2)$ under the translation action of Z on $\mathrm{TL}(\mathbf{N})$. From orbital rigidity of p-divisible formal groups we know that the reduced closed formal subscheme $(W \cap Z)_{\mathrm{red}}$ of Z is a union of p-divisible formal subgroups of Z.

(5.2) Theorem. *Notation and assumptions as in the above paragraph.*

(a) *The reduced closed subscheme $(W \cap Z)_{\mathrm{red}}$ of Z is a p-divisible subgroup of Z.*

(b) *The formal subscheme W of $\mathrm{TL}(\mathbf{N})$ is stable under the translation action of the p-divisible subgroup $Z' := (W \cap Z)_{\mathrm{red}}$ of Z.*

Let $\mathbf{Z}'$ be the Tate unipotent subgroup of $\mathbf{Z}$ corresponding to Z'. Let $\mathbf{N}_1 := \mathbf{N}/\mathbf{Z}'$. The quotient $W_1 := W/Z'$ is an irreducible closed formal subscheme of $\mathrm{TL}(\mathbf{N}_1)$, and is stable under natural action of G. Let $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ be the map associated to the quotient map $\mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$.

(c) *The restriction $\bar{\pi}\big|_{W_1} : W_1 \longrightarrow \mathrm{TL}(\mathbf{N}_2)$ of $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ to W/Z_1 is purely inseparable.*

The proof of theorem 5.2 is deferred to §6.

The setup of theorem 5.3 is as follows. Let $\mathbf{N}_1$ be a Tate unipotent group over a field κ of characteristic p. Let G be a p-adic Lie group acting strongly nontrivially on $\mathbf{N}_1$. Let $\mathbf{Z}_1$ be a Tate unipotent normal subgroup of $\mathbf{N}_1$ such that $\mathbf{Z}_1$ is stable under the action of G, and the quotient $\mathbf{N}_2 := \mathbf{N}_1/\mathbf{Z}_1$ is a Tate unipotent group over κ. Assume that $\min \mathrm{slope}(\mathbf{Z}_1) > \max \mathrm{slope}(\mathbf{N}_2)$.

(5.3) Theorem. *Notation and assumptions as in the previous paragraph. Suppose that the projection map $\pi : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ admits a G-equivariant section ξ. Then ξ is a TL morphism, i.e. there exists a G-equivariant group homomorphism $\psi : \mathbf{N}_2 \to \mathbf{N}_1$ of fpqc sheaves on $\mathfrak{Sch}_\kappa$ which splits the quotient homomorphism $\mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$, such that the section ξ of π is equal to the map $\mathrm{TL}(\psi) : \mathrm{TL}(\mathbf{N}_2) \to \mathrm{TL}(\mathbf{N}_1)$ induced by ψ.*

(5.4) Theorems 5.2 and 5.3 imply the main theorem 5.1.

This is completely formal. Given a closed irreducible formal subscheme W of $\mathrm{TL}(\mathbf{N})$ stable under G as in 5.1. Let λ_1, $\mathbf{Z}$ and $Z = \mathrm{TL}(\mathbf{Z})$ be as in the paragraph preceding theorem 5.2. Let $Z' := (W \cap Z)_{\mathrm{red}}$, a p-divisible subgroup of Z stable under G, by orbital rigidity for p-divisible formal groups [5]. Consider $N_1 := \mathbf{N}/Z$, $W_1 := W/Z' \subseteq \mathrm{TL}(\mathbf{N}_1)$, and $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$. By induction card(slope($\mathbf{N}$)), the schematic image of W_1 under the purely inseparable morphism $\bar{\pi}\big|_{W_1} : W_1 \to \mathrm{TL}(\mathbf{N}_2)$ is equal to $\mathrm{TL}(\mathbf{N}_4)$ for a Tate unipotent subgroup $\mathbf{N}_4$ of $\mathbf{N}_2$. Let $\mathbf{N}_3$ be the inverse image of $\mathbf{N}_4$ under the quotient homomorphism $\mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$. Thus we have $W_1 \subseteq \mathrm{TL}(\mathbf{N}_3)$, Since the restriction to W_1 of the projection map $\pi_4 : \mathrm{TL}(\mathbf{N}_3) \to \mathrm{TL}(\mathbf{N}_4)$ is purely inseparable, there exists an isogeny $\alpha : \mathbf{N}_6 \to \mathbf{N}_4$ such that for the Tate unipotent group

$$\mathbf{N}_5 := \mathbf{N}_3 \times_{\mathbf{N}_4,\alpha} \mathbf{N}_6,$$

the projection map π_5 in the diagram

$$\begin{tikzcd}
\mathrm{TL}(\mathbf{N}_5) \arrow[d, "\pi_6"'] & (\mathrm{TL}(\alpha)^* W_1)_{\mathrm{red}} \arrow[l, "\supseteq"'] \arrow[r] \arrow[d, "\pi_5"] & W_1 \arrow[r, "\subseteq"] \arrow[d, "\pi_3"] & \mathrm{TL}(\mathbf{N}_3) \arrow[d, "\pi_4"] \\
\mathrm{TL}(\mathbf{N}_6) & \mathrm{TL}(\mathbf{N}_6) \arrow[l, "="'] \arrow[r, "\alpha"] & \mathrm{TL}(\mathbf{N}_4) \arrow[r, "="] & \mathrm{TL}(\mathbf{N}_4)
\end{tikzcd}$$

is an isomorphism. Let ξ be the inverse of π_5, which is a G-equivariant section of the projection $\pi_6 : \mathrm{TL}(\mathbf{N}_5) \to \mathrm{TL}(\mathbf{N}_6)$.

Theorem 5.3 applied to the section ξ of π_6 says that there exists a G-equivariant homomorphism $\psi : \mathbf{N}_6 \to \mathbf{N}_5$ which is a section of the quotient map $\mathbf{N_5} \twoheadrightarrow \mathbf{N}_6$, such that $\mathrm{TL}(\psi) : \mathrm{TL}(\mathbf{N}_6) \xrightarrow{\sim} (\mathrm{TL}(\alpha)^* W_1)_{\mathrm{red}}$. Let $\mathbf{N}_7$ be the smallest co-torsion free Tate unipotent subgroup of the image of $\psi(\mathbf{N}_6)$ in $\mathbf{N}_3$ under the map $\mathbf{N}_5 = \mathbf{N}_3 \times_{\mathbf{N}_4,\alpha} \mathbf{N}_6 \longrightarrow \mathbf{N}_3$. Let $\mathbf{N}'$ be the inverse image of $\mathbf{N}_7$, which is a sheaf of subgroups of $\mathbf{N}_1$, under the quotient homomorphism $\mathbf{N} \twoheadrightarrow N_1$, i.e. $\mathbf{N}' := \mathbf{N} \times_{\mathbf{N}_1} \mathbf{N}_7$. This Tate unipotent subgroup $\mathbf{N}'$ of $\mathbf{N}$ has the required property that $W = \mathrm{TL}(\mathbf{N}')$. $\square$

(5.5) Theorem. *Let κ be a field of characteristic p. Let $\mathbf{N}, \mathbf{Z}'$ be Tate unipotent groups over κ. Let G be a p-adic Lie group which operates strongly nontrivially on $\mathbf{N}$ and $\mathbf{Z}'$. Assume that* $\max(\mathrm{slope}(\mathbf{N})) < \min(\mathrm{slope}(\mathbf{Z}'))$. *Let $\xi : \mathrm{TL}(\mathbf{N}) \to \mathrm{TL}(\mathbf{Z}')$ be a formal morphism over κ. If ξ is G-invariant, then ξ is the trivial map.*

Proof. An easy induction on the number of slopes of $\mathbf{Z}'$ shows that it suffices to prove 5.3 in the case when $\mathbf{Z}'$ is isoclinic, i.e. $\mathrm{TL}(\mathbf{Z}')$ is an isoclinic p-divisible group with slope λ'. Let $\lambda_1 = \max(\mathrm{slope}(\mathbf{N})) < \lambda'$. Denote by $\eta : G \to \mathrm{Aut}(\mathbf{N})$ and $\rho : G \to \mathrm{Aut}(\mathbf{Z}')$ the actions of G on $\mathbf{N}$ and $\mathbf{Z}'$ respectively.

Enlarge the base field κ if necessary, we may assume that κ is algebraically closed. Composing $\xi : \mathrm{TL}(\mathbf{N}) \to Z'$ with a suitable isogeny $Z' \to Z''$, we may and do assume that there exists a positive integer r such that $r\lambda' \in \mathbb{N}$, and the complete κ-algebra $\Gamma(Z', \mathcal{O}_{Z'})$ is topologically generated by formal functions f on Z' such that $[p^{r\lambda'}]^* f = f^{p^r}$. In other words Z' is the base change to κ of a p-divisible formal group Y over the finite subfield $\mathbb{F}_q$ with $q := p^r$ elements, and there exists an isomorphism $\beta : (Z')^{(p^r)} \xrightarrow{\sim} Z'$ such that

$$[p^{mr\lambda'}]_{Z'} = \beta^m \circ \mathrm{Fr}^{mr}_{Z'} \quad \forall\, m \in \mathbb{N},$$

where $\mathrm{Fr}^{mr}_{Z'} : Z' \to (Z')^{(p^{mr})}$ is the mr-th iterate of the relative Frobenius map for Z', and $\beta^m : (Z')^{(p^{mr})} \xrightarrow{\sim} Z'$ is the m-th iterate of β.

Let B be an element of $\mathfrak{g} := \mathrm{Lie}(G)$ such that $\exp_G(uB) \in G$ for all $u \in \mathbb{Z}_p$, and the element $d\rho(B)$ in $\mathrm{End}(Z') \otimes_{\mathbb{Q}} \mathbb{Q}$ is in $\mathrm{End}(Z')$. We will abuse the notation and write $\rho(B)$ instead of $d\rho(B)$.

We know from the standard exponential series that

$$\mathrm{Ker}(\rho(\exp_G(p^n B))) - [1]_{Z'}) = \mathrm{Ker}(\rho(p^n B))$$

for all $n \geq 2$. In other words there exists an automorphism γ_n of Z' such that

$$\rho(\exp_G(p^n B))) - [1]_{Z'} = \gamma_n \circ \rho(p^n B) \quad \forall\, n \geq 2.$$

Hence

$$\rho(\exp_G(p^{mr\lambda'} B))) - [1]_{Z'} = \gamma_{mr\lambda'} \circ \rho(B) \circ \beta^m \circ \mathrm{Fr}_{Z'}^{mr}$$

for all $m \geq 2$. It follows that for every formal function f in the maximal ideal $\mathfrak{m}_{Z'}$ of $\Gamma(Z', \mathcal{O}_{Z'})$, there exists a unit $u_{f,B,m} \in \Gamma(Z', \mathcal{O}_{Z'})^\times$ such that

$$\rho(\exp_G(p^{mr\lambda'} B)))^* f - f = u_{f,B,m} \cdot \big(\rho(B)^* \big(\gamma^*_{mr\lambda_1} f)\big)^{p^{mr}}\big) \tag{5.5.1}$$

Let $\mathfrak{m} = \mathfrak{m}_{\mathrm{TL}(\mathbf{N})}$ be the maximal ideal of the pointed formal scheme $\mathrm{TL}(\mathbf{N})$. From 4.4.2 we know that there exist constants $c_6, n_6 \in \mathbb{N}$ such that for all natural numbers $n \geq n_6$, we have

$$\eta(\exp_G(p^n B))^* g - g \in \mathfrak{m}^{(p^{\lfloor n/\lambda_1 \rfloor - c_6})} \tag{5.5.2}$$

for all formal functions $g \in \Gamma(\mathrm{TL}(\mathbf{N}, \mathcal{O}_{\mathrm{TL}(\mathbf{N})}))$ on the formal scheme $\mathrm{TL}(\mathbf{N})$. Here $\mathfrak{m}^{(p^{\lfloor n/\lambda_1 \rfloor - c_6})}$ is the ideal of $\Gamma(\mathrm{TL}(\mathbf{N}), \mathcal{O}_{\mathrm{TL}(\mathbf{N})})$ generated by all $p^{\lfloor n/\lambda_1 \rfloor - c_6}$-th powers of elements of $\mathfrak{m}$.

Let f be any formal function in the maximal ideal $\mathfrak{m}_{Z'}$ of $\Gamma(Z', \mathcal{O}_{Z'})$. Combine (5.5.1) and (5.5.2) with $g = \xi^* f$, $n = mr\lambda'$, we get

$$\begin{aligned} \eta(\exp_G(p^{mr\lambda'} B))^* \xi^* f - \xi^* f &= \xi^* \rho(\exp_G(p^{mr\lambda'} B))^* f - \xi^* f \\ &= (\xi^* u_{f,B,m}) \cdot \big(\xi^* \rho(B)^* (\gamma^*_{mr\lambda_1} f)\big)^{p^{mr}} \;\; \in \;\; \mathfrak{m}^{p^{\lfloor mr\lambda'/\lambda_1 \rfloor - c_6}} \end{aligned}$$

for all $m \geq \lceil n_6 / r\lambda' \rceil$. It follows that there exist constants $m_7, c_7 \in \mathbb{N}$ such that

$$\xi^* \rho(B)^* \gamma^*_{mr\lambda_1} f \in \mathfrak{m}^{p^{\lfloor mr((\lambda'/\lambda_1)-1) \rfloor - c_7}}$$

for all $m \geq m_7$ and all $f \in \mathfrak{m}_{Z'}$. Therefore

$$\xi^* \rho(B)^* f \in \mathfrak{m}^{p^{\lfloor mr((\lambda'/\lambda_1)-1) \rfloor - c_7}}$$

for all $f \in \mathfrak{m}_{Z'}$ and all $m \geq m_7$, since $\gamma_{mr\lambda_1}$ is an automorphism of Z'. It follows that

$$\xi^* \rho(B)^* f \in \bigcap_{m \geq m_7} \mathfrak{m}^{p^{\lfloor mr((\lambda'/\lambda_1)-1) \rfloor - c_7}} = (0) \qquad \forall\, f \in \mathfrak{m}_{Z'}. \tag{5.5.3}$$

Since G operates strongly nontrivially on Z', by [5, 4.1.1] there exist positive integers $b_1, \ldots, b_m$, elements $B_{i,j} \in \mathrm{Lie}(G)$ indexed by pairs (i,j) with $1 \leq i \leq m$ and $1 \leq j \leq b_i$, such that $\exp_G(uB_{ij}) \in G$ for all $u \in \mathbb{Z}_p$ and $\rho(B_{i,j}) \in \mathrm{End}(Z')$ for all pairs (i,j) as above, and the element

$$A := \sum_{i=1}^{m} \rho(B_{i,1}) \circ \cdots \circ \rho(B_{i,b_i}) \;\in\; \mathrm{End}(Z')$$

is an isogeny. It follows from (5.5.3) that

$$\xi^* A^* f = f \qquad \forall\, f \in \mathfrak{m}_{Z'}.$$

In other words the composition $A \circ \xi : \mathrm{TL}(\mathbf{N}) \to Z'$ is trivial, which implies that the map $\xi : \mathrm{TL}(\mathbf{N}) \to Z'$ is also trivial because A is an isogeny. □

(5.6) PROOF OF THEOREM 5.3.

We are given

- a quotient homomorphism $\boldsymbol{\pi} : \mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$ of Tate unipotent groups whose kernel $\mathbf{Z}$ is a Tate unipotent subgroup of $\mathbf{N}_1$ such that every slope of Z is strictly bigger than every slope of $\mathbf{N}_2$,
- a p-adic Lie group acting strongly nontrivially on $\mathbf{N}_1$ and $\mathbf{N}_2$, such that $\boldsymbol{\pi}$ is G-equivariant,
- a G-equivariant section ψ of the TL morphism $\pi = \mathrm{TL}(\boldsymbol{\pi}) : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$.

We want to produce a G-equivariant homomorphism $\xi : \mathbf{N}_2 \to \mathbf{N}_1$ such that $\boldsymbol{\pi} \circ \xi = \mathrm{id}_{\mathbf{N}_2}$ and $\psi = \mathrm{TL}(\xi)$.

The proof consists of a number of steps, eventually reducing 5.3 to the case when $\mathbf{N}_1$ has at most 3 slopes, which is covered by the case when $\mathrm{TL}(\mathbf{N}_1)$ is a biextensions of p-divisible formal groups.

REDUCTION STEPS.

(1) Climbing up along the slope filtration of $\mathbf{Z}$, we may and do assume that $\mathbf{Z}$ is isoclinic of slope $\lambda_1 \in (0,1]$.

(2) It suffices to show that there exists a G-equivariant homomorphism $\xi : \mathbf{N}_2 \to \mathbf{N}_1$ such that $\boldsymbol{\pi} \circ \xi = \mathrm{id}_{\mathbf{N}_2}$, because 5.5 implies that the two G-equivariant sections ξ and $\mathrm{TL}(\xi)$ of $\pi : \mathrm{TL}(\mathbf{N}_2) \to \mathrm{TL}(\mathbf{N}_2)$ are equal.

(3) It suffices to show that there exists a G-equivariant homomorphism $\xi_{\mathbb{Q}} : \mathbf{N}_{\mathbb{Q}} \to \mathbf{N}_1$ such that the composition $\boldsymbol{\pi}_{\mathbb{Q}} \circ \xi_{\mathbb{Q}} = \mathrm{id}_{(\mathbf{N}_2)_{\mathbb{Q}}}$, where $\boldsymbol{\pi}_{\mathbb{Q}} : (\mathbf{N}_1)_{\mathbb{Q}} \to (\mathbf{N}_2)_{\mathbb{Q}}$ is the Mal'cev completion of $\boldsymbol{\pi}$.

(4) Let $\mathfrak{N}_{1,\mathbb{Q}_p}$ and $\mathfrak{N}_{2,\mathbb{Q}_p}$ be the Tate unipotent Lie $\mathbb{Q}_p$-algebras corresponding to $(\mathbf{N}_1)_{\mathbb{Q}}$ and $(\mathbf{N}_2)_{\mathbb{Q}}$ respectively. According to the Mal'cev correspondence, it suffices to show that

the projection map $\mathfrak{N}_{1,\mathbb{Q}_p} \to \mathfrak{N}_{2,\mathbb{Q}_p}$ admits a G-equivariant section $\eta_{\mathbb{Q}_p}$. Equivalently, the central extension

$$0 \to \mathfrak{Z}_{\mathbb{Q}_p} \to \mathfrak{N}_{1,\mathbb{Q}_p} \to \mathfrak{N}_{2,\mathbb{Q}_p} \to 0$$

splits, where $\mathfrak{Z}_{\mathbb{Q}_p}$ is the Tate unipotent Lie $\mathbb{Q}_p$-algebras corresponding to $(\mathbf{Z})_{\mathbb{Q}}$.

(5) Since the base field κ is perfect, both $\mathfrak{N}_{1,\mathbb{Q}_p}$ and $\mathfrak{N}_{2,\mathbb{Q}_p}$ split into direct sums of isotypic components, to the effect that the sheaf of commutative groups underlying $\mathfrak{N}_{1,\mathbb{Q}_p}$ is canonically isomorphic to the sheaf of commutative groups underlying $\mathfrak{N}_{2,\mathbb{Q}_p} \oplus \mathfrak{Z}_{\mathbb{Q}_p}$. Moreover $\mathfrak{Z}_{\mathbb{Q}_p}$ lies in the center of the Lie $\mathbb{Q}_p$-algebra $\mathfrak{N}_{1,\mathbb{Q}_p}$, and the quotient Lie algebra $\mathfrak{N}_{1,\mathbb{Q}_p}/\mathfrak{Z}_{\mathbb{Q}_p}$ is canonically isomorphic to $\mathfrak{N}_{2,\mathbb{Q}_p}$. We want to show that the subsheaf $\mathfrak{N}_{2,\mathbb{Q}_p}$ of $\mathfrak{N}_{1,\mathbb{Q}_p}$ is closed under the Lie bracket of $\mathfrak{N}_{1,\mathbb{Q}_p}$.

(6) For each $s \in \mathrm{slope}(\mathbf{N}_2)$, denote by $\mathfrak{N}_{2,s}$ the component of $\mathfrak{N}_{2,s}$ with slope s, so that

$$\mathfrak{N}_{1,\mathbb{Q}_p} = \mathfrak{Z}_{\mathbb{Q}_p} \oplus_{s\in \mathrm{slope}(\mathbf{N}_2)} \mathfrak{N}_{2,s}.$$

We know that for any two elements $s, s' \in \mathrm{slope}(\mathbf{N}_2)$,

$$[\mathfrak{N}_{2,s}, \mathfrak{N}_{2,s'}]_{\mathfrak{N}_{2,\mathbb{Q}_p}} \subseteq \mathfrak{N}_{2,s+s'}.$$

Recall that $\mathbf{Z}$ is isoclinic of slope λ_1, and $\lambda_1 > \max(\mathrm{slope}(\mathbf{N}_2))$. So we are reduced to showing that

$$[\mathfrak{N}_{2,s}, \mathfrak{N}_{2,s'}]_{\mathfrak{N}_{2,\mathbb{Q}_p}} = (0) \qquad \forall\, s, s' \in \mathrm{slope}(\mathbf{N}_2) \text{ with } s + s' = \lambda_1.$$

It remains to show that the statement in the last reduction step (7) holds. For any $s, s' \in \mathrm{slope}(\mathbf{N}_2)$ such that $s + s' = t$, let $\mathfrak{N}_{3,s,s'}$ be the Tate unipotent Lie $\mathbb{Q}_p$-subalgebra

$$\mathfrak{N}_{3,s,s'} := \begin{cases} \mathfrak{Z}_{\mathbb{Q}_p} \oplus \mathfrak{N}_{2,s} \oplus \mathfrak{N}_{2,s'} & \text{if } \ s \neq s' \\ \mathfrak{Z}_{\mathbb{Q}_p} \oplus \mathfrak{N}_{2,s} & \text{if } \ s = s' \end{cases}$$

of $\mathfrak{N}_{1,\mathbb{Q}_p}$, and let $\mathfrak{N}_{4,s,s'}$ be the Tate unipotent Lie $\mathbb{Q}_p$-subalgebra

$$\mathfrak{N}_{4,s,s'} := \begin{cases} \mathfrak{N}_{2,s} \oplus \mathfrak{N}_{2,s'} & \text{if } \ s \neq s' \\ \mathfrak{N}_{2,s} & \text{if } \ s = s' \end{cases}$$

of $\mathfrak{N}_{1,\mathbb{Q}_p}$. Let $\mathbf{N}_{3,s,s',\mathbb{Q}}$ be the sheaf of uniquely divisible subgroups of $(\mathbf{N}_1)_{\mathbb{Q}}$ corresponding to $\mathfrak{N}_{3,s,s'}$ under the Mal'cev correspondence. Define a co-torsion free Tate unipotent subgroup $\mathbf{N}_{3,s,s'}$ of $\mathbf{N}_1$ by

$$\mathbf{N}_{3,s,s'} := \mathbf{N}_1 \cap \mathbf{N}_{3,s,s',\mathbb{Q}}.$$

Similarly, let $\mathbf{N}_{4,s,s',\mathbb{Q}}$ be the uniquely divisible subgroup of $(\mathbf{N}_2)_{\mathbb{Q}}$ corresponding to $\mathfrak{N}_{4,s,s'}$, and let $\mathbf{N}_{4,s,s'}$ be the co-torsion free Tate unipotent subgroup

$$\mathbf{N}_{4,s,s'} := \mathbf{N}_2 \cap \mathbf{N}_{4,s,s',\mathbb{Q}}$$

of $\mathbf{N}_2$. Clearly $\mathbf{N}_{3,s,s'}$ and $\mathbf{N}_{4,s,s'}$ are stable under the action of G, and

$$\mathrm{TL}(\mathbf{N}_{3,s,s'}) = \mathrm{TL}(\mathbf{N}_1) \times_{\mathrm{TL}(\mathbf{N}_2)} \mathrm{TL}(\mathbf{N}_{4,s,s'}).$$

Moreover the restriction $\xi|_{\mathrm{TL}(\mathbf{N}_{4,s,s'})}$ of the section ξ of $\pi : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ to $\mathrm{TL}(\mathbf{N}_{4,s,s'})$ is a G-equivariant section of the projection map $\mathrm{TL}(\mathbf{N}_{3,s,s'}) \to \mathrm{TL}(\mathbf{N}_{4,s,s'})$. As remarked in 5.1.3, this special case of 5.3 is known, because it is a consequence of the orbital rigidity of biextensions of p-divisible formal groups proved in [7, Ch. 10]. So the quotient map $\mathbf{N}_{3,s,s'} \twoheadrightarrow \mathbf{N}_{4,s,s'}$ has a section, therefore $[\mathfrak{N}_{2,s}, \mathfrak{N}_{2,s'}] = 0$. $\square$

§6. Proof via hypocotyl elongation in tempered perfections

(6.1) Notation

(6.1.1) We will use the following notation in this section.

- κ is a field of characteristic p.
- Let $\mathbf{N}$ be a Tate unipotent group over κ.
- Let $\lambda_1 := \max(\mathrm{slope}(\mathbf{N}))$, and $\mathbf{Z} := \mathrm{Fil}_{\mathrm{sl}}^{\lambda_1}\mathbf{N}$.
- Let $\mathbf{N}_2 := \mathbf{N}/\mathbf{Z}$, a Tate unipotent subgroup over κ.
- Denote by $\mathbf{N}_{\mathbb{Q}}$ (respectively $\mathbf{N}_{2,\mathbb{Q}}$ and $\mathbf{Z}$) the Mal'cev completions of $\mathbf{N}$ (respectively $\mathbf{N}_2$ and $\mathbf{Z}$).
- Denote by $\mathfrak{N}$ (respectively $\mathfrak{N}_2$ and $\mathfrak{Z}$) the Tate unipotent Lie $\mathbb{Q}_p$-algebras attached to $\mathbf{N}_{\mathbb{Q}}$ (respectively $\mathbf{N}_{2,\mathbb{Q}}$ and $\mathbf{Z}$). Let

  $$\exp_{\mathbf{N}} : \mathfrak{N} \longrightarrow \mathbf{N}_{\mathbb{Q}} \quad \text{and} \quad \exp_{\mathbf{N}_2} : \mathfrak{N}_2 \longrightarrow \mathbf{N}_{2,\mathbb{Q}}$$

  be the exponential maps for $\mathbf{N}_{\mathbb{Q}}$ and $\mathbf{N}_{2,\mathbb{Q}}$ respectively.
- Let $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}) = \mathrm{Aut}(\mathfrak{N})$ be the p-adic Lie group consisting of all automorphisms of $\mathbf{N}_{\mathbb{Q}}$. Let $\mathrm{Aut}(\mathbf{N})$ be the compact p-adic Lie group consisting of all automorphisms of $\mathbf{N}$. It is a closed subgroup of $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$.
- Let $\mathfrak{g}(\mathfrak{N}) := \mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}))$ (respectively $\mathfrak{g}(\mathfrak{N}_2) := \mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_{2,\mathbb{Q}}))$) be the Lie algebra of $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$ (respectively $\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})$).
- Let $\tau_{\mathbf{N}} : \mathbf{N}_{\mathbb{Q}} \to \mathbf{N}_{\mathbb{Q}}/\mathbf{N} = \mathrm{TL}(\mathbf{N})$ be the quotient map from $\mathbf{N}_{\mathbb{Q}}$ to the Tate-linear formal variety $\mathrm{TL}(\mathbf{N})$. Let $\mathbf{e}_{\mathbf{N}} : \mathfrak{N} \to \mathrm{TL}(\mathbf{N})$ be the composition $\mathfrak{N} \xrightarrow{\exp_{\mathbf{N}}} \mathbf{N}_{\mathbb{Q}} \xrightarrow{\tau_{\mathbf{N}}} \mathrm{TL}(\mathbf{N})$.
- Let G be a compact p-adic Lie group, and let $\rho : G \to \mathrm{Aut}(\mathbf{N})$ be a *strongly nontrivial* continuous group homomorphism.

- Denote by $d\rho : \mathrm{Lie}(G) \to \mathfrak{g}(\mathfrak{N})$ the Lie algebra homomorphism attached to ρ. We often abuse the notation and write ρ instead of $d\rho$.

- Let $\pi : \mathrm{TL}(\mathbf{N}) \to \mathrm{TL}(\mathbf{N}_2)$ be the projection map between Tate linear formal varieties induced by the quotient homomorphism $\mathbf{N} \twoheadrightarrow \mathbf{N}_2$.

- Let $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G, and let $\mathfrak{g}_{\mathbb{Z}_p}$ be a Lie $\mathbb{Z}_p$-subalgebra of the Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\exp_G(\mathfrak{g}_{\mathbb{Z}_p}) \subseteq G$.

(6.2) Definition. Let $\lambda_2 = \max(\mathrm{slope}(\mathbf{N}_2))$. According to 4.4.2, there exist constants $n_0, c_0 \in \mathbb{N}$, $n_0 \geq 2$, such that for every $n \geq n_0$ and every element $v \in \mathfrak{g}_{\mathbb{Z}_p}$, the element $\exp_G(p^n v)$ of G operates trivially on $\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}]$, where

$$\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0} : \mathrm{TL}(\mathbf{N}_2) \to \mathrm{TL}(\mathbf{N}_2)^{(p^{\lfloor n/\lambda_2 \rfloor - c_0})}$$

is the $(\lfloor n/\lambda_2 \rfloor - c_0)$-th iterate of the relative Frobenius morphism on $\mathrm{TL}(\mathbf{N}_2)$.

For every $n \geq n_0$, define

$$\mathfrak{z}_n[v] : \pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}]\big) \longrightarrow Z$$

to be the unique morphism from $\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}]$ to Z such that

$$\rho(\exp(p^n v))\big|_{\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}])} = \mathfrak{z}_n[v] * \mathrm{id}_{\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}])}.$$

In other words,

$$\rho(\exp(p^n v))(x) = \mathfrak{z}_n[v](x) * x$$

for every functorial point x of $\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}])$, where $\mathfrak{z}_n[v](x) * x$ is the translation of x by $\mathfrak{z}_n[v](x)$ for the Z-torsor structure of $\mathrm{TL}(\mathbf{N})$. Let

$$\varsigma_1 := \max(\tfrac{\lambda_1}{2}, \lambda_2) = \max\big(\tfrac{\lambda_1}{2}, \max(\mathrm{slope}(\mathbf{N}_2))\big).$$

Note that

$$\lambda_2 \leq \varsigma_1 < \lambda_1.$$

The statement 6.2.1 is immediate from the definition of $\mathfrak{z}_n[v]$.

(6.2.1) Corollary. *The map $\mathfrak{z}_n[v]$ is compatible with the Z-torsor structure of* $\mathrm{TL}(\mathbf{N})$ *in the following sense: For every functorial point (z, x) of $Z \times \pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_0}])$ and any $v \in \mathfrak{g}_{\mathbb{Z}_p}$, we have*

$$\mathfrak{z}_n[v](z * x) = \big(\exp_G(p^n v)\cdot z\big) * \mathfrak{z}_n[v](x).$$

(6.2.2) Proposition. *We use the notation in 6.2. There exist constants $n_1, c_1 \in \mathbb{N}$ with $n_1 \geq n_0$ such that for all $n \geq n_1$, we have $\lfloor n/\varsigma_1 \rfloor - c_1 \leq \lfloor n/\lambda_2 \rfloor - c_0$, and*

$$\big(\,\mathfrak{z}_{n+1}\,[v] - [p]_Z \circ \mathfrak{z}_n[v]\big)\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/\varsigma_1 \rfloor - c_1}]} = 0$$

for all $v \in \mathfrak{g}_{\mathbb{Z}_p}$.

(6.2.3) Remark. The compatibility statement 6.2.2 will be proved in 6.3.8 as a consequence of a formula 6.3.6 for $\mathfrak{z}_n[v]$. The effect of restricting to $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/\varsigma_1\rfloor - c_1}]$ is to strip all terms of higher order in v in the formula for $\mathfrak{z}_n[v]$, so that $\mathfrak{z}_n[v]$ is equal to its first-order approximation on the infinitesimal neighborhood $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{\lfloor n/\varsigma_1\rfloor - c_1}]$ of the base point of $\mathrm{TL}(\mathbf{N})$. See 6.3.7.

(6.3) A formula for the map $\mathfrak{z}_n[v]$.
Assume that the base field κ is perfect.

(6.3.1) Since κ is perfect, both $\mathfrak{N}$ and $\mathfrak{N}_2$ admit decompositions as direct sums of Tate unipotent Lie $\mathbb{Q}_p$-algebras attached to isoclinic p-divisible groups. In particular we have a canonical decomposition

$$\mathfrak{N} \cong \mathfrak{Z} \oplus \mathfrak{N}_2.$$

Under this decomposition, the Lie bracket $[\ ,\ ]_{\mathfrak{N}}$ on $\mathfrak{N}$ has the form

$$[\,(z_1,u_1),\,(z_2,u_2,)\,]_{\mathfrak{N}} = \big(\beta_{\mathfrak{N}}(u_1,u_2),\,[u_1,u_2]_{\mathfrak{N}_2}\big)\quad \forall\, u_1,u_2\in\mathfrak{N}_2,\ \forall\, z_1,z_2\in\mathfrak{Z},$$

where $[\ ,\ ]_{\mathfrak{N}_2}$ denotes the Lie bracket for $\mathfrak{N}_2$, and

$$\beta_{\mathfrak{N}} : \mathfrak{N}_2\times\mathfrak{N}_2 \longrightarrow \mathfrak{Z}$$

is a $\mathfrak{Z}$-valued skew-symmetric bilinear pairing on $\mathfrak{N}_2$.

Let $\mathfrak{N}_2 = \oplus_{s\in \mathrm{slope}(\mathfrak{N}_2)}\mathfrak{N}_{2,s}$ be the slope decomposition of $\mathfrak{N}_2$, where $\mathfrak{N}_{2,s}$ is the isoclinic component of $\mathfrak{N}_2$ with slope s. Clearly each $\mathfrak{N}_{2,s}$ is stable under the action of G, so is $\mathfrak{Z}$. Moreover the Lie bracket $[\ ,\ ]_{\mathfrak{N}}$ is G-equivariant, or equivalently $[\ ,\ ]_{\mathfrak{N}_2}$ and $\beta_{\mathfrak{N}}$ are both G-equivariant. In addition,

$$[\mathfrak{N}_{2,s},\mathfrak{N}_{2,s'}]_{\mathfrak{N}_2}\ \subseteq\ \begin{cases} \mathfrak{N}_{2,s'_s} & \text{if}\ \ s+s'\in \mathrm{slope}(\mathfrak{N}_2),\\ (0) & \text{if}\ \ s+s'\notin \mathrm{slope}(\mathfrak{N}_2),\end{cases}$$

and

$$\beta_{\mathfrak{N}}\big(\mathfrak{N}_{2,s},\mathfrak{N}_{2,s'}\big) = (0)\quad \text{unless}\ \ s+s'=\lambda_1.$$

The statements in lemma 6.3.2 below follow easily from the above discussion.

(6.3.2) Lemma. *There exists a Tate unipotent Lie $\mathbb{Z}_p$-subalgebra $\mathfrak{N}_{2,\mathbb{Z}_p}$ of $\mathfrak{N}_{2,\mathbb{Q}}$ and a Tate unipotent Lie $\mathbb{Z}_p$-subalgebra $\mathfrak{Z}_{\mathbb{Z}_p}$ of $\mathfrak{Z}$ with the following properties.*

(1) $\mathfrak{N}_{2,\mathbb{Z}_p}\otimes_{\mathbb{Q}}\mathbb{Q} = \mathfrak{N}_2$, *and* $\mathfrak{Z}_{\mathbb{Z}_p}\otimes_{\mathbb{Q}}\mathbb{Q} = \mathfrak{Z}$.

(2) *Both $\mathfrak{N}_{2,\mathbb{Z}_p}$ and $\mathfrak{Z}_{\mathbb{Z}_p}$ are stable under the action of G.*

(3) *The direct sum $\mathfrak{Z}_{\mathbb{Z}_p}\oplus\mathfrak{N}_{2,\mathbb{Z}_p} =: \mathfrak{N}_{\mathbb{Z}_p}$ is a Tate unipotent Lie $\mathbb{Z}_p$-subalgebra of $\mathfrak{N}$. In other words $\beta_{\mathfrak{N}}(\mathfrak{N}_{2,\mathbb{Z}_p},\mathfrak{N}_{2,\mathbb{Z}_p})\subseteq\mathfrak{N}_{\mathbb{Z}_p}$.*

(4) *The image* $\exp_{\mathbf{N}}(\mathfrak{N}_{\mathbb{Z}_p})$ *of* $\mathfrak{N}_{\mathbb{Z}_p}$ *under the exponential map* $\exp_{\mathbf{N}}:\mathfrak{N}\to\mathbf{N}_{\mathbb{Q}}$ *is a Tate unipotent subgroup of* $\mathbf{N}$ *isogenous to* $\mathbf{N}$*. Consequently* $\exp_{\mathbf{N}_2}(\mathfrak{N}_{2,\mathbb{Z}_p})$ *is a Tate unipotent subgroup of* $\mathbf{N}_2$*. Similarly the exponential map* $\exp_{\mathbf{Z}}$ *identifies* $\mathfrak{Z}_{\mathbb{Z}_p}$ *with a Tate unipotent subgroup of* $\mathbf{Z}$*, and the inclusion map* $\exp_{\mathbf{Z}}(\mathfrak{Z}_{\mathbb{Z}_p})\hookrightarrow\mathbf{Z}$ *is an isogeny.*

(5) *For every* $n\in\mathbb{N}$*, the relative Frobenius morphism* $\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}:\mathfrak{N}_{\mathbb{Z}_p}\to\mathfrak{N}_{\mathbb{Z}_p}^{(p^n)}$ *is compatible with the direct sum decomposition*

$$\mathfrak{N}_{\mathbb{Z}_p}=\mathfrak{Z}_{\mathbb{Z}_p}\oplus\mathfrak{N}_{2,\mathbb{Z}_p}$$

and the bilinear pairing $\beta_{\mathfrak{N}}$*. In other words,*

$$\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}\big([u,v]_{\mathfrak{N}_2}\big)=\big[\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}(u),\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}(v)\big]_{\mathfrak{N}_2^{(p^n)}}$$

and

$$\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}\big(\beta_{\mathfrak{N}}([u,v])\big)=\beta_{\mathfrak{N}^{(p^n)}}\big(\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}(u),\mathrm{Fr}^n_{\mathfrak{N}_{\mathbb{Z}_p}/\kappa}(v)\big)$$

for all functorial points (u,v) *of* $\mathfrak{N}_{\mathbb{Z}_p}\times\mathfrak{N}_{\mathbb{Z}_p}$*.*

(6.3.3) We continue with the setup in 6.3.2. Let n_0,c_0 be as in definition 6.2. Let n be a natural number such that $n\geq n_0$. Let (z,u) be a functorial point of $\mathfrak{Z}\oplus\mathfrak{N}_2=\mathfrak{N}$ such that $\mathrm{Fr}^{\lfloor n/\lambda_2\rceil-c_0}_{\mathfrak{N}_{2,\mathbb{Z}_p}}(u)\in\mathfrak{N}_{2,\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2\rceil-c_0})}$, so that the element $\mathbf{e}_{\mathbf{N}}(z,u)=\tau_{\mathbf{N}}(\exp_{\mathbf{N}}(z,u))$ of $\mathrm{TL}(\mathbf{N})$ lies in $\pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rceil-c_0}]\big)$. Let $v=(C,B)$ be an element of $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{N}))$ with components $C\in\mathrm{End}(\mathfrak{Z}_{\mathbb{Z}_p})$ and $B\in\mathrm{End}(\mathfrak{N}_{2,\mathbb{Z}_p})$, so that v is a derivation of the Lie $\mathbb{Z}_p$-algebra $\mathfrak{N}_{\mathbb{Z}_p}$. Shrinking $\mathfrak{N}_{\mathbb{Z}_p}$ if necessary, we may and do assume that $\exp_{\mathrm{Aut}(\mathfrak{N})}(\mathbb{Z}_p\cdot v)\in\mathrm{Aut}(\mathbf{N})$. We would like to compute $\mathfrak{z}_n[v](\mathbf{e}_{\mathbf{N}}(z,u))$, which is equal to the restriction to $\pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rceil-c_0}]\big)$ of the image of

$$\begin{aligned}&\exp_{\mathbf{N}}(z,u)^{-1}\cdot\exp_{\mathbf{N}}(\exp_G(p^nv)\cdot(z,u))\\ &=\ \exp_{\mathbf{N}}(z,u)^{-1}\cdot\exp_{\mathbf{N}}\big(\exp_G(p^nC)\cdot z,\ \exp_G(p^nB)\cdot u\big)\\ &=\ \exp_{\mathbf{Z}}((\exp_G(p^nC)-1)z)\cdot\exp_{\mathbf{N}}(u)^{-1}\cdot\exp_{\mathbf{N}}(\exp_G(p^nB)u)\end{aligned}$$

under the map $\tau_{\mathbf{N}}:\mathbf{N}_{\mathbb{Q}}\to\mathrm{TL}(\mathbf{N})$. So $\mathfrak{z}_n[v](\mathbf{e}_{\mathbf{N}}(z,u))\cdot\exp_{\mathbf{Z}}(-(\exp_G(p^nC)-1)z)$ is equal to the restriction to $\pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rceil-c_0}]\big)$ of the image of

$$\exp_{\mathbf{N}}(-u)\cdot\exp_{\mathbf{N}}\big(\exp_G(p^nB)u\big)=\exp_{\mathbf{N}}(-u)\cdot\exp\big(u+p^nBu+\textstyle\sum_{m\geq2}\frac{p^{mn}}{m!}B^mu\big).$$

Note that $\mathrm{ord}_p(\frac{p^{mn}}{m!})\geq m(n-1)\geq 2$ for all $m\geq 2$.

The BCH formula applied to $\exp(-x)\cdot\exp(x+y)$ tells us that there exist elements $\theta_{r,s}(x,y)$ in the free Lie algebra $L(x,y)$ over $\mathbb{Q}$ with free generators $\{x,y\}$, where (r,s) ranges through all pairs with $r,s\in\mathbb{N}$, $s\geq1$, such that $\theta_{r,s}(x,y)$ is homogeneous of bi-degree (r,s) for all $r,s\in\mathbb{N}$, such that

$$\exp(-x)\cdot\exp(x+y)=\exp(\theta(x,y))=\exp\big(\textstyle\sum_{r\geq0,\,s\geq1}\theta_{r,s}(x,y)\big),\tag{6.3.3.1}$$

where $\theta(x,y) := \sum_{r,s} \theta_{r,s}(x,y)$. We are interested in the part $\theta'(x,y) := \sum_{r,1} \theta_{r,1}(x,y)$ of $\theta(x,y)$ which is linear in y. Note that there exists an infinite power series $f(t) \in \mathbb{Q}[[t]]$ such that $\theta'(x,y) = f(\mathrm{ad}x)(y)$.

(6.3.4) Definition. Let $\theta_{r,s}(x,y)$ be the homogeneous elements of bi-degree (r,s) in the free Lie algebra $L(x,y)$ over $\mathbb{Q}$ in (6.3.3.1). For any functorial point u of $\mathfrak{N}_2 \subseteq \mathfrak{N}$, any $v = (C,B) \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{N}))$ with components $C \in \mathrm{End}(\mathfrak{Z}_{\mathbb{Z}_p})$ and $B \in \mathrm{End}(\mathfrak{N}_{2,\mathbb{Z}_p})$, consider the functorial point $\theta'(u, p^n Bu))$ of $\mathfrak{N}$, where $\theta'(x,y) = \sum_{r,1} \theta_{r,1}(x,y)$ is the element of $L(x,y)^\wedge$ defined in the preceding paragraph.

Define functorial points $\gamma(u,Bu) \in \mathfrak{Z}$ (respectively $\delta(u,Bu) \in \mathfrak{N}_2$) as the $\mathfrak{Z}$-component (respectively $\mathfrak{N}_2$-component) of $\theta'(u,Bu)$. In other words

$$\theta'(u,Bu) = \gamma(u,Bu) + \delta(u,Bu), \quad \gamma(u,Bu) \in \mathfrak{Z},\ \delta(u,Bu) \in \mathfrak{N}_2.$$

For each $n \in \mathbb{N}$, let

$$\gamma(u,p^nBu) = [p^n]_{\mathfrak{Z}}(\gamma(u,Bu)), \quad \delta(u,p^nBu) = [p^n]_{\mathfrak{N}_2}(\delta(u,Bu)).$$

Remark. Since the inverse image of $\mathfrak{N}_{\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2 \rceil - c_0})}$ under the relative Frobenius map

$$\mathrm{Fr}_{\mathfrak{N}}^{\lfloor n/\lambda_2 \rceil - c_0} : \mathfrak{N} \to \mathfrak{N}^{(p^{\lfloor n/\lambda_2 \rceil - c_0})}$$

is stable under the Lie bracket, the functoriality of the Frobenius maps implies that

$$\mathrm{Fr}_{\mathfrak{N}}^{\lfloor n/\lambda_2 \rceil - c_0}\gamma(u,Bu) \in \mathfrak{Z}_{\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2 \rceil - c_0})}, \quad \mathrm{Fr}_{\mathfrak{N}}^{\lfloor n/\lambda_2 \rceil - c_0}\delta(u,Bu) \in \mathfrak{N}_{2,\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2 \rceil - c_0})}.$$

(6.3.5) Lemma. *We keep the notation in* 6.3.3–6.3.4. *There exist natural numbers $n_2 \geq n_0$ and $c_2 \geq c_0$, for all functorial points u of $\mathfrak{N}_2 \subseteq \mathfrak{N}$ with $\mathrm{Fr}_{\mathfrak{N}_2}^{\lfloor n/\lambda_2 \rceil - c_2}(u) \in \mathfrak{N}_{2,\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2 \rceil - c_2})}$ and all element $v = (C,B) \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{N}))$ with components $C \in \mathrm{End}(\mathfrak{Z}_{\mathbb{Z}_p})$ and $B \in \mathrm{End}(\mathfrak{N}_{2,\mathbb{Z}_p})$, the following statements hold for all $n \geq n_2$.*

(a) $p^{\lfloor n((\lambda_1/\lambda_2)-1) \rfloor} \cdot \gamma(u,p^nBu) \in \mathfrak{Z}_{\mathbb{Z}_p}$, *and* $\delta(u,p^nBu) \in \mathfrak{N}_{2,\mathbb{Z}_p}$.

(b) *The restriction to* $\pi^{-1}(\mathrm{TL}(\mathbf{N}_2))[\mathrm{Fr}^{\lfloor n/\lambda_2 \rfloor - c_2}]$ *of the image of*

$$\exp_{\mathbf{N}}(-\gamma(u,p^nBu)) \cdot \exp_{\mathbf{N}}(-u) \cdot \exp\left(u + p^nBu + \sum\nolimits_{m \geq 2} \tfrac{p^{mn}}{m!} B^m u\right)$$

in $\mathrm{TL}(\mathbf{N})$ *is* 0, *namely the* 0*-element of* Z.

Proof. The statement (a) follow from the remark after 6.3.4, that

$$\mathrm{Fr}_{\mathfrak{N}}^{\lfloor n/\lambda_2 \rceil - c_0}\gamma(u,Bu) \in \mathfrak{Z}_{\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2 \rceil - c_0})} \quad \text{and} \quad \mathrm{Fr}_{\mathfrak{N}}^{\lfloor n/\lambda_2 \rceil - c_0}\delta(u,Bu) \in \mathfrak{N}_{2,\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2 \rceil - c_0})},$$

because $\mathfrak{Z}$ is isoclinic of slope λ_1, and $\lambda_2 = \max(\mathrm{slope}(\mathfrak{N}_2)$.

(b) By the BCH formula, there exist elements $\eta_{m,j}(x_0, x_1, \ldots, x_m)$ in the free Lie algebra over $\mathbb{Q}$ with free generators $\{x_0, \ldots, x_m\}$, where m, d ranges through all positive integers, with the following properties: Each $\eta_{m,d}(x_0, x_1, \ldots, x_m)$ is a sum of elements homogeneous elements $\eta_{m,d,i}$ such that

$$\deg_{x_1}\theta_{m,d,i} + 2\deg_{\{x_2,\ldots,x_m\}}\theta_{m,d,i} \geq 2, \quad \deg_{\{x_1,\ldots,x_m\}}\theta_{m,d,i} = d,$$

and

$$\begin{aligned}&\exp_{\mathbf{N}}(-\gamma(u, p^n Bu)) \cdot \exp_{\mathbf{N}}(-u) \cdot \exp\left(u + p^n Bu + \sum\nolimits_{m\geq 2} \tfrac{p^{mn}}{m!} B^m u\right) \\ &= \exp\left(\delta(u, p^n B) + \sum\nolimits_{m,d\geq 1} \eta_{m,d}(u, p^n B, p^{2n} B^2 u, \ldots, p^{mn} B^m u)\right)\end{aligned}$$

Note that $\eta_{m,d}(u, p^n B, p^{2n} B^2 u, \ldots, p^{mn} B^m u) = 0$ if $\mathbf{N}$ is nilpotent of class at most $d-1$.

Since the inverse image of $\mathfrak{N}_{2,\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2\rceil - c_2})}$ under the relative Frobenius map

$$\mathrm{Fr}_{\mathfrak{N}_2}^{\lfloor n/\lambda_2\rceil - c_2} : \mathfrak{N}_2 \to \mathfrak{N}_2^{(p^{\lfloor n/\lambda_2\rceil - c_2})}$$

is stable under the Lie bracket,

$$\eta_{m,d}(u, p^n B, p^{2n} B^2 u, \ldots, p^{mn} B^m u) \in \mathfrak{Z}_{\mathbb{Z}_p} \oplus \mathfrak{N}_{2,\mathbb{Z}_p}$$

for all m, d, after passing to a bigger constant c_2 if necessary. The statement 6.3.5 follows from the assumption that $\exp_{\mathbf{N}}(\mathfrak{Z}_{\mathbb{Z}_p} \oplus \mathfrak{N}_{2,\mathbb{Z}_p}) \subseteq \mathbf{N}$. $\square$

We summarize the above calculation in proposition 6.3.6 below.

(6.3.6) Proposition. *Let n_2, c_2 be as in 6.3.5. Let (z, u) be a functorial point of $\mathfrak{Z} \oplus \mathfrak{N}_2 = \mathfrak{N}$ with $\mathrm{Fr}_{\mathfrak{N}_2}^{\lfloor n/\lambda_2\rceil - c_2}(u) \in \mathfrak{N}_{2,\mathbb{Z}_p}^{(p^{\lfloor n/\lambda_2\rceil - c_2})}$. Let $v = (C, B)$ be an element of $\mathfrak{Z}_{\mathbb{Z}_p} \oplus \mathfrak{N}_{2,\mathbb{Z}_p}$. Then the restriction*

$$\mathfrak{z}_n[v](\mathbf{e}_{\mathbf{N}}(z, u))\big|_{\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rfloor - c_2}])}$$

of $\mathfrak{z}_n[v](\mathbf{e}_{\mathbf{N}}(z, u))$ to $\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rfloor - c_2}])$ is equal to the image in $Z = \mathbf{Z}_{\mathbb{Q}}/\mathbf{Z}$ of the element

$$\exp_{\mathbf{Z}}\left((\exp_G(p^n C) - 1)\cdot z + \gamma(u, p^n Bu)\right)$$

of $\mathbf{Z}_{\mathbb{Q}}$. In other words

$$\left[\, \mathfrak{z}_n\, [v](\mathbf{e}_{\mathbf{N}}(z, u)) * \mathbf{e}_{\mathbf{Z}}\big((1 - \exp_G(p^n C))\cdot z - \gamma(u, p^n Bu)\big)\right]\big|_{\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rfloor - c_2}])} = 0.$$

The following corollary 6.3.7 of proposition 6.3.6 gives an approximation of the map $\mathfrak{z}_n[v]$ which is linear in v.

(6.3.7) Corollary. *There exist constants $n_1 \geq n_2$ and $c_1 \geq c_2$ such that*

$$\left[\, \mathfrak{z}_n\, [v](\mathbf{e}_{\mathbf{N}}(z, u)) -_Z \mathbf{e}_{\mathbf{Z}}\big(p^n C z + \gamma(u, p^n Bu)\big)\right]\big|_{\pi^{-1}(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rfloor - c_2}])} \equiv 0 \pmod{\mathfrak{m}_Z^{p^{\lfloor n/\varsigma_1\rfloor - c_1}}}$$

for all $n \geq n_1$. In other words

$$\mathfrak{z}_n[v](\mathbf{e}_{\mathbf{N}}(z, u))\big|_{\mathfrak{m}_Z^{p^{\lfloor n/\varsigma_1\rfloor - c_1}}} = \mathbf{e}_{\mathbf{Z}}\big(p^n C z + \gamma(u, p^n Bu)\big)\big|_{\mathfrak{m}_Z^{p^{\lfloor n/\varsigma_1\rfloor - c_1}}}$$

(6.3.8) Proof of 6.2.2. Clearly it suffice to prove the statement 6.2.2 after extending the base field κ to the perfect closure of κ. So we may and do assume that κ is perfect. Proposition 6.2.2 follows immediately from the approximate formula 6.3.7 for $\mathfrak{z}_n[v]$ and $\mathfrak{z}_{n+1}[v]$. □

(6.4) Assume from now on till the end of §6 that the base field κ is algebraically closed.

(6.4.1) Let Z be the isoclinic p-divisible group $\mathrm{TL}(\mathbf{Z})$ over κ. After a suitable modification of $\mathbf{N}$ by an isogeny, and passing to a suitable open subgroup of G and a suitable open Lie $\mathbb{Z}_p$-subalgebra of $\mathfrak{g}$, we may and do assume that there exists a positive integer r_0 such that $r_0\lambda_1 \in \mathbb{N}$

$$\mathrm{Ker}(\mathrm{Fr}_Z^{r_0}) = [p^{r_0\lambda_1}]_Z$$

and the properties in 6.3.2 are satisfied. *This assumption on Z in force throughout the rest of* §6.

It follows that there exists an isomorphism $Z^{(p^{r_0})} \xrightarrow[\sim]{\alpha} Z$ such that $\alpha \circ \mathrm{Fr}_Z^{r_0} = [p^{r_0\lambda_1}]_Z$. Then the affine coordinate ring of Z is topologically generated by formal functions f on Z such that $\alpha^* f = f \otimes 1 \in \mathcal{O}_Z \otimes_{(\kappa,\sigma_\kappa^{r_0})} \kappa$. Such formal functions form the affine coordinate ring of a model of Z over the finite field $\mathbb{F}_{p^{r_0}}$.

Choose a positive integer multiple r_1 of r_0 and an integer s_1 such that

$$r_1 < s_1, \quad s_1\varsigma_1 \in \mathbb{N}, \quad \text{and} \quad s_1\varsigma_1 < r_1\lambda_1.$$

(6.4.2) Corollary. *There exists a constant m_1 such that for all $m \geq m_1$, we have $mr_1 - c_0 \leq ms_1$, and*

$$\left(\mathfrak{z}_{(m+1)r_1\lambda_1}[v] - [p^{r_1\lambda_1}]_Z \circ \mathfrak{z}_{mr_1\lambda_1}[v]\right)\Big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]} = 0$$

for all $v \in \mathfrak{g}_{\mathbb{Z}_p}$.

Note that $ms_1 < \lfloor mr_1\lambda_1/\varsigma_1 \rfloor - c_1$ for all m sufficiently large, because $s_1\varsigma_1 < r_1\lambda_1$, hence $\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}] \subseteq \mathrm{TL}[\mathrm{Fr}^{\lfloor mr_1\lambda_1/\varsigma_1\rfloor - c_1}]$.

(6.4.3) Remark. (a) Corollary 6.4.2 is a weaker form of proposition 6.2.2. It says that for any v in the lattice $\mathfrak{g}_{\mathbb{Z}_p}$ of the Lie algebra $\mathfrak{g}$ of G, the sequence of maps

$$\left(\mathfrak{z}_{mr_1\lambda_1}[v] : \mathrm{TL}[\mathrm{Fr}^{ms_1}] \to Z\right)_{m \geq m_1}$$

indexed by integers $m \geq m_1$ is $[p^{r_1\lambda_1}]$*-compatible with respect to* ϕ^{s_1} in the sense of [7, §10.7].
(b) The congruence in 6.4.2 becomes stronger for larger values of $\frac{s_1}{r_1}$. Note that

$$\frac{s_1}{r_1} < \frac{\lambda_1}{\varsigma_1} = \min\Big(2, \frac{\lambda_1}{\lambda_2}\Big).$$

(6.5) Review of tempered perfections.
In 6.5, κ denotes a field of characteristic p.

(6.5.1) In [7, Ch. 10.7], given a complete augmented noetherian local integral domain $(R,)$ over κ, we introduced a family

$$\big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}\,\big)_{r,s,i_0}$$

of non-noetherian complete augmented local domains over κ sandwiched between $(R,\mathfrak{m})$ and the completion $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$ its perfection $(R,\mathfrak{m})^{\mathrm{perf}}$, with integer parameters r,s,i_0 satisfying

$$0<r<s,\quad i_0\geq 0.$$

Recall that the perfection R^{perf} of $(R,\mathfrak{m})$ has a decreasing filtration $\mathrm{Fil}^{\bullet}_{\mathrm{deg}}R^{\mathrm{perf}}$ indexed by real numbers, defined by

$$\mathrm{Fil}^u_{\mathrm{deg}}R^{\mathrm{perf}} := \begin{cases} \big\{x\in R^{\mathrm{perf}}\mid \exists j\in\mathbb{N}\text{ s.t. } x^{p^j}\in\mathfrak{m}^{\lceil u\cdot p^j\rceil}\big\} & \text{if } u\geq 0\\ R^{\mathrm{perf}} & \text{if } u\leq 0\end{cases}$$

The completed perfection $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$ of R is the completion of R^{perf} with respect to the above filtration. The filtration $\mathrm{Fil}^{\bullet}_{\mathrm{deg}}$ on R^{perf} induces a decreasing filtration on $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$, denoted again by $\mathrm{Fil}^{\bullet}_{\mathrm{deg}}$.

By definition, $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ is the completion of the subring

$$\sum_{n\geq 1}\phi^{-nr}\big(\mathfrak{m}^{ns-i_0}\big)$$

of R^{perf} with respect to the filtration given by powers of the ideal generated by $\mathfrak{m}$.

Each ring $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ is called a *tempered perfection* of $(R,\mathfrak{m})$. This family of complete augmented local domains over κ is *filtered* in the following sense: given any two rings R_1,R_2 in this family, there is a third ring R_3 in the family which contains both R_1 and R_2. The union

$$(R,\mathfrak{m})^{\mathrm{tmp\,perf}} := \bigcup_{r,s,i_0}(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

is a subring of $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$ which contains R^{perf}, but strictly smaller than $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$. Elements of $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ will be called *tempered* elements of the completed perfection $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$ of R. The filtration $\mathrm{Fil}_{\mathrm{deg}}$ on the completed perfection $(R^{\mathrm{perf}})^\wedge$ induces a filtration on each tempered perfection of $(R,\mathfrak{m})$.

(6.5.2) Given a complete augmented noetherian local domain $(R,\mathfrak{m})$ over κ, there are other versions of families

$$\big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}\,\big)_{r,s,i_0},\quad \big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\,\big)_{A,b,d}\quad\text{and}\quad \big(\,(R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\,\big)_{A,b,d}$$

of tempered perfections of R, indexed by parameters (r,s,i_0) and (A,b,d) respectively.

Each of these three families is cofinal with the family $\big(\,(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}\,\big)_{r,s,i_0}$. For instance each ring $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ is contained in $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ for a suitable (A,b,d), and each $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$

is contained in a $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$. Therefore the union of rings in each of these families is equal to the subring $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ of all tempered elements of $((R,\mathfrak{m})^{\mathrm{perf}})^\wedge$. Any ring in one of the above families is said to be a *tempered perfection* of $(R,\mathfrak{m})$. It is instructive to regard elements of $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ as some sort of "tempered generalized functions on the formal scheme $\mathrm{Spf}(R)$. Since "generalized" is an overused word, we will call elements of $(R,\mathfrak{m})^{\mathrm{tmp\,perf}}$ *tempered virtual functions* on $\mathrm{Spf}(R)$.

(6.5.3) For fixed parameters (r,s,i_0), the assignment

$$(R,\mathfrak{m}) \rightsquigarrow (R,\mathfrak{m})^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

is functorial in $(R,\mathfrak{m})$. Moreover the continuous κ-algebra homomorphism

$$h^\flat : (R_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]} \to (R_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

induced by a continuous κ-algebra homomorphism

$$h : (R_1,\mathfrak{m}_1) \to (R_2,\mathfrak{m}_2)$$

is surjective (respectively injective) if h is surjective (respectively injective). The same is true for the formations of $(R,\mathfrak{m})^{\mathrm{perf},\#}_{s:\phi^r;[i_0]}$, $\big((R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}\big)_{A,b,d}$ and $\big((R,\mathfrak{m})^{\mathrm{perf},\#}_{A,b;d}\big)_{A,b,d}$.

(6.5.4) We illustrate the general idea of tempered virtual functions with the family

$$\big(\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}\big)_{E,c,d}$$

of tempered perfections of power series ring $\kappa[[t_1,\ldots,t_m]]$, depending on parameters (E,c,d), where E,c,d are real numbers, $E>0, C>0, d\geq 0$. This family of tempered perfections of $\kappa[[t_1,\ldots,t_m]]$ is cofinal with each of the four families of tempered perfections of $\kappa[[t_1,\ldots,t_m]]$ mentioned in 6.5.1.

By definition $\kappa\langle\langle t_1^{p^{-\infty}},\ldots,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C;\,d}$ consists of all formal power series of the form

$$\sum\nolimits_{I\in\mathrm{supp}(m:\flat:E;C,d)} b_I\,\underline{t}^I$$

such that $b_I\in\kappa$ for all $I\in\mathrm{supp}(m:\flat:E;C,d)$. Here

- $\mathrm{supp}(m:\flat:E;C,d)$ is the sub-semigroup of $(\mathbb{N}[\frac{1}{p}]^m,+)$ given by

$$\mathrm{supp}(m:\flat:E;C,d) := \big\{\, I\in\mathbb{N}[\tfrac{1}{p}]^m \;\big|\; |I|_p \leq \max\big(C\cdot(|I|_\sigma+d)^E,1\big)\,\big\}.$$

- $\mathbb{N}[\frac{1}{p}]^m$ is the sub-semigroup of $(\mathbb{Z}[\frac{1}{p}]^m,+)$ consisting of all m-tuples $(i_1,\ldots,i_m)$ in $\mathbb{Z}[\frac{1}{p}]^m$ with all entries $i_j\geq 0$.

- For each $I = (i_1, \ldots, i_m) \in \mathbb{N}[\frac{1}{p}]^m$, $|I|_p$ is the usual p-adic norm of I given by

$$|I|_p := p^{-\mathrm{ord}_p(\max(i_1,\ldots,i_m))},$$

while $|I|_\sigma$ is the archimedean norm of I given by

$$|I|_\sigma := i_1 + \cdots + i_m.$$

In particular the p-adic norm of I is bounded by a polynomial $f_{E,c,d}(|I|_\sigma)$ of the archimedean norm of I, for all I in $\mathrm{supp}(m : \flat : E; C, d)$.

If we replace the archimedean norm $|I|_\sigma$ on $\mathbb{N}[\frac{1}{p}]^m$ by the max norm

$$|I|_\infty := \max(i_1, \ldots, i_p),$$

we get a ring $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\#}$. The resulting family

$$\left(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\#}\right)_{E,c,d}$$

of tempered perfections $\kappa[[t_1, \ldots, t_m]]$ is cofinal with the family $\left(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\,\flat}\right)_{E,c,d}$ as well.

(6.6) How tempered virtual formal functions are used to prove rigidity

The notion of tempered perfections is critically important to the proof of rigidity of biextensions of p-divisible formal groups, and it plays a similar role in the proof of orbital rigidity of Tate-linear formal varieties. Both of the two components of proof of theorem 5.1, described in 6.6.1 and 6.6.3 below, use tempered formal functions $\mathrm{TL}(\mathbf{N})$ in an essential way.

(6.6.1) For each element $v \in \mathrm{Lie}(\mathrm{Aut}(\mathbf{N}))$, define a *tempered virtual morphism*

$$\tilde{\mathfrak{z}}[v] : \mathrm{TL}(\mathbf{N}) \dashrightarrow Z$$

which interpolates the action of $\exp(p^{mr_1\lambda_1}v)$ on $\mathrm{TL}(\mathbf{N})$ for all $m \gg 0$.

Technically, such a tempered virtual morphism means a continuous κ-linear ring homomorphism

$$R_Z \xrightarrow{\tilde{\mathfrak{z}}[v]^*} \left(R_{\mathrm{TL}(\mathbf{N})}\right)_{A,b;d}^{\mathrm{perf},\,\flat}$$

from the affine coordinate ring $R_Z := \Gamma(Z, \mathcal{O}_Z)$ of Z to a tempered perfection $\left(R_{\mathrm{TL}(\mathbf{N})}\right)_{A,b;d}^{\mathrm{perf},\,\flat}$ of $R_{\mathrm{TL}(\mathbf{N})} := \Gamma(\mathrm{TL}(\mathbf{N}), \mathcal{O}_{\mathrm{TL}(\mathbf{N})})$ with suitable parameters A, b, d. Such a tempered virtual morphism is constructed from the compatibility statement 6.2.2 of the maps $\mathfrak{z}_n[v]$, as the limit

$$\lim_{m\to\infty} [p^{-r_1t_1}] \circ \mathfrak{z}_{mr_1\lambda_1}[v]$$

in a suitable sense. This tempered virtual morphism $\tilde{\mathfrak{z}}[v]$ can be regarded as a substitute for the "derivative" of the action on $\mathrm{TL}(\mathbf{N})$ of the "one-parameter subgroup" $\exp(p^n v)$ with discrete parameter $n \in \mathbb{N}$, $n \gg 0$. It has the following properties:

(a) The restriction of $\tilde{\mathfrak{z}}[v]$ to the closed formal subscheme $Z \subseteq \mathrm{TL}(\mathbf{N})$ is equal to the restriction $v|_Z$ of v to Z, regarded as an element of $\mathrm{End}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(b) More generally, $\tilde{\mathfrak{z}}[v]$ is compatible with the Z-torsor structure of $\mathrm{TL}(\mathbf{N})$.

(6.6.2) Remark. (i) In the setting of 6.1.1, there isn't any natural map from $\mathrm{TL}(\mathbf{N})$ to $Z = \mathrm{TL}(\mathbf{Z})$ in the category of formal schemes over κ. However as indicated in (a) above, there are many tempered virtual morphisms from $\mathrm{TL}(\mathbf{N})$ to $Z = \mathrm{TL}(\mathbf{Z})$, one for each discrete one parameter subgroup in $\mathrm{Aut}(\mathrm{TL}(\mathbf{N}))$ associated to elements of $\mathrm{Lie}(\mathrm{Aut}(\mathrm{TL}(\mathbf{N}))$. In the special case when $v = \frac{1}{\lambda_1} D_{\mathrm{Euler}}$ in the notation of 3.4.6, we get a tempered virtual morphism from $\mathrm{TL}(\mathbf{N})$ to Z which is a *projection*, i.e. its restriction to Z is the identity map id_Z on Z.
(ii) The restriction $\tilde{\mathfrak{z}}[v]\big|_Z$ of $\tilde{\mathfrak{z}}[v]$ to Z corresponds to the composition

$$R_Z \xrightarrow{\tilde{\mathfrak{z}}[v]^*} \big(R_{\mathrm{TL}(\mathbf{N})}\big)^{\mathrm{perf},\flat}_{A,b;d} \xrightarrow{q_Z^\flat} \big(R_Z\big)^{\mathrm{perf},\flat}_{A,b;d},$$

where $q_Z^\flat$ is the homomorphism between tempered perfections induced by the quotient homomorphism $q_Z : R_{\mathrm{TL}(\mathbf{N})} \to R_Z$.
(iii) The property (b) above should mean, in spirit, that $\tilde{\mathfrak{z}}[v](z * x) = \tilde{\mathfrak{z}}[v]\big|_Z(z) +_Z \tilde{\mathfrak{z}}[v](x)$ for all functorial points (z, x) of $Z \times \mathrm{TL}(\mathbf{N})$, if $\tilde{\mathfrak{z}}[v]$ is a morphism of formal schemes. At present (before a general theory of "tempered formal schemes" is available), it means that the following diagram

$$\begin{array}{ccc}
R_Z & \xrightarrow{\quad\mu^*\quad} & R_Z \hat{\otimes}_\kappa R_{\mathrm{TL}(\mathbf{N})} \\
\downarrow{\scriptstyle \tilde{\mathfrak{z}}[v]^*} & & \downarrow{\scriptstyle (\tilde{\mathfrak{z}}[v]|_Z)^* \otimes j_{R_{\mathrm{TL}(\mathbf{N})}}} \\
\big(R_{\mathrm{TL}(\mathbf{N})}\big)^{\mathrm{perf},\flat}_{A,b;d} & \xrightarrow{\quad(\mu^*)^\flat\quad} & \big(R_Z\big)^{\mathrm{perf},\flat}_{A,b;d} \hat{\otimes}_\kappa \big(R_{\mathrm{TL}(\mathbf{N})}\big)^{\mathrm{perf},\flat}_{A,b;d}
\end{array}$$

commutes, where $\mu^* : R_Z \to R_Z \hat{\otimes}_\kappa R_{\mathrm{TL}(\mathbf{N})}$ corresponds to the Z-torsor structure on $\mathrm{TL}(\mathbf{N})$ and $j_{R_{\mathrm{TL}(\mathbf{N})}}$ is the inclusion map of $R_{\mathrm{TL}(\mathbf{N})} \hookrightarrow \big(R_{\mathrm{TL}(\mathbf{N})}\big)^{\mathrm{perf},\flat}_{A,b;d}$.

(6.6.3) Given a reduced irreducible closed formal subscheme W of $\mathrm{TL}(\mathbf{N})$ stable under the action of G as in 5.2, we will use the method of hypocotyl elongation in tempered perfections to show that W is closed under the translation action of schematic image of $\tilde{\mathfrak{z}}[v]\big|_W : W \dashrightarrow Z$ for the Z-torsor structure on $\mathrm{TL}(\mathbf{Z})$.

The schematic image of the restriction $\tilde{\mathfrak{z}}[v]\big|_W$ to W of $\tilde{\mathfrak{z}}[v]$ refers to the closed subscheme of Z corresponding to ideal

$$I_{v,W} := \mathrm{Ker}\Big(\, R_Z \xrightarrow{\tilde{\mathfrak{z}}[v]^*} \big(R_{\mathrm{TL}(\mathbf{N})}\big)^{\mathrm{perf},\flat}_{A,b;d} \xrightarrow{q_W^\flat} \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d} \Big)$$

of $R_Z = \Gamma(Z, \mathcal{O}_Z)$, where q_W^* is the homomorphism induced from the quotient homomorphism $q_W : R_{\mathrm{TL}(\mathbf{N})} \to R_W := \Gamma(W, \mathcal{O}_W)$. Translated into algebra, the above assertion means that for every formal function $f \in R_Z$ which lies in the prime ideal corresponding to W, the image of f under the composition

$$R_{\mathrm{TL}(\mathbf{N})} \xrightarrow{\mu^*} R_Z \hat{\otimes}_\kappa R_{\mathrm{TL}(\mathbf{N})} \xrightarrow{(q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W} \big(R_W\big)_{A,b;d}^{\mathrm{perf},\flat} \hat{\otimes}_\kappa R_W$$

is 0.

Such an element $((q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W)(\mu^* f)$ is a tempered virtual function on $W \times W$, in two sets of variables, one from each of the two factors of $W \times W$. We want to conclude that it is 0. The available information is an infinite family of congruences, from the interpolation property of $\tilde{\mathfrak{z}}[v]$. This is where the method of hypocotyl elongation comes in. This method was first used in [5], in the context of augmented complete noetherian local domains over κ. We need a version for tempered perfection, summarized in 6.7 below.

(6.7) Hypocotyl elongation in tempered perfections

(6.7.1) The statement 6.7.2, which follows from propositions 2.1 and 3.1 of [5], embodies the method of hypocotyl elongation for commutative noetherian local domains over perfect fields of characteristic p. It provides a way to establish a power series relation $f(u_1, \ldots, u_a, v_1, \ldots, v_b)$ between functions on a product formal scheme $\mathrm{Spf}(R) \times \mathrm{Spf}(R)$ of the form $\mathrm{pr}_1^* g_1, \ldots, \mathrm{pr}_1^* g_a$ and $\mathrm{pr}_2^* h_1, \ldots, \mathrm{pr}_2^* h_b$. Proposition 6.7.3 extends 6.7.2 to tempered perfections, and allows $g_1, \ldots, g_a, h_1, \ldots, h_b$ to be tempered virtual functions on $\mathrm{Spf}(R)$.

(6.7.2) Proposition. *Let κ be a perfect field of characteristic p. Let $\mathbf{u} = (u_1, \ldots, u_a)$, $\mathbf{v} = (v_1, \ldots, v_b)$ be two tuples of variables, and let $f(\mathbf{u}, \mathbf{v}) \in \kappa[[\mathbf{u}, \mathbf{v}]]$ be a formal power series in variable $u_1, \ldots, u_a, v_1, \ldots, v_b$ with coefficients in κ. Let $(R, \mathfrak{m})$ be an augmented Noetherian complete local domain R over κ such that $\kappa \xrightarrow{\sim} R/\mathfrak{m}$. Let $g_1, \ldots, g_a, h_1, \ldots, h_b$ be elements of the maximal ideal $\mathfrak{m}$. Let $n_0 \in \mathbb{N}$ and let $(d_n)_{n \geq n_0}$ be a sequence of positive integers and let q be a power of p such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$. Suppose that*

$$f(g_1, \ldots, g_a, h_1^{q^n}, \ldots, h_b^{q^b}) \equiv 0 \pmod{\mathfrak{m}^{d_n}}$$

for all $n \geq n_0$. Then

$$f(g_1 \otimes 1, \ldots, g_a \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_b) = 0$$

in the completed tensor product $R \hat{\otimes}_k R$, where $R \hat{\otimes}_k R$ is the formal completion of the local domain $R \otimes_\kappa R$.

(6.7.3) Proposition. *Let $(R, \mathfrak{m})$ be an augmented complete Noetherian local domain over a perfect field κ of characteristic p.*

- *Let $g_1, \ldots, g_m, h_1, \ldots, h_{m'}$ be elements of the maximal ideal of $(R, \mathfrak{m})_{A,b;d}^{\mathrm{perf},\flat}$.*

- *Let $f(u_1, \ldots, u_m, v_1, \ldots, v_{m'})$ be an element of*

$$\kappa\langle\langle u_1^{p^{-\infty}}, \ldots, u_m^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_{m'}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$$

  which lies in the closure of the image of

$$\kappa\langle\langle \underline{u}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat} \otimes_\kappa \kappa\langle\langle \underline{v}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}.$$

- *Let $q = p^r$ be a power of p for some positive integer r. Let $(d_n)_{n\in\mathbb{N},\, n\geq n_0}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$.*

Suppose that

$$(\dagger) \qquad f(g_1, \ldots, g_m, h_1^{q^n}, \ldots, h_{m'}^{q^n}) \equiv 0 \quad \left(\mathrm{mod}\ \mathrm{Fil}_{\deg}^{d_n}(R, \mathfrak{m})_{A',b';d'}^{\mathrm{perf},\flat}\right)$$

in $(R, \mathfrak{m})_{A',b';d'}^{\mathrm{perf},\flat}$ for all sufficiently large natural numbers n. Then

$$f(g_1 \otimes 1, \ldots, g_m \otimes 1, 1 \otimes h_1, \ldots, 1 \otimes h_{m'}) = 0$$

in the completed tempered perfection $\left(R\hat{\otimes}_\kappa R, \mathfrak{m}_{R\hat{\otimes}_\kappa R}\right)_{A',b';d'}^{\mathrm{perf},\flat}$ of $R\hat{\otimes}_\kappa R$.

(6.7.4) Remark. (a) The condition in 6.7.3 that the relation $f(u_1, \ldots, u_a, v_1, \ldots, v_b)$ lies in the closure of the image of $\kappa\langle\langle \underline{u}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat} \otimes_\kappa \kappa\langle\langle \underline{v}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$ may seem a little odd at first sight. However some subtleties in tensor products of tempered perfections are to be expected, by the analogous situation in the theory of distributions, such as the Schwartz kernel theorem.

(b) The proof of orbital rigidity of Tate-linear formal varieties uses the special case of proposition 6.7.3 when the relation $f(\mathbf{u}, \mathbf{v})$ is an element of $\kappa[[u_1, \ldots, u_a, v_1, \ldots, v_b]]$. We don't know whether there is a natural class of subspaces of $\left(\kappa[[u_1, \ldots, u_a, v_1, \ldots, v_b]]\right)^{\mathrm{tmp\,perf}}$, larger than the class of all closures of images of $\kappa\langle\langle \underline{u}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat} \otimes_\kappa \kappa\langle\langle \underline{v}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle_{C;\,d}^{E,\flat}$ for some parameters E, c, d, such that the conclusion of 6.7.2 holds. There are many other questions about tempered perfections that we have not considered with any degree of seriousness, some of which are mentioned in [7, Ch. 10 §7].

(c) Proposition 6.7.5 below is a "coordinate-free" version of 6.7.3. When applied to the proof of orbital rigidity of Tate-linear formal varieties, S_2 is the affine coordinate ring of an isoclinic p-divisible group Z of slope λ_1 such that $Z[p^{\lambda_1 r}] = Z[\mathrm{Fr}^q]$, $q = p^r$.

(6.7.5) Proposition. *Let κ be a perfect field of characteristic p which contains a finite field with $q = p^r$ elements. Let $(R, \mathfrak{m})$, $(S_1, \mathfrak{m}_1)$ and $(S_2, \mathfrak{m}_2)$ be augmented commutative noetherian local domains over κ. Suppose that S_2 has a $\mathbb{F}_q$-model $S_{2,\mathbb{F}_q}$, i.e. an augmented noetherian local subring $S_{2,\mathbb{F}_q}$ over $\mathbb{F}_q$ such that the natural map $S_{2,\mathbb{F}_q}\hat{\otimes}_{\mathbb{F}_q}\kappa \to S_2$ is an isomorphism.*

- *Let $\phi = \phi_q : S_2 \to S_2$ be the κ-linear continuous ring endomorphism of S_2 which sends every element $x \in S_{2,\mathbb{F}_q}$ to x^q.*

- *Let $g_1 : S_1 \to (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ and $g_2 : S_2 \to (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$ be continuous κ-linear ring homomorphisms from S_i to $(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A,b;d}$, $i = 1,2$.*

- *Let f be an element of the completed tensor product $(S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2}$ for parameters (A_1,b_1,d_1) and (A_2,b_2,d_2).*

- *Let $(d_n)_{n\in\mathbb{N},\, n\geq n_0}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$.*

- *Let (A',b',d') be suitable parameters such that the homomorphisms g_i extends to a continuous κ-linear homomorphism $g_i^\flat : (S_i,\mathfrak{m}_i)^{\mathrm{perf},\flat}_{A_i,b_i;d_i} \to (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'}$ for $i = 1,2$.*

Let $(g_1^\flat \cdot g_2^\flat)\circ(1\otimes(\phi_q^\flat)^n)$ be the composition

$$(S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2} \xrightarrow{1\otimes(\phi_q^\flat)^n} (S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2} \xrightarrow{g_1^\flat\cdot g_2^\flat} (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} .$$

Suppose that

$$\big((g_1^\flat \cdot g_2^\flat)\circ(1\otimes(\phi_q^\flat)^n)\big(f\big) \equiv 0 \quad \Big(\mathrm{mod}\ \mathrm{Fil}^{d_n}_{\mathrm{deg}}(R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'}\Big)$$

for all sufficiently large natural numbers n. Then

$$(g_1^\flat \otimes g_2^\flat)(f) = 0 \quad \text{in} \quad (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} \hat{\otimes}_\kappa (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} ,$$

where $g_1^\flat \otimes g_2^\flat$ denotes the composition

$$(S_1,\mathfrak{m}_1)^{\mathrm{perf},\flat}_{A_1,b_1;d_1} \hat{\otimes}_\kappa (S_2,\mathfrak{m}_2)^{\mathrm{perf},\flat}_{A_2,b_2;d_2} \xrightarrow{g_1^\flat\otimes g_2^\flat} (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} \hat{\otimes}_\kappa (R,\mathfrak{m})^{\mathrm{perf},\flat}_{A',b';d'} .$$

Proof. This is an easy consequence of 6.7.3. □

(6.8) Construction of a tempered virtual morphism $\tilde{\mathfrak{z}}[v] : \mathrm{TL}(\mathbf{N}) \dashrightarrow Z$

Suppose we are given an element $v \in \mathrm{Lie}(\mathrm{Aut}(\mathbf{N}))$ satisfying the conditions in 6.3.3. For every $n \geq n_0$, we defined a map

$$\mathfrak{z}_n[v] : \pi^{-1}\big(\mathrm{TL}(\mathbf{N}_2)[\mathrm{Fr}^{\lfloor n/\lambda_2\rfloor - c_0}]\big) \longrightarrow Z$$

in 6.2. According to 6.4.2, the restrictions $\mathfrak{z}_{mr_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}$ satisfy the compatibility condition

$$\mathfrak{z}_{(m+1)r_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]} = [p^{r_1\lambda_1}]_Z \circ \mathfrak{z}_{mr_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]} \quad \forall\, m \geq m_1.$$

Let $R_Z := \Gamma(Z, \mathcal{O}_Z)$ and $R_T := \Gamma(\mathrm{TL}(\mathbf{N}), \mathcal{O}_{\mathrm{TL}(\mathbf{N})})$ be the affine coordinate rings of Z and $\mathrm{TL}(\mathbf{N})$ respectively. We want to produce, out of the family of maps $\mathfrak{z}_{mr_1\lambda_1}[v]$, a continuous κ-linear ring homomorphism

$$\tilde{\mathfrak{z}}[v]^* : R_Z \to (R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{A,b;d}$$

for some parameters (A, b, d). Such an homomorphism is, by definition, a tempered virtual morphism $\tilde{\mathfrak{z}}[v] : \mathrm{TL}(\mathbf{N}) \dashrightarrow Z$ from $\mathrm{TL}(\mathbf{N})$ to Z.

For any $a \in \mathbb{N}$, the κ-linear continuous ring homomorphism $[p^a]^* : R_Z \to R_Z$ is purely inseparable and induces an isomorphism from R_Z^{perf} to itself. Given any element $f \in R_Z$, $\big(\mathfrak{z}_{mr_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}\big)^*(f)$ is an element of $R_T/\mathfrak{m}_T^{p^{ms_1}}$. The compatibility condition of the maps $\mathfrak{z}_{mr_1\lambda_1}[v]$ implies that

$$\big(\mathfrak{z}_{mr_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}\big)^*([p^{r_1\lambda_1}]^* f) \equiv \big(\mathfrak{z}_{(m+1)r_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{(m+1)s_1}]}\big)^*(f)$$

As was shown in [7, Ch. 10 §7], there exists a natural number $i_0 \in \mathbb{N}$, depending only on m_1, such that the congruence classes

$$\big(\mathfrak{z}_{mr_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}\big)^*([p^{mr_1\lambda_1}]^*)^{-1}(f)$$

converges, in an obvious sense, in the tempered perfection $(R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ of R_T. Define $\tilde{\mathfrak{z}}[v]^* : R_Z \to (R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ by

$$\tilde{\mathfrak{z}}[v]^*(f) = \lim_{m\to\infty} \big(\mathfrak{z}_{mr_1\lambda_1}[v]\big|_{\mathrm{TL}(\mathbf{N})[\mathrm{Fr}^{ms_1}]}\big)^*([p^{mr_1\lambda_1}]^*)^{-1}(f) \quad \forall\, f \in R_Z.$$

(6.8.1) Corollary. *We keep the notation and assumptions in* 6.3.6*. Let v be an element of* $\mathrm{Lie}(\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}}))$ *such that* $\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(\mathbb{Z}_p\, v) \subseteq \mathrm{Aut}(\mathbf{N})$*, and let* $\rho_0(\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^{\lambda_1 r_1 m}v))^*$ *be the ring automorphism of* $(R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$ *corresponding to the action of* $\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^{\lambda_1 r_1 m}v)$ *on* $\mathrm{TL}(\mathbf{N})$*.*

(a) *The action of* $\exp(p^{\lambda_1 r_1 m}v)$ *on* $\mathrm{TL}(N)$ *satisfies the congruence relation*

$$\rho_0(\exp_{\mathrm{Aut}(\mathbf{N}_{\mathbb{Q}})}(p^{\lambda_1 r_1 m}v))^* \equiv \mathrm{id} + (\tilde{\mathfrak{z}}[v]^* \circ [p^{\lambda_1 r_1 m}]^*_Z) \mod \mathrm{Fil}^{ms_1}_{\deg}(R_T, \mathfrak{m}_T)^{\mathrm{perf},\flat}_{s:\phi^r;[i_0]}$$

for all sufficiently large natural numbers m.

(b) *The tempered virtual map* $\tilde{\mathfrak{z}}[v] : \mathrm{TL}(\mathbf{N}) \dashrightarrow Z$ *is compatible with the Z-torsor structure in the sense that*

$$\tilde{\mathfrak{z}}[v](z * x) = (d\rho_0(v)(z)) * \tilde{\mathfrak{z}}[v](z)$$

for all functorial points (z, x) of $Z \times \mathrm{TL}(\mathbf{N})$*. Here ρ_0 is the identity map of* $\mathrm{Aut}(\mathbf{N})$ *corresponding to the tautological action of* $\mathrm{Aut}(\mathbf{N})$ *on* $\mathbf{N}$*. In particular the restriction* $\tilde{\mathfrak{z}}[v]\big|_Z$ *of* $\tilde{\mathfrak{z}}[v]$ *to Z is equal to the* $d\rho_0(v)\big|_Z$*, an endomorphism of Z.*

Proof. The statement (a) follows from the construction of $\tilde{\mathfrak{z}}[v]$. The meaning of the statement (b) has been explained in 6.6.2 (ii)–(iii). It follows from 6.2.1 and the construction of $\tilde{\mathfrak{z}}[v]$. □

(6.9) Proof of Theorem 5.2. Let W be a reduced irreducible closed formal subscheme of $T = \mathrm{TL}(\mathbf{N})$ stable under the action of G. Recall that the base field κ is assumed to be algebraically closed, and $\mathrm{Ker}(\mathrm{Fr}_Z^{r_0}) = [p^{r_0\lambda_1}]_Z$. We have also fixed a positive integer multiple r_1 of r, and an integer s_1 such that $r_1 < s_1$, $s_1\varsigma_1 \in \mathbb{N}$, $s_1\varsigma_1 < r_1\lambda_1$.

Step 1. Claim. *Suppose that v is an element of $\mathrm{Lie}(G)$ such that $\exp_G(\mathbb{Z}_p v) \subseteq G$. Then W is stable under the translation action by the schematic image of $\tilde{\mathfrak{z}}[v] : W \dashrightarrow Z$.*

The meaning of this claim, as explained in 6.6.3, is that

$$\big(((q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W) \circ \mu^*\big)(f) = 0$$

for all f in the ideal I_W of R_T which defines W, where $((q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W) \circ \mu^*$ is the composition

$$R_T \xrightarrow{\mu^*} R_Z \hat{\otimes}_\kappa R_T \xrightarrow{(q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W} \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d} \hat{\otimes}_\kappa R_W .$$

Clearly it suffices to show that

$$\big((q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W^\flat\big)(\mu^*(f)) = 0$$

for all $f \in I_W$, because the canonical arrow $j_{R_W} : R_W \to \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d}$ in the commutative diagram below

$$\begin{tikzcd}
R_T \arrow[r, "\mu^*"] & R_Z \hat{\otimes}_\kappa R_T \arrow[r, "(q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W"] \arrow[d, "="] & \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d} \hat{\otimes}_\kappa R_W \arrow[d, "1 \otimes j_{R_W}"] \\
 & R_Z \hat{\otimes}_\kappa R_T \arrow[r, "(q_W^\flat \circ \tilde{\mathfrak{z}}[v]^*) \otimes q_W^\flat"] & \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d} \hat{\otimes}_\kappa \big(R_W\big)^{\mathrm{perf},\flat}_{A,b;d}
\end{tikzcd}$$

is injective. This is a consequence of proposition 6.7.5, where the required family of congruence conditions follows from the fact that W is stable under the action of $\rho(\exp_G(p^{m\lambda_1 r_1}))$ for all sufficiently large natural number m, and the congruence property 6.8.1 (a) of the discrete one-parameter family $\rho(\exp_G(p^{m\lambda_1 r_1}))$ of automorphisms of $\mathrm{TL}(\mathbf{N})$. The claim is proved.

Step 2. Clearly the close formal subscheme $Z' := (W \cap Z)_{\mathrm{red}}$ of Z is stable under the action of G. So by orbital rigidity of p-divisible formal groups [5], Z' is a p-divisible subgroup of Z.

Suppose that $v \in \mathrm{Lie}(G)$ is an element of the Lie algebra of G such that $\exp_G(\mathbb{Z}_p v) \subseteq G$. Since the restriction of $\tilde{\mathfrak{z}}[v]$ to Z is equal to the action of $d\rho(v)$ on Z according to 6.8.1, schematic image of $\tilde{\mathfrak{z}}[v] : W \dashrightarrow Z$ contains $d\rho(v)(Z)$. So step 1 tells us that W is stable under the translation action by the p-divisible formal subgroup $d\rho(v)(Z')$.

Since G operates strongly nontrivially on Z', we know from [5, Lemma 4.1.1] that there exist elements $v_{i,j} \in \mathrm{Lie}(G)$, indexed by pairs (i,j), $i = 1, \ldots, a$, $j = 1, \ldots, b_i$, such that $\exp_G(\mathbb{Z}_p v_{i,j}) \subseteq G$ for all (i,j), and $\big(\sum_{i=1}^a d\rho(v_{i,1}) \circ \cdots \circ d\rho(v_{i,b_i})\big)\big|_{Z'}$ is an isogeny from Z' to itself. In particular $\sum_{i=1}^a d\rho(v_{i,1})(Z') = Z'$. Therefore W is stable under the translation action by Z'. We have proved the statements (a) and (b) of 5.2. It remains to prove 5.2 (c), which asserts that the formal morphism $\bar{\pi}\big|_{W_1} : W_1 \to \mathrm{TL}(\mathbf{N}_2)$ is purely inseparable. Here $\mathbf{N}_1 = \mathbf{N}/\mathbf{Z}'$, $\mathbf{N}_2 = \mathbf{N}/\mathbf{Z}$, $\bar{\pi} : \mathrm{TL}(\mathbf{N}_1) \to \mathrm{TL}(\mathbf{N}_2)$ is the morphism attached to the quotient map $\mathbf{N}_1 \twoheadrightarrow \mathbf{N}_2$, $W_1 = W/Z' \subseteq \mathrm{TL}(\mathbf{N}_1)$, and $\bar{\pi}\big|_{W_1}$ is the restriction of $\bar{\pi}$ to W_1.

Step 3. To prove the statement 5.2 (c), we may and do assume that $Z' = (W \cap Z)_{\mathrm{red}} = (0)$. We need to show that the formal morphism $\pi|_W : W \to \mathrm{TL}(\mathbf{N}_2)$, which is finite because $(W \cap Z)_{\mathrm{red}} = (0)$, is purely inseparable.

Suppose that $\pi|_W$ is not purely inseparable. Then there exist two $\kappa[[u]]$-valued points of W, $\xi_1, \xi_2 : \mathrm{Spf}(\kappa[[u]]) \to W$, such that $\pi \circ \xi_1 = \pi \circ \xi_2$ and $\xi_1 \neq \xi_2$. Let $\delta : \mathrm{Spf}(\kappa[[u]]) \to Z$ be the $\kappa[[u]]$-valued point of Z such that $\xi_1 = \delta * \xi_2$, i.e. ξ_1 is the translation of ξ_2 by δ with respect to the Z-torsor structure of $\pi : \mathrm{TL}(\mathbf{N}) \to \mathrm{TL}(\mathbf{N}_2)$. The condition that $\xi_1 \neq \xi_2$ means that $\delta \neq 0$.

Let v be any element of $\mathrm{Lie}(G)$ such that $\exp_G(\mathbb{Z}_p v) \subseteq G$, and let Z_v be the smallest subgroup of Z containing the schematic image of $\tilde{\mathfrak{z}}[v]$. According to step 1, W is stable under translation by the schematic image of $\tilde{\mathfrak{z}}[v]$, therefore W is stable under the translation by Z_v. Corollary 6.8.1 (b) tells us that

$$\tilde{\mathfrak{z}}[v](\xi_1) = \tilde{\mathfrak{z}}[v](\delta * \xi_2) = (d\rho(v)(\delta)) * \tilde{\mathfrak{z}}[v](\xi_2).$$

So $d\rho(v)(\delta)$ is a $\kappa[[u]]$-valued point of Z_v, and W is stable under translation by $d\rho(v)(\delta)$.

Choose elements $v_{ij} \in \mathrm{Lie}(G)$, $i = 1, \ldots, a$, $j = 1, \ldots, b_i$ with $\exp_G(\mathbb{Z}_p v_{ij}) \subseteq G$, such that the endomorphism

$$\alpha := \Big(\sum_{i=1}^{a} d\rho(v_{i,1}) \circ \cdots \circ d\rho(v_{i,b_i}) \Big)\Big|_Z$$

of Z is an isogeny from Z to itself, just as in the argument of step 2. Then $\alpha(\delta)$ is a non-trivial $\kappa[[u]]$-valued point of Z, and W is stable under translation by $\alpha(\delta)$. It follows that $\alpha(\delta)$ is a non-trivial $\kappa[[u]]$-valued point of $(W \cap Z)_{\mathrm{red}}$, which is a contradiction. So the formal morphism $\pi\big|_W : W \to \mathrm{TL}(\mathbf{N})_2$ is purely inseparable. We have finished the proof of theorem 5.2, and also the proof of theorem 5.1. $\square$

Remark. The proof of 5.2 (c) in step 3 of 6.9 was presented in a geometric language. The readers may want to compare with the more algebraic, and possibly more convincing version of this argument in [7, Ch. 10 §7]. The argument there for the pure inseparability of $\bar{\pi}\big|_W$ in the case when $\mathrm{TL}(\mathbf{N})$ is a biextension of p-divisible formal groups works for all Tate linear formal varieties.

References

[1] M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, vol. 269, 270, 305 of Lecture Notes in Math., 1972/3.

[2] G. Baumslag, *Lecture notes on nilpotent groups*, Regional Conference Series in Mathematics, No. 2, American Mathematical Society, Providence, R.I., 1971.

[3] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie*, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1349, Hermann, Paris, 1972.

[4] C.-L. Chai, *Families of ordinary abelian varieties: canonical coordinates, p-adic monodromy, tate-linear subvarieties and hecke orbits.* `https://www2.math.upenn.edu/~chai/papers_pdf/fam_ord_av.pdf`. July 30, 2003, 53 pp.

[5] ——, *A rigidity result for p-divisible formal groups*, Asian J. Math., 12 (2008), pp. 193–202.

[6] ——, *Sustained p-divisible groups and a foliation on moduli spaces of abelian varieties*, in Proceedings of the International Consortium of Chinese Mathematicians 2018, Int. Press, Boston, MA, [2020] ©2020, pp. 3–30.

[7] C.-L. Chai and F. Oort, *Hecke orbits.* Draft, 2022, 700+ pp.

[8] ——, *Sustained p-divisible groups, a foliation retraced*, Chapter 7 in [23].

[9] A. E. Clement, S. Majewicz, and M. Zyman, *The theory of nilpotent groups*, Birkhäuser/Springer, Cham, 2017.

[10] M. D'Addezio and P. van Hoften, *Hecke orbits on shimura varieties of hodge type.* `https://arxiv.org/abs/2205.10344`, 2022.

[11] P. Hall, *The Edmonton notes on nilpotent groups*, Queen Mary College Mathematics Notes, Queen Mary College, Mathematics Department, London, 1969.

[12] S. M. Hedayatzadeh, *Exterior powers of Barsotti–Tate groups*, PhD thesis, ETH Zürich, 2010. arXiv:1009.2460.

[13] S. M. H. Hedayatzadeh, *Exterior powers of π-divisible modules over fields*, J. Number Theory, 138 (2014), pp. 119–174.

[14] P. Hilton, *Localization and cohomology of nilpotent groups*, Cahiers Topologie Géom. Différentielle, 14 (1973), pp. 341–369, 480.

[15] ——, *Remarks on the localization of nilpotent groups*, Comm. Pure Appl. Math., 26 (1973), pp. 703–713. Collection of articles dedicated to Wilhelm Magnus.

[16] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. Sci. Ecole Norm. Sup. (3), 71 (1954), pp. 101–190.

[17] A. I. Mal'cev, *Nilpotent torsion-free groups*, Izvestiya Akad. Nauk. SSSR. Ser. Mat., 13 (1949), pp. 201–212.

[18] ——, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat., 13 (1949), pp. 9–32.

[19] ——, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation, 1951 (1951), p. 33.

[20] B. Moonen, *Serre-Tate theory for moduli spaces of PEL type*, Ann. Sci. École Norm. Sup. (4), 37 (2004), pp. 223–269.

[21] D. Mumford, *Bi-extensions of formal groups*, in Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 307–322.

[22] F. Oort, *Foliations in moduli spaces of abelian varieties*, J. Amer. Math. Soc., 17 (2004), pp. 267–296.

[23] ——, *Open Problems in Arithmetic Algebraic Geometry*, vol. 46 of Advanced Lectures in Mathematics, Higher Education Press, International Press of Boston, 2018, 2019.

[24] R. Pink, *Finite group schemes*. Lecture course in WS 2004/05, ETH Zürich.

[25] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2), 90 (1969), pp. 205–295.

[26] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68, Springer-Verlag, New York-Heidelberg, 1972.

[27] The Stacks Project Authors, *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[28] J.-P. Serre, *Lie algebras and Lie groups*, vol. 1500 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, second ed., 1992. 1964 lectures given at Harvard University.

[29] T. Song, *On the orbital rigidity conjecture and sustained p-divisible groups*, PhD thesis, University of Pennsylvania, 2022.

[30] R. B. Warfield, Jr., *Nilpotent groups*, Lecture Notes in Mathematics, Vol. 513, Springer-Verlag, Berlin-New York, 1976.