§1. Introduction

Let $p$ be a prime number, fixed throughout this article.

Given three commutative group schemes $X, Y, Z$ over a base field $k$, a biextension of $X \times Y$ by $Z$ is a morphism $E \to X \times Y$ plus two relative group laws. The first group law, for $E \to Y$, makes $E \to Y$ an extension of $X_Y := X \times Y$ by $Z_Y := Z \times Y$, while the second group law, for $E \to X$, makes $E \to X$ and extension of $Y_Y$ by $Z_Y$. The best-known example is the Poincare bundle for an abelian variety $A$; it is a biextension of $A \times A'$ by $\mathbb{G}_m$, where $A'$ is the dual abelian variety of $A$. Mumford invented the concept of bi-extension in [6] to treat deformation and lifting problems for polarized abelian varieties. In standard applications of biextensions the “fiber group” $Z$ is usually $\mathbb{G}_m$.

Biextensions also arise when one tries to deform a $p$-divisible group in such a way that all $p$-adic invariants of the deformed $p$-divisible group are fixed. Suppose that $U_1, U_2, U_3$ are three isoclinic $p$-divisible formal groups over a perfect field $k \supseteq \mathbb{F}_p$, such that

$$\text{slope}(U_1) > \text{slope}(U_2) > \text{slope}(U_3).$$

The equi-characteristic-$p$ deformation space $D = \text{Def}(U_1 \times U_2 \times U_3)$ of the product $U_1 \times U_2 \times U_3$ is a smooth formal scheme. There exists a closed formal subscheme $S = S(U_1 \times U_2 \times U_3)$ of $D$ such that the restriction to $S$ of the universal $p$-divisible group $\mathcal{U}$ is sustained, and every closed subscheme $S'$ with this property is contained is $S$. That $\mathcal{U}|_S$ is sustained means that for every $n \in \mathbb{N}$, there exists a faithfully flat cover $\mathcal{T} \to S$ such that $\mathcal{U}|_S[p^n] \times_S \mathcal{T}$ is isomorphic to $(U_1 \times U_2 \times U_3) \times_{\text{Spec}(k)} \mathcal{T}$. Similarly one has the maximal sustained locus $S(U_1 \times U_j)$ in the deformation space $D(U_1 \times U_j)$ for any pair $(i, j)$ with $1 \leq i < j \leq 3$. We will call $S(U_1 \times U_2 \times U_3)$ the (central) leaf in the deformation space $\text{Def}(U_1 \times U_2 \times U_3)$ which passes through the closed point. Similarly $S(U_1 \times U_j)$ is the leaf in $\text{Def}(U_1 \times U_j)$ through the closed point.

It turn out that in the two-slope case, the leaf $S(U_i \times U_j)$ has a natural structure as an isoclinic $p$-divisible group whose slope is equal to $\text{slope}(U_i) - \text{slope}(U_j)$ for any pair $(i, j)$ with $1 \leq i \leq j \leq 3$. In the three-slope case, the leaf $S(U_1 \times U_2 \times U_3)$ is not a $p$-divisible group, but it has a natural structure as a biextension of $p$-divisible formal groups: there exists a canonical morphism

$$\pi : S(U_1 \times U_2 \times U_3) \to S(U_1 \times U_2) \times S(U_2 \times U_3)$$

plus two relative group laws

$$+_{1} : S(U_1 \times U_2 \times U_3) \times \times S(U_2 \times U_3) S(U_1 \times U_2 \times U_3) \to S(U_1 \times U_2 \times U_3)$$

and

$$+_{2} : S(U_1 \times U_2 \times U_3) \times \times S(U_1 \times U_2) S(U_1 \times U_2 \times U_3) \to S(U_1 \times U_2 \times U_3),$$

making $S(U_1 \times U_2 \times U_3)$ a biextension of $S(U_1 \times U_2) \times S(U_2 \times U_3)$ by $S(U_1 \times U_3)$.
Suppose that $G$ is a closed subgroup of the group $\text{Aut}(U_1 \times U_2 \times U_3)$ of automorphisms of the $p$-divisible group $U_1 \times U_2 \times U_3$. By functoriality the group $G$ also acts on $S(U_1 \times U_2 \times U_3)$. We assume that the action of $G$ on $S(U_1 \times U_2 \times U_3)$ is strongly nontrivial, in the sense that there is no open subgroup of $G$ which fixes all points of a non-trivial $p$-divisible subgroup of $S(U_i \times U_j)$ for some pair $(i,j)$ with $1 \leq i < j \leq 3$. The goal of the local rigidity problem for this biextension is:

**Question (local rigidity for the biextension $S(U_1 \times U_2 \times U_3)$).** Find a sharp constraint on formal subvarieties of the biextension $S(U_1 \times U_2 \times U_3)$ which are stable under a strongly non-trivial action by a $p$-adic Lie group $G$.

More generally one can ask the local rigidity question for general biextensions of $p$-divisible formal groups. One can also ask the (easier) local rigidity question for the leaves $S(U_i \times U_j)$, $1 \leq i \leq j \leq 3$. Recall that $S(U_i \times U_j)$ is a $p$-divisible formal group, and local rigidity question for $p$-divisible groups has a clean answer; see [3, Thm. 4.3].

**Theorem** (local rigidity for $p$-divisible formal groups). Suppose that $G$ is a $p$-adic Lie group acting strongly nontrivially on a $p$-divisible formal group $V$ over a base field $k \supset \mathbb{F}_p$. Every formal subvariety of $V$ which is stable under the action of $G$ is a $p$-divisible subgroup of $V$.

For a long time it was unclear whether there is a good answer to the local rigidity question for biextensions of $p$-divisible formal groups. It turns out that the outline of the argument in [3] can be followed, but new ideas are needed to analyze the asymptotic behavior of the action on the biextension by elements sufficiently close to 1 in a one-parameter subgroup.

Suppose that a $p$-adic Lie group $G$ acts on a $p$-divisible group $V$, and $w$ is an element of the Lie algebra of $G$ which operates on $V$ through an endomorphism $C \in \text{End}(V)$. Let $V_1$ be the largest among isoclinic subgroups of whose slope $\mu_1$ is bigger than other slopes of $V$. Assume that $V$ is the product of $V_1$ and another $p$-divisible subgroup $V_2$ of $V$, and let $\text{pr}_{V_1}: V \to V_1$ be the projection to $V_1$. Then for all $n \gg 0$, the action of the element $\exp(p^n w) \in G$ on $V$ is very closely approximated by $\text{Id}_V + p^n \cdot C|_{V_1} \circ \text{pr}_{V_1}$. The precise meaning of “very closely approximated” is provided by the following estimates for the “main term” $p^n \cdot C|_{V_1} \circ \text{pr}_{V_1}$ and the difference of $\exp(p^n w)$ and $\text{Id}_V + p^n \cdot C|_{V_1} \circ \text{pr}_{V_1}$. The size of the main term and the error term will be estimated by powers of the maximal ideal $m = m_V$.

- There are constants $c_1, c_2 \in \mathbb{N}_{>0}$ such that the main term $p^n C|_{V_1} \circ \text{pr}_{V_1}$ has coordinates in $m^{c_1 \cdot p^{|n/\mu_1|} + c_2}$ and is non-zero modulo $m^{c_1 \cdot p^{|n/\mu_1|} + c_2}$, for all $n \gg 0$.

- There is a constant $\mu_2$ with $0 < \mu_2 < \mu_1$ such that the error term

$$\exp(p^n w) - \text{Id}_V - p^n \cdot C|_{V_1} \circ \text{pr}_{V_1}$$

is congruent to 0 modulo $m^{c_1 \cdot p^{|n/\mu_2|}}$ for all $n \gg 0$.

With the above analysis of the action by a one-parameter subgroup, one is in a position to apply the identity principle [3, 3.1], recalled in 6.1.1, to conclude that the given formal subvariety of $V$ stable under $G$ is stable under translation by elements of the $p$-divisible subgroup $C \cdot V_1$ of $V$. 

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For the case of a biextension \( \pi : E \to X \times Y \) of two \( p \)-visible formal groups \( X, Y \) by a \( p \)-divisible formal group \( Z \), one does not have an analog of “the projection from \( E \) to the isoclinic factor of \( Z \) of maximal slope”, nor an analog of “the projection to \( Z \)” for that matter, no matter how one modifies \( E \) by isogenies. The first step to deal with this difficulty is the construction of a morphism \( \eta_n : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z \) in 2.7. After modifying \( Z \) by a suitable isogeny so that \( Z \) is a product of \( Z_1 \) with a \( p \)-divisible subgroup \( Z_2 \) of \( Z \), we can compose \( \eta_n \) with the projection \( \text{pr}_1 \) from \( Z \) to \( Z_1 \), and obtain a morphism \( \rho'_n : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z_1 \). This morphism \( \rho'_n \) is an analog of \( p^n \cdot \text{pr}_1 \) (but not \( \text{pr}_1 \)).

To make use of the maps \( \rho'_n \), one needs the existence of a single object \( \bar{\rho} \) such that each \( \rho'_n \) “is” \( p^n \cdot \bar{\rho} \) in a suitable sense. To figure out where this animal \( \bar{\rho} \) might be found, we first remind ourselves that if we choose a coordinate system for \( Z_1 \), a map from \( E \) to \( Z_1 \) is determined by a sequence of functions on \( E \), one for each coordinate of \( Z_1 \). If we think of the coordinate ring of \( E \), which is isomorphic to a power series ring \( k[[t_1, \ldots, t_m]] \), as functions on \( E \), what we need is to introduce a suitable ring of “generalized functions”. Then we can define the sought-after object \( \rho \) as a “generalized map” whose components with respect to the chosen coordinate system for \( Z_1 \) are generalized functions.

There are a few related procedures, each depending on some parameters, which take Noetherian complete equi-characteristic-\( p \) local domains as input, and produce “generalized functions” as outputs. We call the resulting rings \textit{complete restricted perfections} of the input, because they are completions of suitable subrings of the perfection of the input complete local domains. Here we provide a sample, denoted by \( k\langle \langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle \rangle_{C; d} \), the where \( E > 0, C \geq 1, d \geq 0 \) are real parameters. The \textit{support set} \( \text{supp}(m : E; C, d) \) for this ring is a subset of \( \mathbb{Z}[1/p]_{\geq 0} \), defined by

\[
\text{supp}(m : E; C, d) = \{ I \in \mathbb{Z}[1/p]_{\geq 0} : |I|_p \leq C \cdot (|I|_\sigma + d)^E \}.
\]

Here for any \( I = (i_1, \ldots, i_m) \in \mathbb{Z}[1/p]_{\geq 0} \), \(|I|_\sigma := i_1 + \cdots + i_m \) is the usual archimedean norm of \( I \), \(|I|_p = \max(|i_1|_p, \ldots, |i_m|_p) \) is the normalized \( p \)-adic norm of \( I \), and \(|\cdot|_p \) is the normalized \( p \)-adic absolute value on \( \mathbb{Q} \) with \(||p|_p = 1/p \). By definition

\[
k\langle \langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle \rangle_{C; d} := \left\{ \sum_{I \in \text{supp}(m : E; C, d)} a_I \cdot t^I \mid a_I \in k \ \forall I \in \text{supp}(m : E; C, d) \right\},
\]

where \( t^I \) stands for the monomial \( t_1^{i_1} \cdots t_m^{i_m} \) for every \( I = (i_1, \ldots, i_m) \in \mathbb{Z}[1/p]_{\geq 0} \). The standard formula for multiplication of formal series make sense in the ring \( k\langle \langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle \rangle_{C; d} \). One can also compose generalized function, and substitute the variables \( t_1, \ldots, t_m \) of an element

\[
f \in k\langle \langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle \rangle_{C; d}
\]

by elements \( g_1, \ldots, g_m \in k\langle \langle u_1^{p^{-m}}, \ldots, u_n^{p^{-m}} \rangle \rangle_{E_1; d_1} \). The result is a function

\[
f(g_1(u), \ldots, g_m(u)) \in k\langle \langle u_1^{p^{-m}}, \ldots, u_n^{p^{-m}} \rangle \rangle_{E_2; d_2}
\]

for suitable parameters \( E_2, C_2, d_2 \). Details about the construction of the coordinates of \( \bar{\rho} \) in suitable complete restricted perfections are explained in §3. Basic properties of complete restricted perfections are in §3 and §4. The identity principle in [3, 3.1] is extended to complete restricted perfection of complete equi-characteristic-\( p \) local rings in 6.4.
We don’t know whether the rings \( \kappa\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{E,C,d} \) have applications to other problems. These local rings are not Noetherian, but smaller and more manageable than the completion of the perfection \( \kappa[t_1^{-\infty}, \ldots, t_m^{-\infty}] \) with respect to the filtration given by the total degree of monomials. We provide a form of the Weierstrass preparation theorem for these rings and compute the integral closure of \( \kappa\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C,d} \) in its field of fractions, in 4.4.2 and 4.5 respectively. These two results are not needed for rigidity of biextensions. Most of the basic algebraic properties of these rings are still unexplored.

Armed with the above tools, the same train of thoughts in the proof of local rigidity for \( p \)-divisible group leads to a satisfactory answer of the local rigidity question for biextensions of \( p \)-divisible formal groups, theorems 7.2 and 7.5. The latter is easy to state: in a biextension \( E \) of \( p \)-divisible formal groups \( X \times Y \) by \( Z \) such that \( X, Y, Z \) have mutually distinct slopes, every formal subvariety of \( E \) which is stable under a strongly non-trivial action of a \( p \)-adic Lie group is a sub-biextension.

From the perspective of the Hecke orbit problem, a good answer to the local rigidity question for leaves in deformation spaces of \( p \)-divisible groups is quite useful. It provides a tight structural constraint on what the Zariski closure of a Hecke orbit can possibly be, when examined at any \( \overline{F}_p \)-point of the intersection of the Zariski closure of the given Hecke orbit with the leaf containing the Hecke orbit. It is hoped that the tools introduced to solve the three-slope case will bring us closer to the answer of the general local rigidity problem for leaves in deformation spaces of \( p \)-divisible groups.

\section{2. Biextension basics}

The notion of biextensions of commutative groups was first introduced by Mumford in [6] and further developed by Grothendieck in expositions VI, VII of [5].

\begin{definition}
\begin{align*}
(\pi : E \rightarrow X \times_SY, +_1 : E \times_Y E \rightarrow E, +_2 : E \times_X E \rightarrow E, \varepsilon_1 : Y \rightarrow E, \varepsilon_2 : X \rightarrow E)
\end{align*}
\end{definition}

where \( E \) is the formal spectrum of a Noetherian complete local ring formally smooth over \( R \), \( \pi \) is an \( S \)-morphism, \( +_1 \) and \( \varepsilon_1 \) are \( Y \)-morphisms, \( +_2 \) and \( \varepsilon_2 \) are \( X \)-morphisms. In addition the following properties are satisfied.

(0) The morphism \( \pi \) is formally smooth and faithfully flat.

(1a) The pair \( (+_1, \varepsilon_1) \) makes \( E \) a \( p \)-divisible group (resp. commutative smooth formal group) over \( Y \).

(1b) The projection map \( \pi : E \rightarrow X \times_SY \) is a group homomorphism for the group law \( +_1 \) and the base change to \( Y \) of the group law \( +_X : X \times_SY \rightarrow X \) of the \( p \)-divisible group \( X \).
2a) The pair $(+,e_2)$ makes $E$ a $p$-divisible group (resp. commutative smooth formal group) over $X$.

2b) The projection map $\pi : E \to X \times_S Y$ is a group homomorphism for the group law $+_2$ and the base change to $X$ of the group law $+_Y : Y \times_S Y \to Y$ of the $p$-divisible group $Y$.

3a) The $S$-morphism
\[ Z \times_S Y \to E, \quad (z,y) \mapsto z +_2 e_1(y) \]
defines an $S$-isomorphism from $Z \times_S Y$ to $E \times_{(X \times_S Y)} (0_X \times_S Y)$.

3b) The $S$-morphism
\[ Z \times_S X \to E, \quad (z,x) \mapsto z +_1 e_2(x) \]
defines an $S$-isomorphism from $Z \times_S X$ to $E \times_{(X \times_S Y)} (X \times_S 0_Y)$.

4) (compatibility of the two relative group laws) For any formal scheme $T$ over $S$ and any four $T$-valued points $w_{11}, w_{12}, w_{21}, w_{22}$ of $E$ such that
\[ \pi_1(w_{11}) = \pi_1(w_{12}), \quad \pi_1(w_{21}) = \pi_1(w_{22}), \quad \pi_2(w_{11}) = \pi_2(w_{21}), \quad \pi_2(w_{12}) = \pi_1(w_{22}) \]
where $\pi_1 := \text{pr}_1 \circ \pi$ and $\pi_2 := \text{pr}_2 \circ \pi$ are the two projections from $E$ to $X$ and $Y$ respectively, the equality
\[ (w_{11} +_2 w_{12}) +_1 (w_{21} +_2 w_{22}) = (w_{11} +_1 w_{21}) +_2 (w_{12} +_1 w_{22}) \]
holds.

(2.1.1) **Remark.** Conditions (1a) and (1b) assert that the relative group law $+_1$ on $E$ over $Y$ is an extension of (the base change to $Y$ of) $X$ by (the base change to $Y$ of) $Z$. Similarly (2a) and (2b) say that the relative group law $+_2$ on $E$ over $X$ is an extension of (the base change to $X$ of) $Y$ by (the base change to $X$ of) $Z$.

(2.1.2) **Remark.** Of course the definition 2.1 of biextension works in other contexts, for instance sheaves of commutative groups for the fppf site for a general scheme $S$. For our purpose the case when $X$, $Y$ and $Z$ are all $p$-divisible groups will be sufficient. For the main result on local rigidity for $p$-divisible groups, $S$ will be the spectrum of a field $k$ of characteristic $p > 0$ and $X$, $Y$, $Z$ are $p$-divisible formal groups over $k$.

(2.1.3) **Remark.** The following properties are easily verified.

(i) For any formal scheme $T$ over $S$, any $T$-valued points $y_1,y_2$ of $Y$ and any $T$-valued points $x_1,x_2$ of $X$, we have
\[ e_1(y_1) +_2 e_1(y_2) = e_2(y_1 + y_2), \quad e_2(x_1) +_1 e_2(x_2) = e_2(x_1 + x_2). \]

(ii) For any formal scheme $T$ over $S$, any $T$-valued points $z$ of $Z$ and any $T$-valued point $w$ of $E$, we have
\[ (z +_1 e_2(\pi_1(w))) +_2 w = (z +_2 e_1(\pi_2(w))) +_1 w. \]
This equality means that the $Z$-actions on $E$ induced by the relative group laws $+_{1}$ and $+_{2}$ are equal, given $\pi : E \to X \times_{S} Y$ a natural structure as a $Z$-torsor. Let 

$$*: Z \times_{S} E = (Z \times_{S} (X \times_{S} Y)) \times_{(X \times Y)} E \to E$$

be the morphism defining this $Z$-torsor structure on $E$.

(iii) The restriction of $+_{1}$ to $Z \times_{S} Z \subset E \times_{Y} E$ is equal to the group law of $Z$. Similarly for the restriction of $+_{2}$ to $Z \times_{S} Z \subset E \times_{X} E$.

(iv) The $S$-isomorphism $(z, y) \mapsto z +_{2} \varepsilon_{1}(y)$ in (3a) is a group isomorphism from the product group $Z \times_{S} Y$ to the group law on $E \times_{(X \times Y)} (0_{X} \times Y)$ induced by $+_{2}$. In other words the restriction to $0_{X} \subset X$ of the extension of $Y$ by $Z$ over $X$, given by the partial group law $+_{2}$, splits canonically. Similarly for the $S$-isomorphism $(z, x) \mapsto z +_{1} \varepsilon_{2}(x)$ in (3b) is a group isomorphism from the product group $Z \times_{S} X$ to the group law on $E \times_{(X \times Y)} (X \times 0_{Y})$ induced by $+_{1}$.

(v) The restriction of $\varepsilon_{1}$ to $0_{Y}$ is equal to the restriction of $\varepsilon_{2}$ to $0_{X}$. Over the scheme-theoretic union $\Delta$ of the images of $X \times_{S} 0_{Y}$ and $0_{X} \times_{S} Y$, i.e. the smallest closed subscheme of $X \times_{S} Y$ containing both, we have an $S$-morphism $\varepsilon : \Delta \to E$ such that $\pi \circ \Delta = \text{id}_{\Delta}$ which is equal to $\varepsilon_{2}$ on $X \times_{S} 0_{Y}$ and equal to $\varepsilon_{1}$ on $0_{X} \times_{S} Y$. Because $\pi : E \to X \times_{S} Y$ is formally smooth, there exists a section $s : X \times_{S} Y \to E$ of $\pi$ which extends $\varepsilon$.

(2.2) The biextension structure can be made explicit in terms of cocycles as follows.

(2.2.1) Definition. Let $\pi : E \to X \times_{S} Y$ be a biextension of $X \times_{S} Y$ by $Z$ as in 2.1, and let $s : X \times_{S} Y \to E$ be a section of $\pi$ which extends both $\varepsilon_{1}$ and $\varepsilon_{2}$ as in 2.1.3 (v). Define $S$-morphisms

$$\tau : (X \times_{S} X) \times_{S} Y \to Z \quad \text{and} \quad \sigma : X \times_{S} (Y \times_{S} Y) \to Z$$

by the following formulas expressed in terms of $T$-valued points $x, x_{1}, x_{2}, y, y_{1}, y_{2}$ in $X$ and $Y$ for formal schemes $T$ over $S$:

(a) $s(x_{1}, y) +_{1} s(x_{2}, y) = \tau(x_{1}, x_{2}; y) * s(x_{1} +_{2} x_{2}, y)$

(b) $s(x, y_{1}) +_{2} s(x, y_{2}) = \sigma(x; y_{1}, y_{2}) * s(x, y_{1} +_{2} y_{2})$

(2.2.2) Cocycle identities. The $S$-morphisms $\tau$ and $\sigma$ satisfy properties (1)–(5) below, for all formal schemes $T$ over $S$, all $T$-valued points $x, x_{1}, x_{2}, x_{3}$ of $X$ and all points $y, y_{1}, y_{2}, y_{3}$ of $Y$. Identities (1) and (2) are consequences of the fact that the section $s$ of $\pi$ extends $\varepsilon_{1}$ and $\varepsilon_{2}$. Identities (3) and (4) hold because the two relative group laws $+_{1}$ and $+_{2}$ are commutative and associative. The identity (5) follows from the compatibility of the two relative group laws.

1. $\sigma(x; 0, y_{2}) = 0 = \sigma(x; y_{1}, 0), \tau(0, x_{2}; y) = 0 = \tau(x_{1}, 0; y)$.

2. $\sigma(0; y_{1}, y_{2}) = 0, \tau(x_{1}, x_{2}; 0) = 0$.

3. (symmetry) $\sigma(x; y_{1}, y_{2}) = \sigma(x; y_{2}, y_{1}), \tau(x_{1}, x_{2}; y) = \tau(x_{2}, x_{1}; y)$
(4) (associativity)

\[ \sigma(x;y_1,y_2) + \sigma(x;y_1 + y_2,y_3) = \sigma(x;y_1,y_2 + y_3) + \sigma(x;y_2,y_3) \]
\[ \tau(x_1,x_2;y) + \tau(x_1 + x_2,x_3;y) = \tau(x_1,x_2 + x_3;y) + \tau(x_2,x_3;y) \]

(5) (compatibility)

\[ \sigma(x_1 + x_2;y_1,y_2) - \sigma(x_1;y_1,y_2) - \sigma(x_2;y_1,y_2) \]
\[ = \tau(x_1,x_2;y_1 + y_2) - \tau(x_1,x_2;y_1) - \tau(x_1,x_2;y_2) \]

(2.2.3) Coboundary. If we replace \( s(x,y) \) by a another section

(2.2.3.1) \[ s'(x,y) = f(x,y) * s(x,y), \]

where \( f(x,y) : X \times_S Y \to Z \) is an \( S \)-morphism such that \( f(x,0) = f(0,y) \) (so that \( s' \) extends \( \varepsilon_1 \) and \( \varepsilon_2 \)), then the resulting maps \( \tau' : (X \times_S X) \times_Y \to Z \) and \( \sigma' : X \times_S (Y \times_S Y) \to Z \) are related to the maps \( \sigma \) and \( \tau \) by

(2.2.3.2) \[ \tau'(x_1,x_2;y) - \tau(x_1,x_2;y) = f(x_1,y) + f(x_2,y) - f(x_1 + x_2,y), \]

(2.2.3.3) \[ \sigma'(x,y_1,y_2) - \sigma(x,y_1,y_2) = f(x,y_1) + f(x,y_2) - f(x,y_1 + y_2). \]

(2.2.4) Conversely given a pair \( (\alpha, \beta) \) of \( S \)-morphisms satisfying equations (1)–(5) in 2.2.2, there exists a biextension of \( X \times_S Y \) by \( Z \) naturally attached to the cocycle \( (\alpha, \beta) \). Moreover the biextensions attached to two cocycles \( (\alpha, \beta), (\alpha', \beta') \) are isomorphic as biextensions of \( X \times_S Y \) by \( Z \) in the sense of 2.3.1 (c) below if and only if the two cocycles differ by a coboundary in the sense that there exists an \( S \)-morphism \( f : X \times_S Y \to Z \) such that 2.2.3.2 and 2.2.3.3 hold.

(2.3) Homomorphisms between biextensions

(2.3.1) Definition. Let \( X, Y, Z, X', Y', Z' \) be \( p \)-divisible groups (resp. commutative smooth formal groups) over \( S = \text{Spf}(R) \) as in 2.1. Let \( \pi : E \to X \times_S Y \) be a biextension of \( X \times_S Y \) by \( Z \), and \( \pi' : E' \to X' \times_S Y' \) be a biextension of \( X' \times_S Y' \) by \( Z' \).

(a) An \( S \)-homomorphism of biextensions from the biextension \( E \) to the biextension \( E' \) is a quadruple of \( S \)-morphisms

\[ (\psi : E \to E', \alpha : X \to X', \beta : Y \to Y', \gamma : Z \to Z') \]

where \( \alpha, \beta, \gamma \) are \( S \)-homomorphisms of commutative formal groups, and \( \psi \) is compatible with the biextension structure of \( E \) and \( E' \), in the sense that the following properties are satisfied.

(i) \( \pi' \circ \psi = (\alpha \times_S \beta) \circ \pi \),

(ii) \( \psi \circ +_1 = +'_1 \circ (\psi \times_S \psi), \ \psi \circ +_2 = +'_2 \circ (\psi \times_S \psi) \),

(iii) \( \psi \circ \varepsilon_1 = \varepsilon'_1 \circ \beta, \ \psi \circ \varepsilon_1 = \varepsilon'_1 \circ \alpha \).
(b) A homomorphism of biextensions \((\psi, \alpha, \beta, \gamma)\) is an isomorphism of biextensions if \(\psi, \alpha, \beta\) and \(\gamma\) are all isomorphism of formal schemes, in which case \((\psi^{-1}, \alpha^{-1}, \beta^{-1}, \gamma^{-1})\) is a homomorphism of biextensions from \(E'\) to \(E\).

(c) Suppose that \(X' = X, Y' = Y\) and \(Z' = Z\). We say that the \(E\) and \(E'\) are isomorphic as biextensions of \(X \times Y\) by \(Z\) if there exists a isomorphism \((\psi, \pi, \id_Y, \id_Z)\) from \(E\) to \(E'\).

(d) An \(S\)-homomorphism \((\psi, \alpha, \beta, \gamma)\) between biextensions of \(p\)-divisible groups (respectively commutative smooth formal groups) is an isogeny if the homomorphism \(\alpha, \beta\) and \(\gamma\) between \(p\)-divisible groups are all isogenies.

Note that an isomorphism \((\psi, \alpha, \beta, \gamma)\) from \(E\) to \(E'\) as in 2.3.1 (b) above induces an isomorphism \((\psi', \id_X, \id_Y, \id_Z)\) from \(\gamma, E\) to \((\alpha \times \beta, \gamma, E')\), so that the two biextensions \(\gamma, E\) and \((\alpha \times \beta)\, E'\) of \(X \times Y\) by \(Z'\) are isomorphic in the sense of 2.3.1 (c).

\[\text{(2.3.2)}\] It is clear that for a homomorphism \((\psi, \alpha, \beta, \gamma)\) from a biextension \(E\) to a biextension \(E'\) as in 2.3.1, the homomorphisms of formal groups \(\alpha, \beta\) and \(\gamma\) are uniquely determined by the morphism \(\psi\).

Conversely, it is easily seen that if \((\psi_1, \alpha, \beta, \gamma)\) and \((\psi_2, \alpha, \beta, \gamma)\) are two homomorphisms of biextensions from \(E\) to \(E'\) with the same individual components \(\alpha, \beta, \gamma\), then there exists an \(S\)-morphism \(g : X \times_S Y \to Z'\) such that \(\psi_2 = (g \circ \pi') \ast \psi'\). Moreover \(g : X \times_S Y \to Z'\) is a bihomomorphism in the sense that

\[\gamma \circ (x) = (\gamma \circ x_1, y_1) \ast (\gamma \circ x_2, y_2)\]

for all formal scheme \(T\) over \(S\), all \(T\)-valued points \(x, x_1, x_2\) of \(X\) and all \(T\)-valued points \(y, y_1, y_2\) of \(Y\). In 2.3.3 below we will see that such a bihomomorphism \(g : X \times_S Y \to Z'\) is necessarily equal to the zero map if \(X\) and \(Y\) are both \(p\)-divisible groups over \(S\). Therefore the natural map

\[\text{Hom}_{\text{biext}}(E, E') \longrightarrow \text{Hom}(X, X') \times \text{Hom}(Y, Y') \times \text{Hom}(Z, Z')\]

\[(\psi, \alpha, \beta, \gamma) \mapsto (\alpha, \beta, \gamma)\]

is injective when \(X\) and \(Y\) are both \(p\)-divisible groups over \(S\).

\[\text{(2.3.3)}\] It is an easy formal fact that if \(X\) and \(Y\) are both \(p\)-divisible groups over \(S\), then every bihomomorphism \(g : X \times_S Y \to Z\) from \(X \times_S Y\) to a sheaf of groups \(Z\) over \(S\) is identically zero:

- (a) The bi-additivity of \(g\) implies that

\[g([p^n]_X(x_1), [p^n]_Y(y_1)) = [p^{2n}]_Z(g(x_1, y_1)) = 0\]

for all \(S\)-scheme \(T_1\), all \(x_1 \in X[p^{2n}](T_1)\) and all \(y_1 \in Y[p^{2n}](T_1)\).

- (b) Recall that the morphisms \([p^n]_X[p^{2n}] \to X[p^{2n}] : X[p^{2n}] \to X[p^n]\) and \([p^n]_Y[p^{2n}] \to Y[p^{2n}] : Y[p^{2n}] \to Y[p^n]\) induced by “multiplication by \(p^n\)” are both faithfully flat. So for every \(S\)-scheme \(T\), every \(x \in X[p^n](T)\), and every \(y \in Y[p^n](T)\), there exists a faithfully flat morphism \(f : T_1 \to T\), an element \(x_1 \in X[p^{2n}](T_1)\) and an element \(y_1 \in Y[p^{2n}](T_1)\) such that

\[x \circ f = [p^n]_X[p^{2n}] \to X[p^n] \circ x_1\quad\text{and}\quad y \circ f = [p^n]_Y[p^{2n}] \to Y[p^n] \circ y_1.\]

The desired conclusion that \(g : X \times_S Y \to Z\) is equal to the zero map follows immediately from (a) and (b).
(2.3.4) Let $E, E'$ be biextensions as in 2.3.1. Let $s(x, y)$ be a section of $\pi : E \to X \times_S Y$ extending $\varepsilon_1$ and $\varepsilon_2$, and let $\tau, \sigma$ be defined as in 2.2. Similarly let $s'(x', y')$ be a section of $\pi : E' \to X' \times_S Y'$ extending $\varepsilon'_1$ and $\varepsilon'_2$, and define $\tau' : (X' \times_S X') \times_S Y' \to Z'$ and $\sigma' : X' \times_S (Y' \times_S Y') \to Z'$ in the same way. Define an $S$-morphism

$$\mu = \mu_s : X \times_S Y \to Z'$$

by

(2.3.4.1) $\psi(s(x, y)) = \mu(x, y) * s'(\alpha(x), \beta(y))$

for all points $x$ of $X$ and all points $y$ of $Y$ with values in the same formal scheme over $S$. It is easy to verify that

(2.3.4.2) $\gamma(\tau(x_1, x_2; y)) - \tau'(\alpha(x_1), \alpha(x_2); \beta(y)) = \mu(x_1, y) + \mu(x_2, y) - \mu(x_1 + x_2, y)$

(2.3.4.3) $\gamma(\sigma(x_1, y_1, y_2)) - \sigma'(\alpha(x); \beta(y_1), \beta(y_2)) = \mu(x, y_1) + \mu(x, y_2) - \mu(x, y_1 + y_2)$

for all formal schemes $T$ over $S$, all $T$-points $x, x_1, x_2$ of $X$ and all $T$-points $y, y_1, y_2$ of $Y$.

Conversely it is easy to see that every $S$-morphism $\mu : X \times_S Y \to Z'$ satisfying the two displayed equations above defines a homomorphism of biextensions from $E$ to $E'$.

(2.3.5) Remark. Let $E, E'$ be biextensions as in 2.3.1. The set $\text{Hom}_{\text{biext}}(E, E')$ of all biextension homomorphisms from $E$ to $E'$ does not have a natural group structure. Instead there are two relative group laws

$$\text{Hom}_{\text{biext}}(E, E') \times_{\text{Hom}(Y, Y')} \text{Hom}_{\text{biext}}(E, E') \to \text{Hom}_{\text{biext}}(E, E')$$

$$\text{Hom}_{\text{biext}}(E, E') \times_{\text{Hom}(X, X')} \text{Hom}_{\text{biext}}(E, E') \to \text{Hom}_{\text{biext}}(E, E')$$

However even in the case when $X, Y, Z, X', Y', Z'$ are all $p$-divisible, the natural map

$$\text{Hom}_{\text{biext}}(E, E') \to \text{Hom}(X, X') \times \text{Hom}(Y, Y')$$

may not be surjective. So in general the set $\text{Hom}_{\text{biextn}}(E, E')$ does not have a natural structure as a biextension of $\text{Hom}(X, X') \times \text{Hom}(Y, Y')$ by $\text{Hom}(Z, Z')$.

(2.4) Let $R$ be a noetherian complete local ring whose residue field $R/m$ has characteristic $p$. Let $X, Y, Z$ be $p$-divisible groups over $S = \text{Spf}(R)$ as in 2.1.

(2.4.1) The trivial biextension of $X \times_S Y$ by $Z$ is the natural biextension structure on $X \times_S Y \times Z$, where the two relative group laws are given by

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \quad (x_1, y_1, z_1) + 2(x_2, y_2, z_2) = (x_1 + y_2, y_1 + z_2).$$

A biextension $E \to X \times_S Y$ by $Z$ is trivial if there is an biextension isomorphism $\psi$ from the trivial biextension to $E$ which induces $i_1, i_2, \text{id}_Y$ on $X, Y, Z$ respectively. We know from 2.3.3 that such an isomorphism is unique if one exists. The restriction of $\psi$ to $X \times_S Y \times 0_Z$ is called the canonical splitting of a trivial biextension of $X \times_S Y$ by $Z$.

The uniqueness in the previous paragraph implies that for any faithfully flat morphism $T \to S$ and any biextension $E \to X \times_S Y$ by $Z$, the base change of the biextension $E$ to $T$ is trivial if and only if $E$ is trivial.
(2.4.2) For every biextension $E$ of $X \times_S Y$ by $Z$, there is an associated family $\theta_E = (\theta_{n,E})_{n \in \mathbb{N}}$ of bilinear pairings

$$\theta_n = \theta_{n,E} : X[p^n] \times_S Y[p^n] \to Z[p^n], \ n \in \mathbb{N}$$

called the **Weil pairing**, attached to this biextension $E \to X \times_S Y$. A definition of the Weil pairing and its basic properties will be reviewed in 2.7. The bilinear pairings $\theta_n$ are compatible in the sense that

(2.4.2.1) $$\theta_n([p]_X(x_{n+1}), [p]_Y(y_{n+1})) = [p]_Z(\theta_{n+1}(x_{n+1}, y_{n+1}))$$

for all $x_{n+1} \in X[p^{n+1}]$, all $y_{n+1} \in Y[p^{n+1}]$ and all $n \in \mathbb{N}$; or equivalently,

(2.4.2.2) $$\theta_{n+1}(x_{n+1}, y_{n+1}) = \theta_n(x_n, [p]_Y(y_{n+1}))$$

(2.4.2.3) $$\theta_{n+1}(x_{n+1}, y_{n}) = \theta_n([p]_X(x_{n+1}), y_{n})$$

for all $x_n \in X[p^n]$, $x_{n+1} \in X[p^{n+1}]$, $y_n \in Y[p^n]$, $y_{n+1} \in Y[p^{n+1}]$ and all $n \in \mathbb{N}$. See Exp. VIII of [5] for details.

Denote by $\text{Biext}^1(X, Y; Z)$ the set of all biextensions of $X \times_S Y$ by $Z$ up to isomorphisms which induce $\text{id}_X, \text{id}_Y, \text{id}_Z$ on $X, Y$ and $Z$; c.f. 2.3.1 (c). It is shown in [6, Prop. 4, p. 319] and also in Exp. VIII of [5] that the map $E \mapsto \theta_E$ establishes a bijection from $\text{Biext}^1(X, Y; Z)$ to the set of all compatible families of bilinear pairings $(\theta_n : X[p^n] \times Y[p^n] \to Z[p^n])_{n \in \mathbb{N}}$; see also 2.6.3.

**Remark.** One knows from [5, VII 3.6.5] that for sheaves of abelian groups $P, Q, G$ over a topos, the set $\text{Biext}^1(P, Q; G)$ of isomorphism classes of biextensions of $P \times Q$ by $G$ is naturally isomorphic to $\text{Ext}^1(P \otimes^L Q, G)$. On the other hand, for $p$-divisible groups $X, Y$ we have $\text{Tor}^1(X[p^n], Y[p^n]) \cong X[p^n] \otimes Y[p^n]$.

The construction of the Weil pairing attached to a biextension reflects these two facts.

(2.4.3) The functoriality of the Weil pairing is as follows. Let $X, Y, Z, X', Y', Z'$ be $p$-divisible groups over $S$, let $E$ be a biextension of $X \times_S Y$ by $Z$, and let $E'$ be a biextension of $X' \times_S Y'$ by $Z'$. Let $(\theta_{n,E})_{n \in \mathbb{N}}$ and $(\theta_{n,E'})_{n \in \mathbb{N}}$ be the Weil pairings attached to $E$ and $E'$ respectively. Suppose that $(\psi, \alpha, \beta, \gamma)$ is a homomorphism of biextensions from $E$ to $E'$. Then

$$\gamma(\theta_{n,E}(x_n, y_n)) = \theta_{n,E'}(\alpha(x_n), \beta(y_n))$$

for all $x_n \in X[p^n]$ and all $y_n \in Y[p^n]$.

(2.4.4) Let $E \to X \times_S Y$ be a biextension of $X \times_S Y$ by $Z$. For any $p$-divisible formal group $Z'$ over $S$ and any homomorphism $\xi : Z \to Z'$, the standard push-forward construction yields a biextension $\xi_*(E \to X \times_S Y)$ of $X \times_S Y$ by $Z'$, plus a homomorphism $\psi_1$ from $E \to X \times_S Y$ to $\xi_*(E \to X \times_S Y)$, which induces $\text{id}_X, \text{id}_Y, \xi$ on $X, Y, Z$ respectively. In addition $\xi_*(E \to X \times_S Y)$ satisfies the universal property that every biextension homomorphisms $(\psi, \alpha, \beta, \xi)$ from $E$ to a biextension $E'$ of $X' \times Y'$ by $Z'$ factors through $\psi_1$. Similarly for any $p$-divisible groups $X_1, Y_1$ over $S$ and any homomorphisms $\zeta : X_1 \to X, \eta : Y_1 \to Y$, the standard pull-back construction yields a biextension $(\zeta_*, \eta_*)^*(E \to X \times_S Y)$ of $X_1 \times_S Y_1$ by $Z$, which satisfies an obvious universal property among biextension homomorphisms $(\psi_1, \alpha_1, \beta_1, \gamma_1)$ from biextensions $E_1 \to X_1 \times_S Y_1$ to $E$ with $\alpha_1 = \alpha$ and $\beta_1 = \beta$. 

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It is clear from the consideration of associated Weil pairings that for any isogeny \( \xi : Z \to Z' \), the push-forward biextension \( \xi_*(E \to X \times_S Y) \) is trivial if and only if \( E \to X \times_S Y \) is. Similarly for any pair of isogenies \( \zeta : X_1 \to X, \eta : Y_1 \to Y \), the pull-back biextension \( (\zeta, \eta)^*(E \to X \times_S Y) \) is trivial if and only if \( E \to X \times_S Y \) is.

\[ \mathbf{2.4.5} \text{ Lemma.} \] Suppose that \( X, Y, Z \) are \( p \)-divisible groups over a field \( k \supset \mathbb{F}_p \). Let \( E \to X \times_{\text{Spec}(k)} Y \) be a biextension of \( X \times_{\text{Spec}(k)} Y \) by \( Z \). If we have \( \lambda + \mu \neq \nu \) for every slope \( \lambda \) of \( X \), every slope \( \mu \) of \( Y \) and every slope \( \nu \) of \( Z \), then the biextension \( E \) is trivial.

**Proof.** By the last paragraph of 2.4, we may assume that \( k \) is a perfect field. By 2.4.4, we may assume that \( X, \ Y, \ Z \) are all \( p \)-divisible groups after suitable push-forward and pull-back by isogenies. So we are reduced to the case when \( X, \ Y, \ Z \) are all isoclinic with slopes \( \lambda, \mu \) and \( \nu \) respectively. The assumption that \( \nu \neq \lambda + \mu \) implies immediately that the Weil pairing attached to \( E \) vanishes identically. \( \square \)

\[ \mathbf{2.5} \text{ The Weil pairing as descent data over torsion subgroup schemes} \]

We review in 2.5.1

(a) the definition of the Weil pairing attached to a biextension \( E \to X \times Y \) of \( p \)-divisible groups \( X \times Y \) by a \( p \)-divisible group \( Z \), and

(b) how to construct a biextension \( E_n \) of \( X[p^n] \times Y[p^n] \) by \( Z \) by descending the split biextension

\[ Z \times X[p^n] \times Y[p^{2n}] \to X[p^n] \times Y[p^{2n}] \]

along the faithfully flat morphism

\[ 1 \times p^n : X[p^n] \times Y[p^{2n}] \to X[p^n] \times Y[p^n] \]

using the descent datum given by a bihomomorphism \( \theta_n : X[p^n] \times Y[p^n] \to Z[p^n] \).

The descent construction reviewed in 2.5.1 (iii), (iv) has many applications. For instance it implies that if the Weil pairings \( \theta_{n_1,E}, \theta_{n_1,E'} \) attached biextensions \( E, E' \) of \( p \)-divisible groups \( X \times Y \) by \( Z \) at a fixed level \( \{p^{n_1}\} \) coincide, then there exists a canonical isomorphism between the restrictions of the biextensions \( E \) and \( E' \) to \( X[p^{n_1}] \times Y[p^{n_1}] \); see 2.5.3 and its Dieudonné theory version 2.6.3, 2.6.4.

\[ \mathbf{2.5.1} \] We first recall the explicit construction of the Weil pairing \( \theta_n : X[p^n] \times Y[p^n] \to Z[p^n] \) in [6, pp. 320–321].

(i) Construct a natural map

\[ \xi_n : X[p^n] \times Y[p^{2n}] \to E_n \]

such that the diagram

\[
\begin{array}{ccc}
X[p^n] \times Y[p^{2n}] & \xrightarrow{\xi_n} & E_n \\
\downarrow & & \downarrow \pi_n \\
X[p^n] \times Y[p^{2n}] & \xrightarrow{1 \times p^n} & X[p^n] \times Y[p^n]
\end{array}
\]

commutes.
Given any $S$-scheme $T$, any $x \in X[p^n](T)$, any $y \in Y[p^{2n}](T)$, there exist a scheme $T_1$ faithfully flat and locally of finite presentation over $T$ and an element $z_1 \in E(T_1)$ which lies above $(x,y)$ such that when one multiplies $z_1$ by $p^n$ with respect to the first partial group law $+_1$, we have

$$[p^n]_1(z_1) = e_1(y).$$

Such an element $z_1$ is not unique, but any two choices differ by an element of $Z[p^n]$. Define $\xi_n(x,y)$ as $p^n$ times $z_1$ with respect to the second group law $+_2$:

$$\xi_n(x,y) := [p^n]_{+2}(z_1).$$

Clearly the right hand size of the above equality is independent of the choice of the element $z_1$, where we have used the first group law $+_1$ to produce a $Z[p^n]$-torsor lying above the $S$-point $(x,y)$ of $X[p^n] \times Y[p^{2n}]$. By descent we conclude that $\xi_n(x,y) \in E_n(S)$. We have produced the desired morphism $\xi_n : X[p^n] \times Y[p^{2n}] \rightarrow E_n$.

(ii) Define a morphism $\alpha_n : Z \times X[p^n] \times Y[p^{2n}] \longrightarrow E_n = \pi^{-1}(X[p^n] \times Y[p^n])$ by

$$\alpha_n(z,x,y) := z * \xi_n(x,y)$$

for all $S$-scheme $T$, all $z \in Z(T)$, all $x \in X[p^n](T)$ and all $y \in Y[p^{2n}](T)$. It is easy to see that the following commutative diagram

$$
\begin{array}{ccc}
Z \times X[p^n] \times Y[p^{2n}] & \longrightarrow & E_n \\
\downarrow \text{pr}_2 \times \text{pr}_3 & & \downarrow \pi|_{E_n} \\
X[p^n] \times Y[p^{2n}] & \longrightarrow & X[p^n] \times Y[p^n]
\end{array}
$$

is cartesian. So the biextension $\pi_n : E_n \rightarrow X[p^n] \times Y[p^n]$ is descended along the faithfully flat morphism

$$1 \times p^n : X[p^n] \times Y[p^{2n}] \longrightarrow X[p^n] \times Y[p^{2n}]$$

from the trivial biextension $\text{pr}_2 : Z \times X[p^n] \times Y[p^{2n}] \longrightarrow X[p^n] \times Y[p^{2n}]$.

(iii) Construct a bihomomorphism

$$\theta_n : X[p^n] \times Y[p^n] \longrightarrow Z[p^n]$$

using the descent datum for $\alpha_n$.

The effect of translation by elements of $Y[p^n]$ to the isomorphism $\alpha_n$ is recorded by a map $\theta'_n : X[p^n] \times Y[p^{2n}] \times Y[p^n] \rightarrow Z$, defined by

$$\alpha_n(\lambda, x, y) = \alpha_n(\lambda + \theta'_n(x, y, b), x, y + b)$$

for all $S$-scheme $T$, all $\lambda \in Z(T)$, all $x \in X[p^n](T)$, all $y \in Y[p^{2n}](T)$ and all $b \in Y[p^n](T)$. An easy calculation shows that $\theta'_n(x, y, b)$ is independent of $y$. In other words there exists an $S$-morphism $\theta_n : X[p^n] \times Y[p^n] \rightarrow Z$ such that the last displayed equation simplifies to

$$\alpha_n(\lambda, x, y, b) = \alpha_n(\lambda + \theta_n(x, b), x, y + b).$$

An easy calculation shows that $\theta_n$ is a bihomomorphism, hence it factors through the closed subgroup scheme $Z[p^n] \hookrightarrow Z$. 

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(iv) Reversing the construction, it is easy to see that $\theta_n$ encodes the descent datum from the trivial biextension $Z \times X[p^n] \times Y[p^{2n}]$ down to $E_n$: the bihomomorphism $\theta_n$ gives an $X[p^n]$-action of the base change to $X[p^n]$ of the group scheme $Y[p^n]$, on the $X[p^n]$-scheme $Z \times X[p^n] \times Y[p^{2n}]$.

**Remark.** The two partial group laws play different roles in the construction the morphisms $\xi_{n}$ and $\theta_n$: for any given $(x,y) \in (X[p^n] \times Y[p^{2n}])_{(T)}$, $[p^n]_{+2}$ sends the whole $Z[p^n]$-torsor $[p^n]_{+1}((x,y))$ to a well-defined element $\xi_{((x,y))} \in E_n(T)$. However one can check that if one interchanges the roles played by the two partial group laws, one obtains the same bihomomorphism $\theta_n : X[p^n] \times Y[p^n] \to Z[p^n]$.

(2.5.2) **Lemma.** Let $\pi : E \to X \times Y$ be a biextension of $p$-divisible groups $X \times S Y$ by a $p$-divisible group $Z$ over a base scheme $Y$. For each positive integer $n$, let $\pi : X[p^n] \times Y[p^n] \to Z[p^n]$ be the canonical bihomomorphism as described in 2.5.1.

1. Suppose that $n_1$ is a positive integer and $\theta_{n_1}$ is equal to the trivial bihomomorphism from $X[p^{n_1}] \times S Y[p^{n_1}]$ to $Z[p^{n_1}]$. Then the biextension $\pi^{-1}(X[p^{n_1}] \times S Y[p^{n_1}])$ of $X[p^{n_1}] \times S Y[p^{n_1}]$ by $Z$ splits canonically. In other words there exists a canonical isomorphism

$$\zeta_{(n_1)}^{\text{can}} : \pi^{-1}(X[p^{n_1}] \times S Y[p^{n_1}]) \sim \to Z \times X[p^{n_1}] \times S Y[p^{n_1}].$$

2. Suppose that $n_2$ is a positive integer, $n_2 > n_1$ and $\theta_{n_2}$ is equal to the trivial bihomomorphism. Then $\theta_n$ is also equal to the trivial biehomomorphism. Moreover the canonical trivializations $\zeta_{(n_1)}^{\text{can}}$ and $\zeta_{(n_2)}^{\text{can}}$ are compatible, i.e. $\zeta_{(n_1)}^{\text{can}}$ is equal to the restriction to $\pi^{-1}(X[p^{n_1}] \times S Y[p^{n_1}])$ of $\zeta_{(n_2)}^{\text{can}}$.

**Proof.** We saw in 2.5.1 that the pull-back of $\pi^{-1}(X[p^{n_1}] \times S Y[p^{n_1}])$ to $X[p^{n_1}] \times S Y[p^{2n_1}]$ by the faithfully flat morphism $1 \times p^{n_1} : X[p^{n_1}] \times S Y[p^{2n_1}] \to X[p^{n_1}] \times S Y[p^{n_1}]$ is canonically trivial, and the bihomomorphism $\theta_{n_1}$ corresponds to the descent data from the trivial biextension $Z \times X[p^{n_1}] \times S Y[p^{2n_1}]$ down to $\pi_{n_1}$ along the morphism $1 \times p^{n_1} : X[p^{n_1}] \times S Y[p^{2n_1}] \to X[p^{n_1}] \times S Y[p^{n_1}]$. So if $\theta_{n_1}$ is the trivial homomorphism, then this descent datum defines a canonical isomorphism between the $\pi^{-1}(X[p^{n_1}] \times S Y[p^{n_1}])$ and the trivial biextension $Z \times X[p^{n_1}] \times Y[p^{n_1}]$. We have proved statement (1).

The first part of (2) follows from the compatibility of Weil pairings 2.4.2.2 and 2.4.2.3. The compatibility statement (2) follows from the same descent argument used in the proof of (1). □

Proposition 2.5.3 and Corollary 2.5.4 below are applications of 2.5.1 (iv). It enables us to determine the restriction of a homomorphism between two biextensions to torsion subgroups schemes $X[p^n] \times Y[p^n]$.

(2.5.3) **Proposition.** Let $\pi : E \to X \times S Y$ and $\pi' : E' \to X \times S Y$ be two biextensions of $p$-divisible groups $X \times S Y$ by a $p$-divisible group $Z$ over $S$. Let $\theta_n, \theta_{n'} : X[p^n] \times Y[p^n] \to Z[p^n]$ be the bihomomorphisms attached to the biextensions $E$ and $E'$ respectively.

1. If $n_1$ is a positive integer and $\theta_{n_1} = \theta_{n_1}'$, then there exists a canonical isomorphism

$$\zeta_n : \pi^{-1}(X[p^{n_1}] \times Y[p^{n_1}]) \sim \to (\pi')^{-1}(X[p^{n_1}] \times Y[p^{n_1}])$$

determined by $\theta_n$ and $\theta_{n'}$.  

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(2) Suppose that \( n_2 > n_1 \) and \( \theta_{n_2} = \theta_{n_2}' \). Then \( \theta_{n_1} = \theta_{n_1}' \) and the canonical isomorphism
\[
\zeta_{n_1} : \pi^{-1}(X[p^{n_1}] \times Y[p^{n_1}]) \xrightarrow{\sim} (\pi')^{-1}(X[p^{n_1}] \times Y[p^{n_1}])
\]

is compatible with the canonical isomorphism
\[
\zeta_{n_2} : \pi^{-1}(X[p^{n_2}] \times Y[p^{n_2}]) \xrightarrow{\sim} (\pi')^{-1}(X[p^{n_2}] \times Y[p^{n_2}]).
\]

(3) Suppose that \( \theta_n = \theta_n' \) for all \( n \in \mathbb{N} \). Then the collection of canonical isomorphisms
\[
\zeta_n : \pi^{-n}(X[p^n] \times Y[p^n]) \xrightarrow{\sim} (\pi')^{-n}(X[p^n] \times Y[p^n]), \quad n \in \mathbb{N}
\]
defines an isomorphism from the biextension \( E \) to the biextension \( E' \) which induces \( \text{id}_X, \text{id}_Y \) and \( \text{id}_Z \) on the \( p \)-divisible groups \( X, Y \) and \( Z \).

(4) Suppose that \( \zeta : E \rightarrow E' \) is an isomorphism of biextensions which induces \( \text{id}_X, \text{id}_Y \) and \( \text{id}_Z \) on the \( p \)-divisible groups \( X, Y \) and \( Z \). Then \( \theta_n = \theta_n' \) for all \( n \in \mathbb{N} \), and the restriction of \( \zeta \) to \( \pi^{-n}(X[p^n] \times Y[p^n]) \) is equal to the canonical isomorphism
\[
\zeta_n : \pi^{-n}(X[p^n] \times Y[p^n]) \xrightarrow{\sim} (\pi')^{-n}(X[p^n] \times Y[p^n])
\]
attached to \( \theta_n \) and \( \theta_n' \), for all \( n \in \mathbb{N} \).

**Proof.** The biextension structures on \( E \) and \( E' \) endow the \( Z \)-torsor \( E \wedge Z([-1]_Z) \wedge E' \) over \( X \times Y \) a structure of a biextension of \( X \times Y \) by \( Z \). The statements (1), (2) follow from 2.5.2 applied to \( E \wedge Z([-1]_Z) \wedge E' \). The statement (3) follows from (2).

To prove the statement (4), we observe first that the functoriality of the Weil pairings tell us that \( \theta_n = \theta_n' \) for all \( n \). By (3), the canonical isomorphisms \( \zeta_n \) are compatible and define an isomorphism of biextensions \( \zeta' : E \rightarrow E' \) over \( X \times Y \). There exists a unique morphism
\[
b : X \times_S Y \rightarrow Z
\]
such that
\[
\zeta'(e) = b(\pi(e)) \ast \zeta(e)
\]
for all \( S \)-scheme \( T \) and all \( e \in E(T) \). Clearly \( b : X \times_S Y \rightarrow Z \) is a bihomomorphism in the sense that
\[
b(x_1 + x_2, y) = b(x_1, y) \quad \text{and} \quad b(x, y_1 + y_2) = b(x, y_1) + b(x, y_2)
\]
for all \( S \)-schemes \( T \), all \( x, x_1, x_2 \in X(T) \) and all \( y, y_1, y_2 \in Y(T) \). We know from 2.3.3 that such a bihomomorphism is necessarily zero. We have shown that \( \zeta' = \zeta \). \( \square \)

(2.5.4) **Corollary.** Let \( X, Y, Z, X', Y', Z' \) be \( p \)-divisible groups over \( S \). Let \( E \) be a biextension of \( X \times_S Y \) by \( Z \), and let \( E' \) be a biextension of \( X' \times_S Y' \) by \( Z' \). There is a natural bijection from the set \( \text{Hom}_{\text{biext}}(E, E') \) of all \( S \)-bihomomorphisms from \( E \) to \( E' \), to the set of all triples \( (\alpha, \beta, \gamma) \in \text{Hom}_S(X, X') \times \text{Hom}_S(Y, Y') \times \text{Hom}_S(Z, Z') \) such that
\[
\gamma(\theta_{n,E}(x_n, y_n)) = \theta_{n,E'}(\alpha(x_n), \beta(y_n))
\]
for all \( n \in \mathbb{N} \), all schemes \( T \) over \( S \), all \( x_n \in X[p^n](T) \), and all \( y_n \in Y[p^n](T) \).
(2.6) Dieudonné theory for biextensions

Suppose that $k$ is a perfect field of characteristic $p > 0$. We review the covariant Dieudonné theory for biextensions of $p$-divisible groups over $k$ the associated Weil pairings. Let $W = W(k)$ be the ring of all $p$-adic Witt vectors with entries in $k$. It is well-known that $W(k)$ is a complete discrete valuation ring of mixed characteristics $(0, p)$, $pW(k)$ is the maximal ideal of $W(k)$ and $W(k)/pW(k)$ is naturally isomorphic to $k$. Let $\sigma : W(k) \to W(k)$ be the map

$$x = (x_0, x_1, x_2, \ldots) \mapsto \sigma_x = (x_0^p, x_1^p, x_2^p, \ldots),$$

and let $V : W(k) \to W(k)$ be the map

$$x = (x_0, x_1, x_2, \ldots) \mapsto V_x = (0, x_0, x_1, x_2, \ldots)$$
on $W(k)$. It is well-known that $\sigma$ is a ring automorphism of $W(k)$ (because the field $k$ is assumed to be perfect), $V$ is an additive endomorphism of $W(k)$, and

$$V(\sigma_x) = px = \sigma^V(x) \quad \forall x \in W(k).$$

(2.6.1) The classical covariant Dieudonné theory attaches to every $p$-divisible formal group $X$ over $k$ a free $W(k)$-module $M_*(X)$ whose rank is equal to height($X$), together with additive endomorphisms

$$F, V : M_*(X) \longrightarrow M_*(X)$$
of $M_*(X)$ such that

$$F(ax) = aF(x), \quad V(\sigma ax) = aV(x) \quad \text{and} \quad F(V(x)) = px = V(F(x))$$

for all $a \in W(k)$ and all $x \in M_*(X)$. A triple $(M, F, V)$, where $M$ is a free $W(k)$-module of finite rank, and $F, V$ are additive endomorphisms of $M$ satisfying the conditions in the above displayed formula, is called a Dieudonné module for $k$.

The main theorem of the classical covariant Dieudonné theory asserts that the assignment

$$X \mapsto M_*(X)$$
establishes an equivalence of categories from the additive category of $p$-divisible groups over $k$ to the additive category of Dieudonné modules for the perfect base field $k$.

(2.6.2) Let $X, Y, Z, X', Y', Z'$ be $p$-divisible groups and let $M_*(X), M_*(Y), \ldots, M_*(Z')$ be their covariant Dieudonné modules.

We have seen in 2.5.3 and 2.5.4 that the map which to every biextension $E$ of $X \times Y$ associates the compatible family of Weil pairing $(\theta_n, E)_{n \in \mathbb{N}}$ establishes an equivalence of categories, from the category of biextensions of $X \times Y$ by $Z$, to the category of compatible families of bilinear pairings

$$(b_n : X[p^n] \times Y[p^n] \to Z[p^n])_{n \in \mathbb{N}}.$$

Moreover the set of all bihomomorphisms $\psi : E \to E'$ from a biextension $E$ of $X \times Y$ by $Z$ to a biextension $E'$ of $X' \times Y'$ by $Z'$ is in natural bijection with the set of all triples

$$(\alpha, \beta, \gamma) \in \text{Hom}_k(X, X') \times \text{Hom}_k(Y, Y') \times \text{Hom}_k(Z, Z')$$

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such that
\[ \gamma(\theta_{n,E}(x_n, y_n)) = \theta_{n,E'}(\alpha(x_n), \beta(y_n)) \]
for all \( k \)-schemes \( T \), all \( x_n \in X[p^n](T) \) and all \( y_n \in Y[p^n](T) \). We explain how to express these statements in terms of Dieudonné modules.

\textbf{(2.6.3) Proposition.} Notation as above.

(i) To every biextension \( E \) of \( X \times Y \) by \( F \), there is an associated \( W(k) \)-bilinear map

\[
\Theta_E : M_*(X) \times M_*(Y) \rightarrow M_*(Z)
\]
such that

\[
\Theta_E(F_{M_*(X)}(x), y) = F_{M_*(Z)}(\Theta(x, V_{M_*(Y)}(y))), \quad \Theta_E(x, F_{M_*(Y)}(y)) = F_{M_*(Z)}(\Theta(V_{M_*(X)}x, y))
\]

and

\[
\Theta_E(V_{M_*(X)}x, V_{M_*(Y)}y) = V_{M_*(Z)}(B(x, y))
\]

for all \( x \in M_*(X) \) and all \( y \in M_*(Y) \).

(ii) For every \( W(k) \)-bilinear map

\[
\Theta : M_*(X) \times M_*(Y) \rightarrow M_*(Z)
\]
satisfying the conditions that

\[
\Theta(F_{M_*(X)}(x), y) = F_{M_*(Z)}(\Theta(x, V_{M_*(Y)}(y))), \quad \Theta(x, F_{M_*(Y)}(y)) = F_{M_*(Z)}(\Theta(V_{M_*(X)}x, y))
\]

and

\[
\Theta(V_{M_*(X)}x, V_{M_*(Y)}y) = V_{M_*(Z)}(B(x, y))
\]

for all \( x \in M_*(X) \) and all \( y \in M_*(Y) \), there exists a biextension \( E \) of \( X \times Y \) by \( Z \) such that \( B = B_E \). Moreover such a biextension \( E \) is unique up to unique isomorphism.

(iii) Given a biextension \( E \) of \( X \times Y \) by \( Z \) and a biextension \( E' \) of \( X' \times Y' \) by \( Z' \), the natural map from the set of all homomorphisms of biextensions

\[
(\psi : E \rightarrow E', \alpha : X \rightarrow X', \beta : Y \rightarrow Y', \gamma : Z \rightarrow Z') \in \text{Hom}_{\text{biext}}(E, E')
\]
to the set of all triples \((f, g, h)\) satisfying the conditions

- \( f \in \text{Hom}_{W(k),F,V}(M_*(X), M_*(X')) \),
- \( g \in \text{Hom}_{W(k),F,V}(M_*(Y), M_*(Y')) \),
- \( h \in \text{Hom}_{W(k),F,V}(M_*(Z), M_*(Z')) \),
- \( h(\Theta_E(x, y)) = \Theta'_E(f(x), g(y)) \quad \forall x \in M_*(X), \forall y \in M_*(Y) \)

is a bijection.
(2.6.4) **Corollary.** Notation as in 2.6.3. In particular $E \to X \times Y$ is a biextension of $X \times Y$ by $Z$ and $\Theta_E$ is the $W(k)$-bilinear map from $M_+(X) \times M_+(Y)$ to $M_+(Z)$ attached to the biextension $Z$.

1. The group $\text{Aut}_{\text{biext}}(E)$ of all automorphisms of the biextension $E$ has a natural structure as a compact $p$-adic Lie group. It is naturally isomorphic to the closed subgroup of

$$\text{Aut}_{W,F,V}(M_+(X)) \times \text{Aut}_{W,F,V}(M_+(Y)) \times \text{Aut}_{W,F,V}(M_+(Z))$$

consisting of all triples $(\alpha, \beta, \gamma) \in \text{Aut}_{W,F,V}(M_+(X)) \times \text{Aut}_{W,F,V}(M_+(Y)) \times \text{Aut}_{W,F,V}(M_+(Z))$ such that

$$\gamma(\Theta_E(x,y)) = \Theta_E(\alpha(x), \beta(y)) \quad \forall x \in M_+(X), \forall y \in M_+(Y).$$

Here $\text{Aut}_{W,F,V}(M_+(X))$ denotes the compact $p$-adic Lie group consisting of all $W(k)$-linear automorphisms of $M_+(X)$ which commute with $F_{M_+(X)}$ and $V_{M_+(X)}$; it is naturally isomorphic to the group $\text{Aut}(X)$ of all automorphisms of the $p$-divisible group $X$. The same notation scheme is applied to $\text{Aut}_{W,F,V}(M_+(Y))$ and $\text{Aut}_{W,F,V}(M_+(Z))$.

2. The Lie algebra of the compact $p$-adic Lie group $\text{Aut}_{\text{biext}}(E)$ is naturally isomorphic to the Lie subalgebra of $\text{End}_{K,F,V}(M_+(X)_Q) \oplus \text{End}_{K,F,V}(M_+(Y)_Q) \oplus \text{End}_{K,F,V}(M_+(Z)_Q)$ consisting of all triples

$$(A, B, C) \in \text{End}_{W,F,V}(M_+(X)_Q) \oplus \text{End}_{K,F,V}(M_+(Y)_Q) \oplus \text{End}_{K,F,V}(M_+(Z)_Q)_Q$$

which satisfy the condition that

$$C(\Theta_E(x,y)) = \Theta_E(Ax,y) + \Theta_E(x,By) \quad \forall x \in M_+(X), \forall y \in M_+(Y).$$

Here $K := W(k)[1/p] = W(k) \otimes_Z Q$, $M_+(X)_Q := M_+(X)[1/p]$ and $\text{End}_{K,F,V}(M_+(X))$ denotes the set of all $K$-linear endomorphisms of $M_+(X)_Q$ which commute with $F$ and $V$; it is naturally isomorphic to the Lie algebra of the compact $p$-adic Lie group $\text{Aut}_{W,F,V}(M_+(X)) \cong \text{Aut}(X)$.

(2.6.5) **Definition.** Let $G$ be a compact $p$-adic Lie group, which is closed subgroup of the group of all $\mathbb{Q}_p$-points of a linear algebraic group over $\mathbb{Q}_p$. Let $k \supset \mathbb{F}_p$ be a perfect field of characteristic $p$. Let $W = W(k)$ be the ring of $p$-adic Witt vectors with entries in $k$, and let $K = W[1/p]$ be the fraction field of $W$.

(a) Let $U$ be a $p$-divisible group over $k$, and let $M(U)$ be the covariant Dieudonné module of $U \times_{\text{Spec}(k)} \text{Spec}(k)$. Let $\zeta : G \to \text{Aut}(U)$ be a continuous homomorphism. We say that the action of $G$ on $U$ is strongly non-trivial if there does not exist a pair $N_1 \subsetneq N_2$ of $K$-vector subspaces of $M \otimes_W K$ stable under the action of $\text{Lie}(G)$ such that the induced action of $\text{Lie}(G)$ on $N_2/N_1$ is trivial.

(b) Let $X, Y, Z$ be $p$-divisible groups over $k$. Let $E \to X \times_{\text{Spec}(k)} Y$ be a biextension of $X \times_{\text{Spec}(k)} Y$ by $Z$. Let $\rho : G \to \text{Aut}_{\text{biext}}(E)$ be a continuous action of $G$ on $E$ which respects the biextension structure of $E$. Let $\alpha : G \to \text{Aut}(X)$, $\beta : G \to \text{Aut}(Y)$, and $\gamma : G \to \text{Aut}(Z)$ be the continuous actions of $G$ on $X$, $Y$ and $Z$ induced by $\rho$. We say that the action of $G$ on $E$ is strongly non-trivial if the actions $\alpha, \beta, \gamma$ of $G$ on $X, Y, Z$ are all strongly non-trivial, or equivalently if the action of $G$ on $X \times Y \times Z$ is strongly non-trivial.
**Remark.** In the definition (a) above, if we require in addition that \(N_1, N_2\) are both stable under \(F\) and \(V\), the resulting new definition of the notion "strongly non-trivial", though apparently weaker, is actually equivalent to the definition in (a). The proof is left as an exercise.

(2.7) Canonical trivializations over torsion subgroup schemes.

Let \(X, Y, Z\) be \(p\)-divisible groups over a base scheme \(S\). Let \(\pi : E \to X \times_S Y\) be a biextension of \(X \times_S Y\) by \(Z\). Let \(E_n := \pi^{-1}(X[p^n] \times_S Y[p^n])\) be the restriction of the biextension \(E\) to \(X[p^n] \times_S Y[p^n]\); it is a biextension of \(X[p^n] \times_S Y[p^n]\) by \(Z\). The push-forward \([p^n]_Z, E_n\) of \(E_n\) by the homomorphism \([p^n]_Z : Z \to Z\) is again a biextension of \(X[p^n] \times_S Y[p^n]\) by \(Z\). In this subsection we will construct a natural splitting of the biextension \([p^n]_Z, E_n\).

(2.7.1) **Definition** of \(\eta : E_n \to Z\). Let \([(p^n]_Z, (E)\) be the push-forward of the biextension \(\pi : E \to X \times_S Y\) by \([p^n]_Z\), and let \(f_n : E \to [(p^n]_Z, (E)\) be the tautological map from \(E\) to its push-forward by \([p^n]_Z\). Clearly bihomomorphism \(\theta_{n, ([p^n]_Z, (E))} : X[p^n] \times_S Y[p^n] \to Z[p^n]\) attached to the biextension \([(p^n]_Z, (E)\) is equal to 0, by the functoriality of the Weil pairings. Let

\[
\xi_{can}^{\ast} : ([p^n]_Z, (E)) \to Z \times_S X[p^n] \times_S Y[p^n]
\]

be the canonical splitting as in 2.5.2. Let \(pr_1 : Z \times_S X[p^n] \times_S Y[p^n] \to Z\) be the projection to \(Z\). Define \(\eta_n : E_n \to Z\) to be the composition

\[
\eta_n := pr_1 \circ \xi_{can}^{\ast} \circ f_n.
\]

(2.7.2) **Alternative Definition** of \(\eta_n\). One can also define \(\eta_n\) directly using the construction of the biextension \(E_n \to X[p^n] \times_S Y[p^n]\) by descent in 2.5, with the descent datum given by the Weil pairing \(\theta_n\) of the biextension \(E\). We will use the notation in 2.5.

Let

\[
\eta_n' := [p^n]_Z \circ pr_1 : Z \times_S X[p^n] \times_S Y[p^{2n}] \to Z
\]

be the composition of the projection \(pr_1 : Z \times_S X[p^n] \times_S Y[p^{2n}] \to Z\) with the endomorphism \([p^n]_Z : Z \to Z\) of \(Z\). Obviously \(\eta_n'((\lambda, x, y)) = \eta_n'((\lambda + \theta_n(x, y, b), x, y + b)\) for all \(S\)-scheme \(T\), all \(x \in X[p^n](T)\), all \(y \in Y[p^n](T)\) and all \(b \in Y[p^n](T)\). Therefore \(\eta_n'\) factors through the faithfully flat morphism \(\alpha_n : Z \times X[p^n] \times_S Y[p^{2n}] \to E_n\). Here \(\alpha_n\) is the faithfully flat morphism in 2.5 which expresses the biextension \(E_n \to X[p^n] \times_S Y[p^n]\) as a descent of the trivial biextension \(Z \times_S X[p^n] \times_S Y[p^{2n}]\), with the descent datum encoded by the Weil pairing \(\theta_n\). By descent there exists a unique morphism \(\eta_n : E_n \to Z\) such that

\[
\eta_n' = \eta_n \circ \alpha_n.
\]

An easy exercise shows that the morphism \(\eta_n\) defined above coincides with the morphism \(\eta_n\) defined in 2.7.1.

(2.7.3) From the definitions of \(\alpha_n\) and \(\eta_n\) it is not difficulty to verify that the compatibility relation

\[
[p^n]_Z \circ \eta_n = \eta_{n+1} \circ (E_n \hookrightarrow E_{n+1})
\]

holds for all \(n \in \mathbb{N}\).
§3. Complete restricted perfections in characteristics $p, I$

In §2.7 we defined a compatible sequence of morphisms $\{ \eta_n : \pi^{-1} E =: E_n \to Z \}_{n \in \mathbb{N}}$ for any biextension of $E$ of $p$-divisible groups $X, Y$ by another $p$-divisible group $Z$, over an arbitrary base scheme $S$. In this section we will consider the special case when $S$ is the spectrum of a perfect field $k \supset \mathbb{F}_p$.

An interesting phenomenon reveals itself in the special case described in 3.1, and the compatible sequence of morphisms $\{ \eta_n \}$ lead us to families commutative rings, whose elements consists of formal series of the form

$$\sum_{(i_1, \ldots, i_m) \in \mathbb{Z}[1/p]_m \geq 0} \in \mathbb{Z}[1/p]_m a_{i_1, \ldots, i_m} i_1^{i_1} i_2^{i_2} \cdots i_m^{i_m}$$

with coefficients $a_{i_1, \ldots, i_m} \in k$, subject to the condition roughly of the following form

$$|I|_p \leq C \cdot |I|_{\max}^E$$

for every $I$ such that $a_I \neq 0$, where $C, E > 0$ are parameters which define the ring. Here for any multi-index $I = (i_1, \ldots, i_m) \in \mathbb{Z}[1/p]_m \geq 0$, $|I|_p$ is the $p$-adic norm of $I$ and $|I|_{\max}^E$ is the archimedean norm of $I$, defined by

$$|I|_p := \max(p^{-\ord_p(i_1)}, \ldots, p^{-\ord_p(i_m)}), \quad \text{and} \quad |I|_{\max} := \max(i_1, i_2, \ldots, i_m).$$

These rings do not seem to have appeared in the literature, but they hold the key to the local rigidity for biextensions of $p$-divisible groups. In this section we give the motivation and definition of these new rings.

(3.1) Assumptions. To focus on the key features in the proof of local rigidity of biextensions of $p$-divisible formal groups, we make three additional assumptions.

(0) The base field $k$ is algebraically closed.

(1) The $p$-divisible group $Z$ has a slope $\mu_1$ which is strictly bigger than every slope of $X$ and every slope of $Y$.

(2) The $p$-divisible group $Z$ is isomorphic to a product $Z_1 \times Z_2$ of two $p$-divisible formal groups, such that $Z_1$ is isoclinic of slope $\mu_1$.

(3) There exist positive integers $a, r > 0$ such that

$$\mu_1 = \frac{a}{r} \quad \text{and} \quad \text{Ker}([p^a]_{Z_1}) = \text{Fr}_{Z_1/k}^r,$$

where $\text{Fr}_{Z_1/k}^r : Z_1 \to Z_1^{(p^r)}$ is the $r$-th iterate of the relative Frobenius morphism for $Z_1/k$.

It follows from assumptions (0) and (3) that there exists elements $u_1, \ldots, u_b \in \Gamma(Z_1, O_{Z_1})$ such that the affine coordinate ring of $Z_1$ is the formal power series ring $k[[u_1, \ldots, u_b]]$, and

$$[p^a]_{Z_1}(u_i) = u_i^{p^r} \quad \forall i = 1, \ldots, b.$$

Remark. Suppose that $E$ is a biextension of $p$-divisible formal groups $X \times Y$ over $k$ by a $p$-divisible formal group $Z'$ over $k$ such that the $Z'$ has a slope $\mu_1$ which is strictly bigger than all slopes of $X$ and $Y$. There exists an isogeny $\beta : Z' \to Z$ of $p$-divisible groups such that the assumptions (2) and (3) hold for $Z$ and also for the push-forward $\beta_* E'$ of $E$ by $\beta$. 

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Similarly we have Artinian subschemes 
\[ X_n \] for all \( n \). Let (3.3) Choose and fix a positive rational number \( \mu_0 < \frac{a}{r} \) such that \( \mu_0 \) is strictly bigger than every slope of \( Z_2 \times X \times Y \). Multiplying both \( a \) and \( r \) by a suitable positive integer, we may and do assume that \( \mu_0 \) has the form 
\[ \mu_0 = \frac{a}{s}, \quad s < r, \quad s \in \mathbb{N}_{>0} \]
From the general properties of slopes we know that there exists a constant \( m_0 \) such that
(3.2.1) \[ X[p^m] \supset \text{Ker}(\text{Fr}_{X/k}^m) \quad \text{and} \quad Y[p^m] \supset \text{Ker}(\text{Fr}_{Y/k}^m) \]
for all \( m \geq m_0 \). Therefore
(3.2.2) \[ X[p^{na}] \supset \text{Ker}(\text{Fr}_{X/k}^{na}) \quad \text{and} \quad Y[p^{na}] \supset \text{Ker}(\text{Fr}_{Y/k}^{na}) \]
for all \( n \geq n_0 = \lceil \frac{ma}{a} \rceil \). On the other hand, assumption (3) implies that
\[ \text{Ker}([p^{na}]_{Z_1}) = \text{Ker}(\text{Fr}_{Z_1/k}^{na}) \]
for all \( n \in \mathbb{N} \).

**Remark.** (a) In practice we will choose \( \mu_0 \) to be “just a tiny bit bigger than the maximum of the slopes of \( X \) and \( Y \”).
(b) If we choose \( \mu_0 \) to be the maximum of the slopes of \( X \) and \( Y \), then the estimate in 3.2.2 needs to be changed to: there exists a constant \( e \) (depending on \( X \) and \( Y \)) such that
(3.2.3) \[ X[p^{na}] \supset \text{Ker}(\text{Fr}_{X/k}^{na-e}) \quad \text{and} \quad Y[p^{na}] \supset \text{Ker}(\text{Fr}_{Y/k}^{na-e}) \]
for all \( n \geq n_0 = \lceil \frac{ma}{a} \rceil \).

(3.3) Let \( R = R_E \) be the coordinate ring of \( E \), let \( m = m_E \) be the maximal ideal of \( R \). Let \( \phi = \phi_R \) be the absolute Frobenius endomorphism of \( R \) which sends every element \( x \in R \) to \( x^p \). For every \( n \in \mathbb{N} \), let
\[ m(p^n) := \phi^n(m) \cdot R \]
be the ideal of \( R \) generated by \( \phi^n(m) \). Clearly \( m(p^n) \subset m(p^n) \) for all \( n \in \mathbb{N} \), where \( m(p^n) \) is the \( p^n \)-th power of the maximal ideal \( m \). In other words \( m(p^n) \) is the ideal of \( R \) generated by all products of the form \( \prod_{1 \leq j \leq p^n} x_j \), with \( x_j \in m \) for all \( j \). If \( t_1, \ldots, t_m \) is a regular system of parameters of the complete regular local ring \( R \), then \( m(p^n) \) is the ideal \( (t_1^{p^n}, \ldots, t_m^{p^n}) \) in \( R = k[[t_1, \ldots, t_m]] \), while \( m(p^n) \) is the ideal generated by all monomials of the form \( \prod_{1 \leq j \leq m} t_j^{i_j} \) with \( i_1, \ldots, i_m \in \mathbb{N} \) and \( i_1 + \cdots + i_m \geq p^n \). It is clear from the above that
\[ m(p^n) \subset m(p^{n+a}) \quad \text{if} \quad p^n \geq m, \]
because if a multi-index \( I = (i_1, \ldots, i_m) \in \mathbb{N}^m \) has the property that \( ||I|| := \sum_{j=1}^m i_j \geq p^n \), then at least one of the indices \( i_1, \ldots, i_m \) is \( \geq \frac{p^n}{m} \). Let
\[ E[F^n] := \text{Spec} \left( R/m(p^n) \right) = \text{Spec} \left( R/\phi^n(m)R \right), \quad E \mod m(p^n) := \text{Spec} \left( R/m(p^n) \right). \]
Similarly we have Artinian subschemes \( X[F^n] \), \( X \mod m(p^n) \subset X \), \( Z_1[F^n] \), \( Z_1 \mod m(p^n) \subset Z_1 \), etc.
In 2.7 we constructed a family of morphisms \( \eta_n : E_n = \pi^{-1}(X[p^n] \times Y[p^n]) \to Z \) such that 
\( [p] \circ \eta_n = \eta_{n+1} | E_n \) for all \( n \). From 3.2.2 we know that \( E[F^{ns}] \subset E_{na} \) for all \( n \geq n_0 \), where \( a \) and \( s \) are the positive integers chosen in 3.2 so that \( \frac{a}{s} \) is strictly bigger than every slope of \( X \) or \( Y \) and both \( s \) and \( a \) are sufficiently divisible. The restriction of \( \eta_{na} \) to \( E[F^{ns}] \) makes sense for all \( n \geq n_0 \) because \( E[F^{ns}] \subset E_{na} \). This restriction is a morphism from \( E[F^{ns}] \) to \( Z[F^{ns}] \). The projection \( \text{pr}_{Z_1} : Z = Z_1 \times Z_2 \to Z_1 \) induces a morphism \( \text{pr}_{Z_1} : Z[F^{ns}] \to Z_1[F^{ns}] \).

**Definition.** Define a morphism \( \rho_{na} \) by composing \( \eta_{na} \) with \( \text{pr}_{Z_1} : \)

\[ \rho_{na} := \text{pr}_{Z_1} \circ \eta_{na} | E[F^{ns}] : E[F^{ns}] \to Z_1[F^{ns}] \quad \forall n \geq n_0 \]

The morphisms \( \rho_{na} \) satisfy the following compatibility relations

\[ [p^n]^n Z_1 \circ \rho_{na} = \rho_{(n+1)a} | E[F^{ns}] \quad \forall n \geq n_0 \]

Let \( u_1, \ldots, u_b \) be a regular system of parameters of the coordinate ring of \( Z_1 \) as in 3.1, so that \( Z_1 = \text{Spf}(k[[u_1, \ldots, u_b]]) \) and

\[ [p^n]^n Z_1(u_j) = u_j^{p^r} \quad \forall j = 1, \ldots, b. \]

**Definition.** Define elements \( a_{j,n} \in R_E / m_E^{(p^n)} \) for all \( j = 1, \ldots, b \) and all \( n \geq n_0 \), by

\[ a_{j,n} = \rho_{na}^* (u_j) \in R_E / m_E^{(p^n)}. \]

The compatibility relations for the \( \rho_n \)'s in 3.3.2 and the fact that \( [p^n]^n Z_1(u_j) = u_j^{p^r} \) imply that

\[ a_{j,n}^{p^r} \equiv a_{j,n+1} \pmod{m_E^{(p^{n+r})}} \]

for all \( n \geq n_0 \) and all \( j = 1, \ldots, b \).

**Remark.**
(a) The Frobenius map \( \phi^r \) on \( R_E \) induces injective ring homomorphisms

\[ \phi^r : R_E / m_E^{(p^n)} \to R_E / m_E^{(p^{n+r})} \]

for all \( n \). In particular the element \( a_{j,n}^{p^r} \) on the left hand side of (†) is an element of \( R_E / m_E^{(p^{n+r})} \) uniquely determined by the element \( a_{j,n} \in R_E / m_E^{(p^n)} \).

(b) The compatibility relation (†) makes the limit procedure in 3.4.2 a little neater than it would have been, had we used the slightly coarser congruence

\[ a_{j,n}^{p^r} \equiv a_{j,n+1} \pmod{m_E^{p^r}} \]

instead of (†).
We saw in 3.3.3 that the compatible family of morphisms \( \rho_n : E[F^m] \to Z_1[F^m] \) is given by \( b \) sequences
\[
\left( a_{j,n} \in R_E/m_E^{(p^m)}) \right)_{n \geq n_0}
\]
of elements in Artinian local rings \( R_E/m_E^{(p^m)} \) which satisfy the relation (\( \dagger \)) in 3.3.3. Each of the chosen coordinates \( u_1, \ldots, u_b \) of the \( p \)-divisible formal group \( Z_1 \) gives rise to a compatible sequence of elements in \( R_E/m_E^{(p^m)} \).

It is natural to try to formulate a convenient version of the “limit” of a given compatible sequence of elements in \( R_E/m_E^{(p^m)} \) (other than just the sequence itself). We record the definition of \( \phi^r \)-compatibility in 3.4.1 (a) below, together with a variant coarser version in 3.4.1 (b).

The following notations will be used in 3.4.1. Let \( \kappa \supset \mathbb{F}_p \) be a perfect field. Let \( n_0 \) be a natural number. Let \( r < s \) be positive integers. Let \( t_1, \ldots, t_m \) be \( m \) variables. We adopt the notation \( t := (t_1, \ldots, t_m) \) and \( \kappa[[t]] := \kappa[[t_1, \ldots, t_m]] \). Let
\[
(t^{p^m}) = (t^{p^m}) := (t_1^{p^m}, \ldots, t_m^{p^m}).
\]

Let \( (t)^{p^m} \) be the ideal of \( \kappa[[t]] \) generated by all monomials \( t^I := t_1^{i_1} \cdots t_m^{i_m} \), where \( I = (i_1, \ldots, i_m) \in \mathbb{N}^m \) ranges through all \( m \)-tuples in \( \mathbb{N}^m \) with \( |I| := i_1 + \cdots + i_m = n \). Let \( \phi \) be the Frobenius map on \( \kappa[[t]] \) which sends every element of \( \kappa[[t]] \) to its \( p \)-th power.

**3.4.1 Definition.** We follow the notation in the previous paragraph.

(a) A sequence of elements \( (a_n)_{n \geq n_0} \) with \( a_n \in \kappa[[t]]/(t^{p^m}) \) for all \( n \) is \( \phi^r \)-compatible if
\[
a^{p^m}_n \equiv a_{n+1} \pmod{(t^{p^{m+r}})} \quad \forall n \geq n_0
\]

(b) A sequence of elements \( (a_n)_{n \geq n_0} \) with \( a_n \in \kappa[[t]]/(t^{p^m}) \) for all \( n \) is \( \phi^r \)-compatible if
\[
a^{p^m}_n \equiv a_{n+1} \pmod{(t^{p^{m+r}})} \quad \forall n \geq n_0
\]

**Remark.** The version (b) is different from (a) in that the element \( a_n \) is in the congruence class modulo the ideal \( (t)^{p^m} \), which is bigger than the ideal \( (t^{p^m}) \). We will mostly use (a) because this version provides more information. For the proof of local rigidity version (b) will also be adequate.

**3.4.2** Suppose we are given a \( \phi^r \)-compatible sequence \( (a_n)_{n \geq n_0} \) with \( a_n \in \kappa[[t]]/(t^{p^m}) \) for all \( n \geq n_0 \). Formally the compatibility relation suggests that
\[
\phi^{-nr}(a^{p^m}_n) \equiv \phi^{-(n+1)r}(a_{n+1}) \pmod{\phi^{-nr}((t^{p^m})}) \quad \forall n \geq n_0.
\]

Here we have used \( \phi^{-nr}((t^{p^m})) \) instead of \( \phi^{-(n+1)r}((t^{p^m})) \) to make the congruence relation look better and more suggestive. Thus it seems reasonable to try to produce a “limit” of the sequence \( \phi^{-nr}a_n \) as \( n \to \infty \) in some suitable way.

There is an obvious problem: \( a_n \) has representatives in \( \kappa[[t]] \), but in general none of the representatives is in \( \phi^{nr}(\kappa[[t]]) \). We need to use at least some elements in the perfection
\[
\kappa[[t]]^{\text{perf}} = \bigcup_n \kappa[[t^{p^{-n}}]] = \bigcup_n \kappa[[t_1^{p^{-n}}, \ldots, t_m^{p^{-n}}]].
\]
of $\kappa[[t]]$. Note that this perfection is not complete for the topology defined by the filtration by total degree. For our purpose we don’t have to be concerned too much about $\kappa[[t]]^{\text{perf}}$ or its completion. We will focus rather on what comes out of the limit procedure for $\phi^r$-compatible sequences in the Artinian local rings $\kappa[[t]]/(t^{pr^r})$.

(3.4.3) Notation involving multi-indices.

(i) For each index $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$, let

$$t^I := t_1^{i_1} \cdots t_m^{i_m}$$

be the corresponding monomial in the variables $t_1, \ldots, t_m$.

(ii) Among the archimedean norms on $\mathbb{Q}^m$, we will use the following two: for $J = (j_1, \ldots, j_m) \in \mathbb{Q}^m$,

$$J_\infty := \max(|j_1|, \ldots, |j_m|), \quad J_\sigma := |j_1| + \cdots + |j_m|.$$ 

Obviously

$$J_\infty \leq J_\sigma \leq m \cdot J_\infty \quad \forall J \in \mathbb{Q}^m$$

(iii) There is also the following $p$-adic norm on $\mathbb{Q}^m$:

$$|J|_p := \max(|j_1|_p, \ldots, |j_m|_p)$$

where $|\cdot|_p$ is multiplicative $p$-adic absolute value on $\mathbb{Q}$, defined by $|x|_p = p^{-\ord_p(x)}$ for all $x \in \mathbb{Q}$, so that $|p| = \frac{1}{p}$ and $|x|_p = 1$ if both the numerator and denominator of $x$ are prime to $p$.

Define

$$\ord_p(J) := \min(\ord_p(j_1), \ldots, \ord_p(j_m)),$$

define

$$|J|_p = p^{-\ord_p(J)}.$$ 

We will use the restriction of these norms to $\mathbb{N}[1/p]^m := \mathbb{Z}[1/p]_{\geq 0}^m$, the additive semigroup of exponents with $p$-power denominators.

(3.5) We will approach the limit problem in 3.4.2 in a lowbrow fashion first.

(3.5.1) Suppose we are given a $\phi^r$-compatible sequence $(a_n)_{n \geq n_0}$ with $a_n \in \kappa[[t]]/(t^{pr^r})$ for all $n \geq n_0$. For each $n \geq n_0$, write the element $a_n \in \kappa[[t]]/(t^{pr^r})$ as

$$a_n = \sum_{J \in \mathbb{N}^m, |J|_\infty < p^{nr^r}} a_{n,J} t^J \mod (t^{pr^r}).$$

Clearly the coefficients $a_{n,J} \in \kappa$ with $|J|_\infty$ are uniquely determined by $a_n$. The compatibility relation $a_n^{pr} \equiv a_{n+1} (\mod (t^{pr^{r+1}}))$ means that $a_{n+1,J} = a_{n,p^{-r}J}$ for all $J \in \mathbb{N}^m$ with $|J|_\infty < p^{nr^r}$ and all $n \geq n_0$. More precisely,

$$a_{n+1,J} = \begin{cases} 0 & \text{if } |J|_\infty < p^{nr^r}, \ p^{-r}J \not\in \mathbb{N}^m \\ a_n^{pr} & \text{if } |J|_\infty < p^{nr^r}, \ p^{-r}J \in \mathbb{N}^m \end{cases}$$

\(\ddagger\)
for all \( n \geq n_0 \). Thus the among the coefficients \( a_{n,j} \) for a fixed natural number \( n \geq n_0 + 1 \), those with \( |J|_\infty < p^{(n-1)s+r} \) arises from coefficients \( a_{n',j'} \) with \( n' < n \). More precisely suppose that \( n \geq n_0 + 1 \), then the following statements hold.

- If \( |J|_\infty < p^{(n-1)s+r} \) and \( J \) is not divisible by \( p^r \), then \( a_{n,j} = 0 \).

- If \( |J|_\infty < p^{(n-1)s+r} \) and \( J = p^{(n-n')r}J' \), where \( n' < n \) and \( J' \) is not divisible by \( p^r \), then \( a_{n,j} = a_{n',J'} \).

There is no constraint for those \( a_{n,j} \)'s with \( |J|_\infty \geq p^{(n-1)s+r} \); these coefficients will be propagated to coefficients of \( a_{n',J'} \)'s with \( n'' > n \).

(3.5.2) **Construction of the limit.** For each multi-index \( I \in \mathbb{N}[1/p]^m \), define \( b_I \in \kappa \) by

\[
b_I := (a_{n,p^rJ})^{-m} = \phi^{-m}(a_{n,p^rJ}),
\]

where \( n \in \mathbb{N} \) is sufficiently large such that \( p^nI \in \mathbb{N}^m \) and \( |p^nI|_\infty < p^m \), so that \( a_{n,p^rI} \) makes sense. The compatibility relation for the \( a_{n,j} \)'s immediately implies that the above definition does not depend on the choice of \( n \), as long as

\[
n \geq \text{Max} \left( \frac{-\text{ord}_p(J)}{r}, \frac{\log_p(|J|_\infty)}{s-r} \right).
\]

The formal series

\[
\sum_{I \in \mathbb{N}[1/p]^m} b_I t^I = \sum_{(i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m} b_{i_1, \ldots, i_m} t_1^{i_1} \cdots t_m^{i_m}
\]

attached to a given \( \phi^r \)-compatible sequence of elements \( (a_n \in \kappa[[t]]/(t^{p^n}))_{n \geq n_0} \) according to the above construction will be called the limit of the \( \phi^r \)-compatible sequence \( (a_n)_{n \geq n_0} \).

(3.5.3) **Proposition.** The construction described in 3.5.2 establishes a bijection, from the set of all \( \phi^r \)-compatible sequences of elements \( (a_n \in \kappa[[t]]/(t^{p^n}))_{n \geq n_0} \), to the set of all formal series

\[
\sum_{I \in \mathbb{N}[1/p]^m} b_I t^I
\]

such that \( b_I \in \kappa \) for all \( I \in \mathbb{N}[1/p]^m \), and

\[
(*) \quad -\text{ord}_p(I) \leq \text{Max} \left\{ n_0, r \cdot \left( \left\lfloor \frac{\log_p(|I|_\infty)}{s-r} \right\rfloor + 1 \right) \right\}
\]

for every \( I \in \mathbb{N}[1/p]^m \) with \( b_I \neq 0 \).

**Proof.** Although the estimate in the statement of 3.5.3 looks complicated, the proof is completely straight-forward from the construction explained in 3.5.2.

Suppose that \( \sum_{I \in \mathbb{N}[1/p]^m} b_I t^I \) is attached to a \( \phi^r \)-compatible sequence \( (a_n)_{n \geq n_0}, a_n \in \kappa[[t]]/(t^{p^n}) \) for all \( n \geq n_0 \). Let \( I \in \mathbb{N}[1/p]^m \) be an index in the support of the above formal series, i.e. \( b_I \neq 0 \). We need to show that the inequality \((*)\) holds. Let \( n_1 \) be the smallest natural number such that
\( p^{n_1}I \in \mathbb{N}^m. \) There is nothing to prove if \( n_1 \leq n_0, \) so we may assume that \( n_1 \geq n_0 + 1. \) In particular \( \text{ord}_p(I) < 0, \) and \( n_1 = \left\lceil -\frac{\text{ord}_p(I)}{r} \right\rceil. \)

From the definition of \( n_1, \) we know that \( p^{n_1}I \) is not divisible by \( p^r. \) If \( |p^{n_1}I|_\infty < p^{(n_1-1)s+r}, \) we get from 3.5.1 (‡) that \( b_I = 0, \) a contradiction. We have shown that

\[
|p^{n_1}I|_\infty \geq p^{(n_1-1)s+r}.
\]

The last inequality is equivalent to

\[
\left\lceil -\frac{\text{ord}_p(I)}{r} \right\rceil = n_1 \leq \frac{\log_p |I|_\infty}{s-r} + 1,
\]

which is easily seen to be equivalent to the asserted inequality (*)

It remains to show that every formal series \( \sum_{I \in \mathbb{N}[1/p]^m} b_I t^I \) whose support satisfies the inequality (*) arises from a \( \phi^r \)-compatible sequence \( (a_n)_{n \geq n_0}. \) This statement is not difficult to see: one verifies using the inequality (*) that for every natural number \( n \geq n_0, \) the truncated series

\[c_n := \sum_{I \in \mathbb{N}[1/p]^m, |I|_\infty < p^n} b_I t^I \in \kappa[[t]]\]

Let \( a_n := c_n \mod (t^{p^n}). \) It is easily verified that \( (a_n)_{n \geq n_0} \) is a \( \phi^r \)-compatible sequence, whose limit is the given formal series \( \sum_{I \in \mathbb{N}[1/p]^m} b_I t^I. \)

**Remark.** For \( \phi^r \) compatible sequences of elements \( (a_n \in \kappa[[t]]/(t)^{p^n}) \), the procedure 3.5.2 for constructing limits also works if the norm \( |\cdot|_\infty \) for multi-indices is replaced by the norm \( |\cdot|_\sigma. \) The corresponding results are similar, so are the proofs: in the statements and proofs of 3.5.1, 3.5.2 and 3.5.3, we need to replace \( \kappa[[t]]/(t)^{p^n} \) by \( \kappa[[t]]/(t)^{p^n} \) and replace \( |\cdot|_\infty \) by \( |\cdot|_\sigma. \)

**Definition.** Let \( \kappa \supset \mathbb{F}_p \) be a perfect field of characteristic \( p > 0, \) and let \( t = (t_1, \ldots, t_m) \) be \( m \) variables. Let \( r, s \in \mathbb{Z}_{>0} \) be two positive integers with \( r < s, \) and let \( n_0 \) be a natural number.

(a) Denote by \( \kappa\langle (t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}) \rangle_{s: \phi^r, \geq n_0} \) the commutative \( \kappa \)-algebra consisting of all formal series

\[
\sum_{I \in \mathbb{N}[1/p]^m} b_I t^I
\]

such that \( b_I \in \kappa \) for all \( I \in \mathbb{N}[1/p]^m, \) and

(*) \[ -\text{ord}_p(I) \leq \text{Max} \left\{ n_0, r \left( \left\lceil \frac{\log_p |I|_\infty}{s-r} \right\rceil + 1 \right) \right\} \]

for every \( I \in \mathbb{N}[1/p]^m \) such that \( b_I \neq 0. \)

(b) Denote by \( \kappa\langle (t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}) \rangle_{s: \phi^r, \geq n_0} \) the commutative \( \kappa \)-algebra consisting of all formal series

\[
\sum_{I \in \mathbb{N}[1/p]^m} b_I t^I
\]
such that $b_I \in \kappa$ for all $I \in \mathbb{N}[1/p]^m$, and

\[
(*) \quad -\text{ord}_p(I) \leq \text{Max} \left\{ n_0, r \cdot \left( \frac{\log_p(|I| \sigma)}{s-r} \right) + 1 \right\} 
\]

for every $I \in \mathbb{N}[1/p]^m$ such that $b_I \neq 0$.

(c) Let $\text{supp}(\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0}))$ be the subset of $\mathbb{N}[1/p]^m$ consisting of all multi-indices $I \in \mathbb{N}[1/p]^m$ such that the inequality $(*)$ holds. Similarly let $\text{supp}(\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0}))$ be the subset of $\mathbb{N}[1/p]^m$ consisting of all multi-indices $I \in \mathbb{N}[1/p]^m$ such that the inequality $(**)$ holds.

**Remark.** (i) The two support sets

\[
\text{supp}(\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0})) \quad \text{and} \quad \text{supp}(\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0}))
\]

are sub-semigroups of $\mathbb{N}[1/p]^m$. Moreover for every $M > 0$, there are only a finite number elements $I$ in either sub-semigroup such that $|I| \leq M$. The last property implies that for each $I$, there are only a finite number of pairs $(I_1, I_2)$ of elements in either sub-semigroup such that $I_1 + I_2 = I$. Therefore the standard formula for multiplication of formal power series defines multiplication on $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0})$ and $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0})$, making them augmented local domains over $\kappa$.

(ii) Let $m \geq 1$ be a positive integer. It is easy to see that the rings $\langle t_1, \ldots, t_m \rangle_{s;\phi^r, \geq n_0}$ and $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0})$ are non-Noetherian local domains. In can be shown that neither of the two local domains is normal. Moreover the integral closure of $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0})$ (respectively $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0}$) in its own fraction field is not a finitely generated module over $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0})$ (respectively $\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, \geq n_0}$), because the fraction field of either ring contains $t_i^j$ for any $j \in \mathbb{N}[1/p]$ and any $i = 1, \ldots, m$. However these integral closures can be described explicitly.

Below is a slightly different version of the rings defined in 3.5.4.

**Definition.** Let $\kappa \subset \mathbb{F}_p$ be a perfect field. Let $r < s$ be two positive integers, and let $i_0 \in \mathbb{N}$ be a natural number. The perfection of the formal power series $\kappa[[t_1, \ldots, t_m]]$ is naturally isomorphic to

\[
\bigcup_{n \in \mathbb{N}} \kappa[[t_1^{p^n}, \ldots, t_m^{p^n}]].
\]

Denote by $\phi$ the Frobenius automorphism of this perfect ring.

(a) Consider the following subring

\[
(\kappa(\langle t_1^{p^n}, \ldots, t_m^{p^n} \rangle_{s;\phi^r, [i_0]}))_{\text{fin}} := \sum_{n \in \mathbb{N}} \phi^{-nr}(t_j^{p^{nr-i_0}})
\]

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of the perfection of the formal power series ring \( \kappa[[t_1, \ldots, t_m]] \), where our convention is that \( (\ell)^{p^{n-i_0}} = R \) if \( ns - i_0 \leq 0 \). Define a decreasing filtration \((\text{Fil}^\#_{s,\varphi',[i_0]})_{s \in \mathbb{Z}}\) on the ring

\[
\left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]} \right)^\#_{\text{fin}}
\]

by

\[
\text{Fil}^\#_{s,\varphi',[i_0]} := \left\{ x \in \left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]} \right)^\# \left| \exists n \in \mathbb{N}_{>0} \text{ s.t. } n + j \geq 0 \text{ and } x^{p^n} \in (\ell)^{p^{n+j}} \right\}.
\]

where \( (\ell = (t_1, \ldots, t_m) \) is the maximal ideal of \( \kappa[[t_1, \ldots, t_m]] \). Define

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]}^\#_{\text{fin}}
\]

to be the completion of \( \left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]} \right)^\#_{\text{fin}} \) with respect to the above decreasing filtration.

(b) Consider the following subring

\[
\left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]}^b \right)_{\text{fin}} := \sum_{n \in \mathbb{N}} \phi^{-nr}\left((\ell)^{p^{n-i_0}}\right)
\]

of the perfection of the formal power series ring \( \kappa[[t_1, \ldots, t_m]] \). In the above our convention is that \( (\ell)^{p^{n-i_0}} = R \) if \( ns - i_0 \leq 0 \). Define a decreasing filtration \((\text{Fil}^b_{s,\varphi',[i_0]})_{s \in \mathbb{Z}[1/p]_{\geq 0}}\) on the ring \( \left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]}^b \right)_{\text{fin}} \) by

\[
\text{Fil}^b_{s,\varphi',[i_0]} := \left\{ x \in \left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]}^b \right)_{\text{fin}} \left| \exists n \in \mathbb{N}_{>0} \text{ such that } p^n u \in \mathbb{N} \text{ and } x^{p^n} \in (\ell)^{u-p^n} \right\}.
\]

Define \( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]}^b \) to be the completion of \( \left( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{s,\varphi';[i_0]}^b \right)_{\text{fin}} \) with respect to the above filtration.

3.6 Definitions of complete restricted perfections

We will introduce in 3.6.1 two other families, \( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C:d}^E, # \) and \( \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C:d}^E, b \) of complete restricted perfections of a given power series ring \( \kappa[[t_1, \ldots, t_m]] \), related to the rings defined in 3.5.4 and 3.5.5. We will also see in 3.6.3 and 3.6.4 that the notion of complete restricted perfection in 3.5.4 and 3.5.5 can be extended to general complete Noetherian local domains of equi-characteristic \( p > 0 \) with perfect residue fields.

3.6.1 Definition. Let \( \kappa \supset \mathbb{F}_p \) be a perfect field and let \( t_1, \ldots, t_m \) be variables. Let \( C > 0, d \geq 0, E > 0 \) be real numbers.

(a) Define a commutative \( \kappa \)-algebra

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C:d}^E, #
\]

whose underlying abelian group is the set of all formal series \( \sum_I b_I t^I \) with \( b_I \in \kappa \) for all \( I \), where \( I \) runs through all elements in \( \mathbb{N}[1/p]^m \) such that

\[
|I|_p \leq \text{Max}(C \cdot (|I|_\infty + d)^E, 1).
\]

The ring structure is given by the standard formula for product of power series.
(b) Define a commutative $\kappa$-algebra

$$
\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}
$$

whose underlying abelian group is the set of all formal series $\sum_I b_I t^I$ with $b_I \in \kappa$ for all $I$, where $I$ runs through all elements in $\mathbb{N}[1/p]^m$ such that

\[ |I|_p \leq \text{Max}(C \cdot (|I|_\sigma + d)^E, 1). \]

The above condition on the support (of elements of this subset) shows that the standard formula for multiplication makes sense and gives $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}$ a natural structure as an augmented commutative algebra over $\kappa$.

Denote by $\text{supp}(m : E; C, d) = \text{supp}(m : b : E; C, d)$ the subset of $\mathbb{N}[1/p]^m$ consisting of all elements $I \in \mathbb{N}[1/p]^m$ satisfying the inequality (b) above.

(3.6.2) **Lemma.** Denote by $\text{Fil}_{t,\text{deg}}$ the decreasing filtration on $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}$ such that

$$
\text{Fil}_{t,\text{deg}}(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}) := \left\{ \sum_{I \in \text{supp}(m; E; C, d), |I|_\sigma \geq u} b_I t^I : b_I \in \kappa \forall I \right\}
$$

for every $u \in \mathbb{R}$. Let

$$
\text{Fil}_{t,\text{deg}}^+(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}) := \bigcup_{e > 0} \text{Fil}_{t,\text{deg}}(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b})
$$

(i) Both $\text{Fil}_{t,\text{deg}}(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b})$ and $\text{Fil}_{t,\text{deg}}^+(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b})$ are ideals of the ring $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}$ for every $u \in \mathbb{R}$.

(ii) Let $\text{gr}^*(\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b})$ be the graded ring attached to the filtration $\text{Fil}_{t,\text{deg}}$ of the ring $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}$. This graded ring is naturally isomorphic to the graded subring

$$
\bigoplus_{I \in \text{supp}(m; E; C, d)} \kappa \cdot t^I
$$

of the perfection

$$
\kappa[t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}] = \bigoplus_{I \in \mathbb{N}[1/p]^m} \kappa \cdot t^I
$$

of the polynomial ring $\kappa[t_1, \ldots, t_m]$, where the latter is graded by the total degree $|I|_\sigma$ of monomials $t^I$.

The proof is easy/obvious, therefore omitted. \[\square\]
(3.6.3) Definition. Let \((R, \mathfrak{m})\) be a complete Noetherian local domain of equi-characteristic \(p > 0\), with perfect residue field \(\kappa\). Let \(R^\text{perf}\) be the perfection of \(R\), and let \(\phi\) be the Frobenius automorphism on \(R\). Let \(r, s, n_0\) be natural numbers, \(0 < r < s, n_0 \geq 0\).

(a) Consider the following subset

\[
\text{def}(R, m)_{s, \phi^r; [i_0]} := \sum_{n \geq 0} \phi^{nr} (m^{p ns - i_0})
\]

of the perfect domain \(R^\text{perf}\). In the above by convention \(m^{p ns - i_0} = R\) if \(ns - i_0 \leq 0\). It is easy to see that this subset is a subring of \(R^\text{perf}\). Define a decreasing filtration \((\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}})^{p^s}_{i_0} \times \mathbb{N}\) on \((R, m)_{s, \phi^r; [i_0]}^{\text{perf}}\) by

\[
\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}} := \left\{ x \in (R, m)_{s, \phi^r; [i_0]}^{\text{perf}} \mid \exists n \in \mathbb{N} \text{ s.t. } x^{p^i} \in m^{p^i + j} \right\},
\]

It is easy to see that each \(\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}}\) is an ideal of \((R, m)_{s, \phi^r; [i_0]}^{\text{perf}}\). Define

\[
(R, m)_{s, \phi^r; [i_0]}^{\text{perf}}
\]

to be the completion of \((R, m)_{s, \phi^r; [i_0]}^{\text{perf}}\) with respect to the above filtration \((\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}})^{p^s}_{i_0} \times \mathbb{N}\).

(b) Consider the following subset

\[
\text{def}(R, m)_{s, \phi^r; [i_0]}^{\text{perf}, b} := \sum_{n \geq 0} \phi^{nr} (m^{p ns - n_0})
\]

of the perfect domain \(R^\text{perf}\). Here \(m^{p ns - n_0} = R\) if \(ns - n_0 \leq 0\). It is easy to see that this subset is a subring of \(R^\text{perf}\). Define a decreasing filtration \((\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}})^{p^s}_{i_0} \times \mathbb{N}\) on \((R, m)_{s, \phi^r; [i_0]}^{\text{perf}, b}\) by

\[
\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}} := \left\{ x \in (R, m)_{s, \phi^r; [i_0]}^{\text{perf}, b} \mid \exists n \in \mathbb{N} \text{ s.t. } x^{p^i} \in m^{p^i + j} \right\},
\]

Define

\[
(R, m)_{s, \phi^r; [i_0]}^{\text{perf}, b}
\]

to be the completion of \((R, m)_{s, \phi^r; [i_0]}^{\text{perf}, b}\) with respect to the above filtration \((\text{Fil}_{s, \phi^r; [i_0]}^{\text{#}})^{p^s}_{i_0} \times \mathbb{N}\).

(3.6.4) Definition. Let \((R, \mathfrak{m})\) be a complete Noetherian local domain of equi-characteristic \(p > 0\) with perfect residue field \(\kappa\). Let \(R^\text{perf}\) be the perfection of \(R\), and let \(\phi\) be the Frobenius automorphism on \(R\). Let \(A, b, d\) be real numbers, with \(A > 0\) and \(d \geq 0\).

(i) Define a decreasing filtrations \((\text{Fil}_{R^\text{perf}, \text{deg}}^{\text{#}})^{u}_{i_0} \times \mathbb{R}_0\) on \(R^\text{perf}\) indexed by real numbers \(u\) by

\[
\text{Fil}_{R^\text{perf}, \text{deg}}^{u} := \left\{ x \in R^\text{perf} \mid \exists j \in \mathbb{N} \text{ s.t. } x^{p^i} \in m^{[u, p^j]} \right\}
\]

if \(u \geq 0\)

and

\[
\text{Fil}_{R^\text{perf}, \text{deg}}^{u} := R^\text{perf}
\]

if \(u \leq 0\)

It is easy to see that \(\text{Fil}_{R^\text{perf}, \text{deg}}^{u}\) is an ideal of \(R^\text{perf}\) for every \(u \in \mathbb{R}\).
(ii) Define a subset \((R, m)_{A,b,d}^{\text{perf}, b}\) of \(R^{\text{perf}}\) by
\[
(R, m)_{A,b,d}^{\text{perf}, b} := \sum_{n \in \mathbb{N}} (\phi^{-n} R \cap \text{Fil}^{b} R_{\text{perf}, \text{deg}}^{n-d})
\]

It is not difficult to see that \((R, m)_{A,b,d}^{\text{perf}, b}\) is a subring of \(R^{\text{perf}}\).

(iii) Define
\[(R, m)_{A,b,d}^{\text{perf}, b}\]
to be the completion of \((R, m)_{A,b,d}^{\text{perf}, b}\) with respect to the filtration induced by the filtration \((\text{Fil}_{A,b,d}^{\text{deg}})\) of \(R^{\text{perf}}\):
\[
(R, m)_{A,b,d}^{\text{perf}, b} = \lim_{u \to \infty} \left( (R, m)_{A,b,d}^{\text{perf}, b} \right)_{\text{fin}} / \left( \text{Fil}_{A,b,d}^{\text{deg}} \cap (R, m)_{A,b,d}^{\text{perf}, b} \right)_{\text{fin}}.
\]

\section{4. Complete restricted perfections, II}

\subsection{4.1 How various complete restricted perfections compare}

In 3.6 we defined three families of rings. Each ring in these families consists of formal series of the form \(\sum_{i \in \mathbb{N}[1/p]} b_{l} t_{i}^{l}\), where \(b_{l} \in \mathbb{R} \forall I\), subject to uniform constraint (depending on parameters) on the support of such series. The three families are:

1. \(\kappa \langle \langle t_{1}^{p^{-\infty}}, \ldots, t_{m}^{p^{-\infty}} \rangle \rangle_{x, \phi^{r} \geq n_{0}}^{s, \phi^{r} \geq n_{0}}\) and \(\kappa \langle \langle t_{1}^{p^{-\infty}}, \ldots, t_{m}^{p^{-\infty}} \rangle \rangle_{x, \phi^{r} \geq n_{0}}^{s, \phi^{r} \geq n_{0}}\)

2. \(\kappa \langle \langle t_{1}^{p^{-\infty}}, \ldots, t_{m}^{p^{-\infty}} \rangle \rangle_{x, \phi^{r} \geq [i_{0}]}^{s, \phi^{r} \geq [i_{0}]}\) and \(\kappa \langle \langle t_{1}^{p^{-\infty}}, \ldots, t_{m}^{p^{-\infty}} \rangle \rangle_{x, \phi^{r} \geq [i_{0}]}^{s, \phi^{r} \geq [i_{0}]}\)

3. \(\kappa \langle \langle t_{1}^{p^{-\infty}}, \ldots, t_{m}^{p^{-\infty}} \rangle \rangle_{E, \#}^{C, d}\) and \(\kappa \langle \langle t_{1}^{p^{-\infty}}, \ldots, t_{m}^{p^{-\infty}} \rangle \rangle_{E, \#}^{C, d}\)

We have defined two additional family of rings, attached to any given equi-characteristic-\(p\) complete Noetherian local domain \((R, m)\):

4. \((R, m)_{x, \phi^{r} \geq [i_{0}]}^{\text{perf}, \#}\) and \((R, m)_{x, \phi^{r} \geq [i_{0}]}^{\text{perf}, b}\)

5. \((R, m)_{A,b,d}^{\text{perf}, b}\)

\subsection{4.1.1 Remark.} (i) The family (1) above was motivated by the compatible sequence of morphisms \((\rho_{n})_{n \geq n_{0}}\) defined in 3.3.2, based on the sequence of morphisms \((\eta_{n})_{n \in \mathbb{N}}\) constructed in 2.7. There are two versions for family of rings. The \#-version is directly tied with compatible families \((\eta_{n})_{n \geq n_{0}}\)'s. The primary parameters are the positive integers \(r < s\). With \(r, s\) fixed, the ring increases as the second parameter \(n_{0}\) increases. The \(b\)-version results from the \# version when one replaces congruences modulo \((t_{1}^{p^{n}}, \ldots, t_{m}^{p^{n}})\) by the coarser congruences modulo \((t_{1}, \ldots, t_{m})^{p^{n}}\).
(ii) The family (2) is a slight variant of the family (1). With the primary parameters \( r < s \) fixed, the rings in the family (2) are closely related to the rings in the family (1); the rings in family (2) increases as the parameter \( i_0 \) increases. In some sense as \( i_0 \) increases, the rings in family (2) increases somewhat faster than the rings in family (1), to the extent that the rings in families (1) and (2) with the same primary parameters \( r, s \) are not co-final with each other as their respective secondary parameters \( n_0 \) and \( i_0 \) vary.

The family (2) is somewhat more convenient than the family (1). Generalization to complete Noetherian local domains 3.6.4 is straight forward. When the primary parameter \( r, s \) are fixed while the secondary parameter \( i_0 \) varies, the \( \# \)-version interlaces with the \( \ast \)-version; see 4.1.2 (1) below.

(iii) In the family (3) the parameters \( E, C > 0 \) and \( d \geq 0 \) are real numbers. The most significant parameter is the “exponent” \( E \); it is written as a superscript in the notation, to indicate that it serves as an exponent in the estimate of \( p \)-adic absolute value in terms of archimedean absolute value for elements in the support of formal series in family (3).

The “multiplicative constant” \( C \) is secondary, while the parameter \( d \) is of least importance among the three. When \( E \) is fixed while \( C \) and \( d \) vary, the \( \# \)-version and the \( \ast \)-version are interlaced; see 4.1.3 (1). Rings in family (2) with primary parameters \( s > r > 0 \) are closely related to rings in family (3) with \( E = \frac{r}{s} \); see 4.1.2 (3) and 4.1.3 (2).

(iv) Clearly the family (2) is a special case of the family (4). This is reflected in the notation for (2) and (4).

(v) The family (5) with real parameters \( A > 0, b > 0, d \geq 0 \) generalizes the family (3). When \( (R, m) = (\kappa[[t_1, \ldots, t_m]], (t_1, \ldots, t_m)) \), the parameters \((A_1, b_1, d_1)\) corresponding to given parameters \((E, C, d)\) are:

\[
A_1 = \frac{1}{E}, \quad b_1 = C^{1/E}, \quad d_1 = d.
\]

When the parameters are related as above, the rings \( \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C; c}^{b} \) and \( (R, m)_{A_1, b_1; d_1}^{\text{perf}} \) are quite close.

(4.1.2) \textbf{Lemma.} Let \( s > r > 0 \) be positive integers, and let \( i_0 \geq 0 \) be a natural number. Let \( \kappa \supset \mathbb{F}_p \) be a perfect field, and let \( t_1, \ldots, t_m \) be variables.

(1) Let \( i_0 \geq 0 \) be a natural number. We have inclusions

\[
\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; [i_0]}^{\#} \subset \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; [i_0]}^{b}
\]

and

\[
\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; [i_0]}^{\#} \subset \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; [i_0]+[\log_p m]}^{\#}
\]

as sets of formal series.

(2) Let \( n_0 \) be a natural number. If \( i_1 \) is a natural number such that \( i_1 \geq \max(s - r, s \cdot \lceil \frac{n_0}{r} \rceil) \), then

\[
\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; \geq n_0}^{\#} \subset \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; [i_1]}^{\#}
\]

and

\[
\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; \geq n_0}^{\#} \subset \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{s; \phi_1^i; [i_1]}^{b}
\].
(3) Let \( i_0 \) be a natural number. We have

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{x,\phi^r[i_0]} \rangle^\# \subset \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{p^{s/r}(s-r);0} \rangle^{r/(s-r),#}
\]

and

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{x,\phi^r[i_0]} \rangle^b \subset \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{p^{s/r}(s-r);0} \rangle^{r/(s-r),b}
\]

PROOF. The first inclusion in (1) is obvious. The second inclusion in (1) holds because

\[
(t_1, \ldots, t_m)^{p^{j+\lceil \log_p m \rceil}} \subset (t_1^{p^j}, \ldots, t_m^{p^j})
\]

for all \( j \in \mathbb{N} \). The statements (2), (3) are easy exercises. \( \square \)

**(4.1.3) Lemma.** Let \( \kappa \supset \mathbb{F}_p \) be a perfect field. Let \( E > 0, C > 0 \) be positive real numbers. Let \( d \geq 0 \) be a non-negative real number as in 3.6.1.

(1) We have natural inclusions

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C; d} \rangle^E, # \subset \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C; d} \rangle^E, b
\]

and

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C; d} \rangle^E, b \subset \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C; d} \rangle^E, #
\]

(2) Let \( r < s \) be positive integers such that

\[
E < \frac{r}{s-r}.
\]

Suppose that \( i_2 \) a sufficiently natural number such that

\[
p^{[m/r] - (s-r)/2} \leq C^{-1/E} \cdot p^m/E - d
\]

for every integer \( m \geq \frac{s-r}{r} i_2 \). Note that such an integer \( i_2 \) exists because \( \frac{s-r}{r} < \frac{1}{E} \). Then

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C; d} \rangle^E, # \subset \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{x,\phi^r[i_2]} \rangle^\#
\]

and

\[
\kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C; d} \rangle^E, b \subset \kappa\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{x,\phi^r[i_2]} \rangle^b
\]

**(4.1.4) Lemma.** Let \((R, \mathfrak{m})\) be an equi-characteristic \( p > 0 \) complete Noetherian local ring. Let \( s > r > 0 \) be positive integers. Let \( i_0 \) be a natural number. We have

\[
(R, \mathfrak{m})^{\text{perf},#}_{x,\phi^r[i_0]} \subset (R, \mathfrak{m})^{\text{perf},b}_{x,\phi^r[i_0]}.
\]

Moreover if the maximal ideal \( \mathfrak{m} \) can be generated by \( n \) elements, then

\[
(R, \mathfrak{m})^{\text{perf},b}_{x,\phi^r[i_0]} \subset (R, \mathfrak{m})^{\text{perf},#}_{x,\phi^r[i_0]+[\log_p n]}.
\]
(4.2) A local homomorphism \( h \) between two equi-characteristic-\( p \) complete Noetherian local domains induces ring homomorphisms between their complete restricted perfections. We show that injective local homomorphisms induce injections on complete restricted perfections.

(4.2.1) Lemma. Let \( (R_1, m_1), (R_2, m_2) \) equi-characteristic-\( p \) complete Noetherian local domains with perfect residue fields \( \kappa_1 \) and \( \kappa_2 \). Let \( h : R_1 \to R_2 \) be a ring homomorphism such that \( h(m_1) \subseteq m_2 \).

(a) Let \( A, b, d \) be real numbers, \( A, b > 0 \), \( d \geq 0 \). Let \( t_1 : R_1 \to (R_1, m_1)^{\text{perf}, b}_{A, b, d} \) be the natural ring homomorphism from \( R_1 \) to its complete restricted completion \( (R_1, m_1)^{\text{perf}, b}_{A, b, d} \). Similarly we have a natural ring homomorphism \( t_2 : R_2 \to (R_2, m_1)^{\text{perf}, b}_{A, b, d} \). The ring homomorphism \( h \) induces a homomorphism from

\[ \tilde{h} : (R_1, m_1)^{\text{perf}, b}_{A, b, d} \to (R_2, m_2)^{\text{perf}, b}_{A, b, d} \]

such that \( \tilde{h} \circ t_1 = t_2 \circ h \).

(b) Let \( r, s, i_0 \in \mathbb{N}, r, s > 0, i_0 \geq 0 \). Let \( t_1 : R_1 \to (R_1, m_1)^{\text{perf}, b}_{r, s, i_0} \) be the natural ring homomorphism from \( R_1 \) to its complete restricted completion \( (R_1, m_1)^{\text{perf}, b}_{r, s, i_0} \). Similarly we have a ring homomorphism \( t_2 : R_2 \to (R_2, m_1)^{\text{perf}, b}_{r, s, i_0} \). The ring homomorphism \( h \) induces a homomorphism from

\[ h^\#: (R_1, m_1)^{\text{perf}, b}_{r, s, i_0} \to (R_2, m_2)^{\text{perf}, b}_{r, s, i_0} \]

such that \( h^\# \circ t_1 = t_2 \circ h \). Similarly \( h \) extends naturally to a ring homomorphism

\[ h^b : (R_1, m_1)^{\text{perf}, b}_{r, s, i_0} \to (R_1, m_1)^{\text{perf}, b}_{r, s, i_0} \].

The proof is easy, therefore omitted. \( \square \)

(4.2.2) Proposition. Let \( (R, m) \) be a Noetherian local domain. Assume that the integral closure \( S \) of \( R \) in the field of fraction of \( R \) is a finite \( R \)-module. There exists a natural number \( n_0 \) such that

\[ \{ x \in R \mid x^a \in m^n \} \subseteq m^{\frac{n}{a} - n_0} \quad \forall a \in \mathbb{N}_{> 0}, \forall n \geq a \cdot n_0. \]

Proof. Let \( \text{Bl}_m(R) = \text{Spec}(\bigoplus_{j \in \mathbb{N}} m^j) \) be the blow-up of \( \text{Spec}(R/m) \subseteq \text{Spec}(R) \), and let \( Y \) be the normalization of \( \text{Bl}_m(R) \). The Noetherian normal domain \( S \) is semi-local; let \( \bar{m}_1, \ldots, \bar{m}_s \) be the maximal ideals of \( S \). The natural morphism \( \pi : Y \to \text{Spec}(R) \) factors through a unique morphism \( f : Y \to \text{Spec}(S) \): \( \pi = g \circ f \), where \( g : \text{Spec}(S) \to \text{Spec}(R) \) corresponds to the inclusion \( R \hookrightarrow S \). We know that \( \Gamma(Y, \mathcal{O}_Y) = S \) because \( S \) is normal.

Let \( \mathcal{L} = \pi^*m = m \cdot \mathcal{O}_Y \) be the pull-back to \( Y \) of the maximal ideal \( m \subseteq R \); it is an invertible sheaf of \( \mathcal{O}_Y \)-ideals on \( Y \) and is an ample invertible \( \mathcal{O}_Y \)-module. The closed subset \( \text{Spec}_y(\mathcal{O}_Y/m \mathcal{O}_Y) \) of \( Y \) is the union of irreducible Weil divisors \( E_1, \ldots, E_r \), where \( r \) is a positive integer. There exist positive integers \( e_1, \ldots, e_r \in \mathbb{N}_{> 0} \) such that

\[ \mathcal{L} = \mathcal{O}_Y\left(-(e_1E_1 + \cdots + e_rE_r)\right). \]
Define for each \( n \in \mathbb{N} \) an ideal \( J_n \subset S \) by

\[
J_n := \Gamma(Y, \mathcal{L}^n) \subseteq \Gamma(Y, \mathcal{O}_Y) = S.
\]

It is clear that \( J_1 \subset \tilde{m}_1 \cap \cdots \cap \tilde{m}_s \), and \( \mathfrak{m}^n S \subseteq J_n \) for all \( n \in \mathbb{N} \).

**Claims.**

1. There exist a positive natural number \( n_1 \in \mathbb{N} \) such that \( J_{n+1} = \mathfrak{m} J_n \) for all integers \( n \geq n_1 \). In particular \( J_n \subseteq \mathfrak{m}^{n-n_1} S \) for all \( n \geq n_1 \).

2. There exists a natural number \( n_2 \in \mathbb{N} \) such that \( R \cap (\mathfrak{m}^{n+n_2} S) \subset \mathfrak{m}^n \) for all \( n \in \mathbb{N} \).

3. We have \( J_{n+n_1+n_2} \cap R \subseteq \mathfrak{m}^n \) for all \( n \in \mathbb{N} \), with the constants \( n_1, n_2 \) in claims 1 and 2 respectively.

4. If \( y \in S, a \in \mathbb{N}_{>0}, n \in \mathbb{N} \) and \( y^a \in J_n \), then \( y \in J_{[n/a]} \).

5. If \( x \in R, a \in \mathbb{N}_{>0}, n \in \mathbb{N} \), and \( x^a \in \mathfrak{m}^n \), then \( x \in \mathfrak{m}^{[n/a]-n_1-n_2} \) for all \( n \geq a(n_1+n_2) \).

Obviously proposition 4.2.2 follows from claim 5, with \( n_0 = n_1+n_2 \). \( J_1 \subset \tilde{m}_1 \cap \cdots \cap \tilde{m}_s \) and \( S \) is Noetherian.

Claim 1 is a consequence of the fact \( \mathcal{L} = \mathfrak{m} \mathcal{O}_Y \) and the general finiteness property for proper morphism [EGA III, §5, Cor. 3.3.2] applied to the proper morphism \( Y \to \text{Spec}(R) \): we see that the graded \( \bigoplus_{i \geq 0} \mathfrak{m}^i \)-module

\[
\bigoplus_{i \geq 0} \Gamma(Y, \mathfrak{m}^i \mathcal{O}_Y) = \bigoplus_{i \geq 0} J_i
\]

is a finitely generated as a graded module, and claim 1 follows.

Claim 2 is the Artin–Rees lemma applied to the finite \( R \)-module \( S \). Claim 3 is a formal consequence of claims 1 and 2, while claim 5 is a formal consequence of claims 3 and 4.

It remains to prove claim 4. Given an element \( y \in S \) such that \( y^a \in J_n \). For each \( i = 1, \ldots, s \), let \( S_i \) be the localization of \( S \) at the generic point of the exceptional divisor \( E_i \). Each \( E_i \) is a discrete valuation ring; let \( \text{ord}_{E_i}(\cdot) \) be associated normalized valuation with value group \( \mathbb{Z} \). The assumption that \( y^a \in J_n \) implies that \( \text{ord}_{E_i}(y^a) \geq n \cdot e_i \) for all \( i \), therefore

\[
\text{ord}_{E_i}(y) \geq \frac{n \cdot e_i}{a} \geq \left\lfloor \frac{n}{a} \right\rfloor e_i
\]

for \( i = 1, \ldots, s \). Therefore there exists an open subset \( U \subset Y \) which contains \( Y \setminus (E_1 \cup \cdots \cup E_s) \) and also the generic point of \( E_i \) for \( i = 1, \ldots, s \), such that \( y \) defines a section of \( \mathcal{L}^{[n/a]} \) over \( U \). Because \( Y \) is normal and the codimension of \( U \) in \( Y \) is at least 2, \( y \) extends uniquely to a section of \( \mathcal{L}^{[n/a]} \) over \( Y \). We have proved claim 4 and proposition 4.2.2. \( \square \)

(4.2.3) **Corollary.** Let \( (R, \mathfrak{m}) \) be a complete Noetherian local domain of equi-characteristic \( p > 0 \), with perfect residue field \( \kappa \).
(i) Let $A, b > 0$, $d \geq 0$ be real numbers. The linear topology on the ring \((R, m)_{A, b; d}^{\text{perf}, b}\) defined by the filtration on \((R, m)_{A, b; d}^{\text{perf}, b}\) induced by the filtration \((\text{Fil}^u_{R_{\text{perf}, d}})\) of $R_{\text{perf}}$ is separated. Therefore the natural ring homomorphism
\[
(R, m)_{A, b; d}^{\text{perf}, b} \rightarrow (R, m)_{A, b; d}^{\text{perf}, b}
\]
from \((R, m)_{A, b; d}^{\text{perf}, b}\) to its completion \((R, m)_{A, b; d}^{\text{perf}, b}\) is an injection.

(ii) Let $r, s, n_0$ be natural numbers, $0 < r < s$. The natural ring homomorphism
\[
(R, m)_{s, \phi^r; [l_0]}^{\text{perf}, #} \rightarrow (R, m)_{s, \phi^r; [l_0]}^{\text{perf}, #}
\]
and
\[
(R, m)_{s, \phi^r; [l_0]}^{\text{perf}, b} \rightarrow (R, m)_{s, \phi^r; [l_0]}^{\text{perf}, b}
\]
are injections.

PROOF. The statements (i) and (ii) are easy consequences of 4.2.2. We note that the statements (i) and (ii) are in fact equivalent. □

(4.2.4) COROLLARY. Notation as in 4.2.1. In particular $h : (R_1, m_1) \rightarrow (R_2, m_2)$ is a ring homomorphism between equi-characteristic-$p$ complete Noetherian local domains. Suppose that $h$ is an injection. Then the induced homomorphisms $h^#$ and $h^b$ in 4.2.1 are also injections.

PROOF. This statement is a corollary of 4.2.3. We explain the proof for $h^#$. The same argument in general topology also proves the statement for $h^b$.

The injective ring homomorphism $h : R_1 \rightarrow R_2$ induces a injective ring homomorphism
\[
h' : (R_1, m_1)_{A, b; d}^{\text{perf}, b} \rightarrow (R_2, m_2)_{A, b; d}^{\text{perf}, b}
\]
According to 4.2.3, we can identify \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\) as a subring of \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\). The injection $h'$ identifies \((R_1, m_1)_{A, b; d}^{\text{perf}, b}\) also as a subring of \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\). (It is actually contained in \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\)). Let \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\) be the closure of \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\) in the topological ring \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\).

The topology on \((R_1, m_1)_{A, b; d}^{\text{perf}, b}\) induced by the filtration \((\text{Fil}^u_{R_{\text{A, b; d}}^{\text{perf}, b}})\) is stronger than the topology \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\). The closure of \((R_1, m_1)_{A, b; d}^{\text{perf}, b}\) with respect to this stronger topology is naturally identified with a subset of \((R_2, m_2)_{A, b; d}^{\text{perf}, b}\). We have shown that $h'$ is an injection. □

(4.3) Let $\kappa$ be a perfect field. Denote by $\sigma$ the Frobenius automorphism on $\kappa$, which sends every element $x \in \kappa$ to $x^p$. Let $u_1, \ldots, u_a$ and $t_1, \ldots, t_m$ be variables. Let $\kappa[u_1^p, \ldots, u_a^p]$ be the perfection of the polynomial ring $\kappa[u_1, \ldots, u_m]$. Elements of $\kappa[u_1^p, \ldots, u_a^p]$ are finite sums of the form
\[
\sum_{J \in \mathbb{N}[1/p]^a} b_J u_J^p,
\]
where \( b_J \in \kappa \) for all \( J \in \mathbb{N}[1/p]^a \), and all \( b_J = 0 \) for all \( J \) outside of a finite subset of \( \mathbb{N}[1/p]^a \).

We observe that for each element \( i \in \mathbb{N}[1/p] \), the \( i \)-th power of an element

\[
\sum_{J \in \mathbb{N}[1/p]^a} b_J u_J^i \in \kappa[u_1^{p^{-\infty}}, \ldots, u_a^{p^{-\infty}}]
\]

is well-defined: write \( i = \frac{r}{p^s} \) with \( r \in \mathbb{Z} \) and \( s \in \mathbb{N} \), and define

\[
\left( \sum_{J \in \mathbb{N}[1/p]^a} b_J u_J^i \right)^{r/p^s} := \left( \sum_{J \in \mathbb{N}[1/p]^a} b_J^{p^{-s} r} \cdot u_J^{p^{-s} i} \right)^{r/p^s}.
\]

Therefore if \( f \in \kappa[u_1^{p^{-\infty}}, \ldots, u_a^{p^{-\infty}}] \) and \( g_1, \ldots, g_a \in \kappa[t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}] \), the composition \( f(g_1, \ldots, g_a) \) is a well-defined element of \( \kappa[t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}] \). It is not difficult to show that the operation of composition extends to complete restricted perfection of power series rings.

(4.3.1) **Lemma.** Let \( \kappa \supset \mathbb{F}_p \) be a perfect field. Let \( u_1, \ldots, u_a \) and \( t_1, \ldots, t_m \) be variables. Suppose that \( f \in \kappa\langle\langle u_1^{p^{-\infty}}, \ldots, u_a^{p^{-\infty}}\rangle\rangle_{C_1:d_1}^{e_1:b} \) and \( g_i \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C_2:d_2}^{E_2:b} \) for \( i = 1, \ldots, a \). Assume for simplicity that \( C_1, C_2, d_1, d_2 \geq 1 \). There exists a positive real number \( d_3 \) such that

\[
f(g_1(t), \ldots, g_a(t)) \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C_3:d_3}^{E_3:b}
\]

where

- \( E_3 = E_1 \cdot E_2 + E_1 + E_2 \),
- \( C_3 = C_2 \cdot C_1^{1+E_2} \cdot (\frac{1}{e_2})^{E_1(1+E_2)} \), and
- \( e_2 := \min \{ |J|_\sigma : J \neq 0 \text{ and } t^J \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C_2:d_2}^{E_2:b} \} \).

A trivial lower bound for \( e_2 \) is

\[
e_2 \geq C_2^{-1}(1 + d_2)^{-E_2}.
\]

**Proof.** Let \( S_2 \subset \mathbb{N}[1/p]^m \) be the set of supports of all formal series in \( \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C_2:d_2}^{E_2:b} \) whose constant terms are 0. Similarly let \( S_1 \subset \mathbb{N}[1/p]^a \) be the set of supports of all formal series in \( \kappa\langle\langle u_1^{p^{-\infty}}, \ldots, u_a^{p^{-\infty}}\rangle\rangle_{C_1:d_1}^{E_1:b} \) whose constant terms are 0. By definition \( e_2 = \min \{ |J|_\sigma : J \in S_2 \} \). Every non-zero element \( K \) in the support of \( f(g_1(t), \ldots, g_a(t)) \) can be written in the following form

\[
K = p^{-r}(J_{1,1} + \cdots + J_{1,i_1} + \cdots + J_{i_1,1} + \cdots + J_{a,i_a}),
\]

where

- \( (i_1, \ldots, i_a) \in \mathbb{N}^a \), \( r = \max(-\text{ord}_p(i_1), \ldots, -\text{ord}_p(i_1), 0) \),
- \( I := p^{-r}(i_1, \ldots, i_a) \in S_1 \), and
- \( J_{v,\mu} \in S_2 \) for all \( v = 1, \ldots, a \) and all \( \mu = 1, \ldots, i_a \).
Clearly the following inequalities hold.

\[(4.3.1.1) \quad |K|_\sigma \geq e_2 \cdot p^{-r} (i_1 e + \cdots + i_s e) = e_2 \cdot |I|_\sigma \]

\[(4.3.1.2) \quad M_\sigma := \text{Max} \{ |J_{\nu,\mu}|_\sigma : 1 \leq \mu \leq i_\nu, 1 \leq \nu \leq a \} \leq p^r \cdot |K|_\sigma \]

\[(4.3.1.3) \quad p^{-r} \cdot |K|_p \leq \text{Max} \{ |J_{\nu,\mu}|_p : 1 \leq \mu \leq i_\nu, 1 \leq \nu \leq a \} =: M_p \]

From the definitions of the rings \(\kappa \langle \langle t_1^{p^\infty}, \ldots, t_m^{p^\infty} \rangle \rangle_{\mathbb{E}^{2};d_2}^{b} \) and \(\kappa \langle \langle u_1^{p^\infty}, \ldots, u_a^{p^\infty} \rangle \rangle_{\mathbb{E}^{1};d_1}^{b} \) we know that

\[(4.3.1.4) \quad p^r \leq C_1 (|I|_\sigma + d_1)^{E_1} \leq C_1 \cdot (\frac{1}{e_2} |K|_\sigma + d_1)^{E_1} \]

\[(4.3.1.5) \quad M_p \leq C_2 (M_\sigma + d_2)^{E_2} \]

Combining the above inequalities, we see that

\[|K|_p \leq p^r \cdot C_2 \cdot (p^r \cdot |K|_\sigma + d_2)^{E_2} \leq C_1 (e_2^{-1} \cdot |K|_\sigma + d_1)^{E_1} \cdot C_2 \cdot (e_2^{-1} \cdot |K|_\sigma + d_1)^{E_1} \cdot (|K|_\sigma + d_2)^{E_2} \]

The last term in the above displayed inequality is a polynomial in \(|K|_\sigma\) of degree

\[E_3 := E_1 + E_2 + E_1 \cdot E_2 \]

whose leading term is

\[C_3 := C_1^{1+E_2} \cdot C_2 \cdot (\frac{1}{e_2})^{E_1(1+E_2)}. \]

Hence for a sufficiently large constant \(d_3\) it is bounded above by \(C_3 (|K|_\sigma + d_3)^{E_3}\) for all \(|K|_\sigma \geq 0\).

We have proved the main assertion of lemma 4.3.1.

To see the trivial lower bound for \(e_2\), we only have to observe that if \(J \in S_2\) and \(|J|_\sigma \leq 1\) and \(J \neq \mathbb{N}^m\), then

\[|J|_\sigma \geq |J|_p^{-1} \geq (C_2 (1+d_2)^{E_2})^{-1}. \]

\[\square \]

**Remark.** Composition can be formulated for complete restricted perfections of general equi-characteristic-\(p\) complete Noetherian local rings.

(4.4) Let \(\kappa \supset \mathbb{F}_p\) be a perfect field. We will generalize the Weierstrass preparation theorem to complete restricted perfections \(\kappa \langle \langle t_1^{p^\infty}, \ldots, t_m^{p^\infty} \rangle \rangle_{\mathbb{E}^{2};d_2}^{b}\) of power series rings.

(4.4.1) **Definition.** Let \(\kappa \supset \mathbb{F}_p\) be a perfect field of characteristic \(p > 0\).

(i) Let \(\kappa \langle \langle t_1^{p^\infty}, \ldots, t_m^{p^\infty} \rangle \rangle_{\mathbb{E}^{2};d_2}^{b}\) be the set of all formal series of the form

\[\sum_{i_1, \ldots, i_m \in \mathbb{N}[1/p]} b_{i_1 \ldots i_m} t_1^{i_1} \cdots t_m^{i_m} \]

where \(b_{i_1 \ldots i_m} \in \kappa\) for all \((i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m\). Note that \(\kappa \langle \langle t_1^{p^\infty}, \ldots, t_m^{p^\infty} \rangle \rangle_{\mathbb{E}^{2};d_2}^{b}\) has a natural structure as a module over the perfection \(\kappa \langle t_1^{p^\infty}, \ldots, t_m^{p^\infty} \rangle_{\mathbb{E}^{1};d_1}^{b}\) of the polynomial ring \(\kappa [t_1, \ldots, t_m]\).
(ii) Let \( e \in \mathbb{Z}[1/p]_{>0} \) be a positive rational number whose denominator is a power of \( p \). An non-zero element \( F(t_1, \ldots, t_m) \) in \( \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle \) is regular of order \( e \) in the variable \( t_m \) if the formal series \( F(0, \ldots, 0, t_m) \) in one variable \( t_m \) has order \( e \). In other words when \( F(t_1, \ldots, t_m) \) is expanded in powers of \( t_m \) with coefficients in formal series of \( t_1, \ldots, t_{m-1} \),

\[
F(t_1, \ldots, t_m) = \sum_{j \in \mathbb{N}[1/p]} F_j(t_1, \ldots, t_{m-1}) t_m^j
\]

we have

\[
F_j(0, \ldots, 0) = 0 \quad \forall \, j < e, \quad \text{and} \quad F_e(0, \ldots, 0) \in \kappa^\times.
\]

(4.4.2) **Proposition.** Let \( F(t_1, \ldots, t_m) \) be a non-zero element of \( \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle_{E, b} \) which is regular of order \( e > 0 \) in the variable \( t_m \).

1. There exist constants \( C', d' > 0 \) depending only on the parameters \( C, d, E, m \) such that for every element \( G \in \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle_{E, b} \), there exists elements \( U, R \in \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle_{E, b} \) such that

\[
G = U \cdot F + R
\]

and for every element \( I = (i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m \in \supp(R) \) in the support of \( R \), the inequalities

\[
i_m < e, \quad i_1 + \cdots + i_{m-1} > 0
\]

hold. Moreover the quotient \( U \) and the remainder \( R \) are uniquely determined by \( G \) and \( F \). The constants \( C' \) and \( d' \) can be taken to be

\[
C' = C \cdot (1 + \varepsilon_0^{-1})^E, \quad d' = \frac{d + e}{1 + \varepsilon_0^e},
\]

where \( \varepsilon_0 \) is defined in 4.4.6.

2. Suppose that \( e = \operatorname{Min}\{ |I|_\sigma : I \in \supp(F) \} \). Then \( U, R \in \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle_{E, b} \).

(4.4.3) The uniqueness part 4.4.2 (1) is easy: suppose that

\[
G = U' \cdot F + R'
\]

with \( U', R' \in \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle_{E, b} \) and \( R' \) satisfies the same condition as \( R \). Then \( (U' - U) \cdot F = R - R' \). Examine the degree in \( t_m \) of monomials appearing on both sides, we see that \( R' - R = 0 \). Therefore \( (U' - U) \cdot F = 0 \). Hence \( U' - U = 0 \) because \( \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle_{E, b} \) is an integral domain.

Our proof of the existence part 4.4.2 is a generalization of the constructive proof of the Weierstrass preparation theorem in [7, p. 139]. The actual proof is in 4.4.5–4.4.8 below; the crucial estimates are in lemma 4.4.7. We will review the argument in [7, p. 139] after recalling the definition of the linear operators used in [7, p. 139].

(4.4.4) **Definition.** Let \( \kappa \supset \mathbb{F}_p \) be a perfect field of characteristic \( p \). Let \( t_1, \ldots, t_m \) be variables. Let \( e > 0 \) be a positive rational number in \( \mathbb{N}[1/p] \). Let \( F \in \kappa\langle t_1^{p^{-m}}, \ldots, t_m^{p^{-m}} \rangle \) be a formal series which is regular of order \( e \) in the variable \( t_m \).
(i) Define \( \kappa \)-linear operators

\[
\eta, \rho : \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle) \rightarrow \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)
\]
depending on \( e \), by

\[
f = t_m^e \cdot \eta(f) + \rho(f)
\]

for every element \( f \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle) \). Clearly every monomial \( t_1^{i_1} \cdots t_m^{i_m} \) with exponent \( (i_1, \ldots, i_m) \in \mathbb{N}[1/p]^m \) and \( \eta(t_1^{i_1} \cdots t_m^{i_m}) \) are given by

\[
\eta(t_1^{i_1} \cdots t_m^{i_m}) = \begin{cases} 
    t_1^{i_1} \cdots t_{m-1}^{i_{m-1}} \cdot t_m^{i_m - e} & \text{if } i_m \geq e \\
    0 & \text{if } i_m < e
\end{cases}
\]

\[
\rho(t_1^{i_1} \cdots t_m^{i_m}) = \begin{cases} 
    0 & \text{if } i_m \geq e \\
    t_1^{i_1} \cdots t_m^{i_m} & \text{if } i_m < e
\end{cases}
\]

For a general element \( f = \sum_{i_1, \ldots, i_m \in \mathbb{N}[1/p]} b_{i_1, \ldots, i_m} t_1^{i_1} \cdots t_m^{i_m} \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle) \), we have

\[
\eta(f) = \sum_{i_1, \ldots, i_m \in \mathbb{N}[1/p]} b_{i_1, \ldots, i_m} \eta(t_1^{i_1} \cdots t_m^{i_m})
\]

\[
\rho(f) = \sum_{i_1, \ldots, i_m \in \mathbb{N}[1/p]} b_{i_1, \ldots, i_m} \rho(t_1^{i_1} \cdots t_m^{i_m})
\]

Note that if \( f \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d} \) for some parameter \( E, C > 0 \) and \( d \geq 0 \), then \( \rho(f) \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d} \) and \( \eta(f) \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d + e} \).

(ii) Suppose that formal series \( F \) is in \( \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d} \) for some constants \( E > 0, C > 0, d \geq 0 \). Define a \( \kappa \)-linear operator

\[
\mu : \bigcup_{C'' > 0} \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C''; d''} \rightarrow \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)
\]
depending on \( e \) and \( F \), by

\[
\mu(f) := \eta(-\eta(F)^{-1} \cdot \rho(F) \cdot f)
\]

for all \( C'', d'' > 0 \) and every element \( f \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C''; d''} \). Note that \( \eta(F) \) is a formal series whose constant term is in \( \kappa^\times \), therefore \( \eta(F)^{-1} \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d + e} \) because \( \eta(F) \in \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d + e} \). The product \( \eta(F)^{-1} \cdot \rho(F) \cdot f \) on the right hand side of the above displayed formula makes sense because both formal series \( \rho(F) \) is also an element of \( \kappa(\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle)^{E, b}_{C; d + e} \).
(4.4.5) The proof in [7, p. 139] is a fixed-point-theorem argument. Suppose that the equation
\[ V = \eta(G) + \mu(V) \]  
has a solution \( V \) in \( \kappa \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{E, b}^{C, d''} \), and the parameters \( C'', d'' \) are such that \( \eta(F) \) and \( \rho(G) \) are also elements of \( \kappa \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C, d''}^{E, b} \). Let
\[ U := \eta(F)^{-1} \cdot V. \]
Then we have
\[ U \cdot \eta(F) = V = \eta(G) - \eta(\eta(F)^{-1} \cdot \rho(F) \cdot V) = \eta(G) - \eta(U \cdot \rho(F)). \]
By the definition of the operators \( \eta \) and \( \rho \), we know that
\[ U \cdot F = t_m^e \cdot U \cdot \eta(F) + U \cdot \rho(F), \]
hence
\[ U \cdot \eta(F) = \eta(U \cdot F) - \eta(U \cdot \rho(F)). \]
From 4.4.5.3 and 4.4.5.4, we see that
\[ \eta(G) = \eta(U \cdot F), \]
hence
\[ G - \rho(G) = U \cdot F - \rho(U \cdot F). \]
In other words
\[ G = U \cdot F + R, \]
where
\[ R = \rho(G) - \rho(U \cdot F). \]
Note that \( U = \eta(F)^{-1} \cdot V \) and \( R \) are both in \( \kappa \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C, d''}^{E, b} \) because we have assumed that \( \eta(F) \) and \( \rho(G) \) are both in \( \kappa \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C, d''}^{E, b} \).

(4.4.6) \textbf{Definition.} Suppose we are given parameters \( e \in \mathbb{N}[1/p]_{>0} \), \( E, C \in \mathbb{R}_{>0} \), and \( d \in \mathbb{R}_{>0} \). Let \( T = T(m : e : E; C, d) \) be the subset of \( \mathbb{N}[1/p]^m \) consisting of all \( m \)-tuples \( (i_1, \ldots, i_m) \) in the support set \( \text{supp}(m : E; C, d + e) \) for \( \kappa \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C, d + e}^{E, b} \) such that \( (i_1, \ldots, i_{m-1}) \neq (0, \ldots, 0) \) \( \mathbb{N}[1/p]^{m-1} \) and \( i_m < e \). Define a positive constant \( \varepsilon_0 \) depending on parameters \( m, e, E; C, d \) by
\[ \varepsilon_0 = \varepsilon_0(m : e : E; C, d) := \min \left\{ \frac{i_1 + \ldots + i_{m-1}}{e} \mid (i_1, \ldots, i_m) \in T(m : e : E; C, d) \right\}. \]
Note that this minimum is attained at some element of \( T(m : e : E; C, d) \).
(4.4.7) Lemma. Let $N \in \mathbb{N}$ be a positive integer. Consider the $\kappa$-linear operator
\[ \mu^n : \mathcal{K}\langle \langle t_1^{p_{\infty}}, \ldots, t_m^{p_{\infty}} \rangle \rangle_{C,d+e}^b \rightarrow \mathcal{K}\langle \langle t_1^{p_{\infty}}, \ldots, t_m^{p_{\infty}} \rangle \rangle_{C,d+(N+1)e}^b, \]
defined in 4.4.4. Let $h$ be an element of $\mathcal{K}\langle \langle t_1^{p_{\infty}}, \ldots, t_m^{p_{\infty}} \rangle \rangle_{C,d+e}^b$. Let $J = (j_1, \ldots, j_{m-1}, j_m)$ be an exponent in the support $\text{supp}(\mu^N(h))$ of the formal series $\mu^N(h)$.

(i) $j_1 + \cdots + j_{m-1} \geq N \cdot \varepsilon_0 \cdot e$.

(ii) $|J|_p \leq C \cdot (1 + \varepsilon_0^{-1})^E \cdot \left( |J|_\sigma + \frac{d+e}{1+\varepsilon_0} \right)^E$.

(iii) Suppose that $e = \text{Min}\{ |I|_\sigma : I \in \text{supp}(F) \}$. Then
\[ |J|_p \leq C \cdot (|J|_\sigma + d + 2e)^E. \]

Proof. The statement (i) is obvious from the definition of the linear operator $\mu$ in 4.4.4. For statement (ii), we know that
\[ |J|_p \leq C \cdot (|J|_\sigma + d + (N+1)e)^E \]
because $\mu^N(\langle \langle t_1^{p_{\infty}}, \ldots, t_m^{p_{\infty}} \rangle \rangle_{C,d+e}^b) \leq \mathcal{K}\langle \langle t_1^{p_{\infty}}, \ldots, t_m^{p_{\infty}} \rangle \rangle_{C,d+(N+1)e}^b$. We know from (i) that
\[ Ne \leq \varepsilon_0^{-1} \cdot (j_1 + \cdots + j_{m-1}) \leq \varepsilon_0^{-1} \cdot |J|_\sigma, \]

hence
\[ |J|_p \leq C \cdot ((1 + \varepsilon_0^{-1}) \cdot |J|_\sigma + d + e)^E \leq C \cdot (1 + \varepsilon_0^{-1})^E \cdot \left( |J|_\sigma + \frac{d+e}{1+\varepsilon_0} \right)^E. \]

It remains to prove (iii). The assumption that $e = \text{Min}\{ |I|_\sigma : I \in \text{supp}(F) \}$ implies that $|I'|_\sigma \geq e$ for every exponent $I' \in \text{supp}(-\eta(F)^{-1} \cdot \rho(F))$. By the definition of the linear operator $\eta$, there exists exponents $I_0, \ldots, I_N$ with $I_0 \in \text{supp}(h)$ and $I_j \in \text{supp}(-\eta(F)^{-1} \cdot \rho(F))$ for $j = 1, \ldots, N$ such that
\[ J = I_0 + I_1 + \ldots + I_N - (0, \ldots, 0, Ne). \]

So we have
\[ |J|_p \leq \text{Max}\{ |I_0|_p, |I_1|_p, \ldots, |I_N|_p, |e|_p \} \]

Because $(0, \ldots, 0, e) \in \text{supp}(F)$, we have
\[ |e|_p \leq C \cdot (d + e)^E < C \cdot (|J|_\sigma + d + 2e)^E. \]

The assumption on $e$ tells us that $|I_j|_\sigma \geq e$ for $j = 1, \ldots, N$, hence
\[ |I_j|_\sigma \leq |J|_\sigma + e \quad \text{for } j = 0, 1, \ldots, N. \]

From $I_0, I_1, \ldots, I_j \in \text{supp}(\mathcal{K}\langle \langle t_1^{p_{\infty}}, \ldots, t_m^{p_{\infty}} \rangle \rangle_{C,d+e}^b)$, we get
\[ |J|_p \leq C \cdot (|J|_\sigma + d + e)^E \leq C \cdot (|J|_\sigma + d + 2e)^E \quad \text{for } j = 0, 1, \ldots, N. \]

It follows that
\[ |J|_p \leq C \cdot (|J|_\sigma + d + 2e)^E. \]

We have proved statement (iii). □
The uniqueness part of 4.4.2(1) has been settled. We will prove the existence part of 4.4.2(1) with
\[ C' = C \cdot (1 + \varepsilon_0^{-1})^E \quad \text{and} \quad d' = \frac{d+e}{1+\varepsilon_0}. \]

As explained in 4.4.5, it suffices to show that there exists an element \( V \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \) such that
\[ V = \eta(G) + \mu(V). \]

By lemma 4.4.7(i), (ii), the limit of
\[ \lim_{N \to \infty} \eta(G) + \mu(G) + \mu^2(G) + \cdots + \mu^N(G) =: V \]
exists in \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d+2e} \). Clearly the element \( V \) given by the above limit satisfies \( V = \eta(G) + \mu(V) \). We have proved 4.4.2(1).

Under the assumption that \( e = \text{Min}\{ |I|_\sigma : I \in \text{supp}(F) \} \), lemma 4.4.7(iii) tells us that
\[ \eta(G) + \mu(G) + \mu^2(G) + \cdots + \mu^N(G) \in \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d+2e} \]
for all \( N \in \mathbb{N} \), and the limit \( V \) exists in \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d+2e} \). We have proved 4.4.2(2).

**Proposition 4.5.** Suppose that \( E > 0, C \geq 1 \) and \( d \geq 1 \). The integral closure of the local domain \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \) in its own field of fractions is
\[ \bigcup_{d' > 0} \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d'}. \]

**Lemma 4.5.1.** Suppose that \( \kappa \supset \mathbb{F}_p \) is an infinite field of characteristic \( p \). Let \( E,C > 0 \) and \( d \geq 0 \) be real parameters. Let \( F(t_1, \ldots, t_m) \) be an element of \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \) whose constant term is 0; i.e. \( F \) is not a unit in \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \). Let \( e = \text{Min}\{ |I|_\sigma : I \in \text{supp}(F) \} \). There exists an element \( L \in \text{GL}_m(\kappa) \) such that the automorphism \( L^* \) of \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \) induced by \( L \) sends \( F \) to an \( t_m \)-regular element of order \( e \) in \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \).

**Proof.** The usual/standard argument works; see the first paragraph of the proof of Lemma 3 on p. 147 of [7].

**Lemma 4.5.2.** Let \( \kappa \supset \mathbb{F}_p \) be a field of characteristic \( p \), and let \( \bar{k} \) be an extension field of \( \kappa \). Let \( G,F \neq 0 \) be element of \( \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \), let \( H \) be an element of \( \bar{k}\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \).

Suppose that \( G = F \cdot H \). Then \( H \in \kappa\langle \langle t_1^{\bar{p}} \cdots t_m^{\bar{p}} \rangle \rangle^E_{C,d} \).

**Proof.** Define a well ordering on \( \text{supp}(m : E; C,d) \) by
\[ (i_1, \ldots, i_m) \preceq (j_1, \ldots, j_m) \iff \text{either } i_1 + \cdots + i_m < j_1 + \cdots + j_m, \text{ or } i_1 + \cdots + i_m = j_1 + \cdots + j_m \text{ and } \exists a, 1 \leq a \leq m, \text{ s.t. } i_\lambda = j_\lambda \text{ for } \lambda = 1, \ldots, a-1 \text{ and } i_a < j_a. \]

Write \( H = \sum_{i_1, \ldots, i_m \in \text{supp}(m : E; C,d)} b_{i_1, \ldots, i_m} t_1^{i_1} \cdots t_m^{i_m} \). An easy induction on \( (i_1, \ldots, i_m) \) with respect to the above well ordering shows that \( b_{i_1, \ldots, i_m} \in \kappa \) for all \( (i_1, \ldots, i_m) \in \text{supp}(m : E; C,d) \).
\textbf{(4.5.3) Definition.} Let $\kappa \supset \mathbb{F}_p$ be a field of characteristic $p > 0$. Let $E, C > 0$, $d \geq 0$ be real numbers. Let $t_1, \ldots, t_m$ be variables. Define an $[0, \infty]$-valued function

$$o_m : \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d} \longrightarrow [0, \infty]$$

by

$$o_m(f) := \inf\{i_m | (i_1, \ldots, i_m) \in \text{supp}(f)\}$$

for every formal series $f$ in $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$. We call $o_m$ the \textit{order} of $f$ in the variable $t_m$. By definition $o_m(0) = \infty$.

\textbf{(4.5.4) Lemma.} The function $f \mapsto o_m(f)$ defined in 4.5.3 satisfies the following properties.

\begin{itemize}
\item[(i)] $o_m(f + g) \geq \min\{o_m(f), o_m(g)\}$ for all $f, g \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$.
\item[(ii)] $o_m(f \cdot g) \geq o_m(f) + o_m(g)$ for all $f, g \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$.
\item[(iii)] $o_m(f^n) = n \cdot o_m(f)$ for all $n \in \mathbb{N}$ and all non-zero formal series $f$ in $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$.
\end{itemize}

\textbf{Proof.} The statements (i) and (ii) are immediate from the definition. To prove (iii), note first that $o_m(f^n) > n \cdot o_m(f)$ by (ii). Again by (ii), if $o_m(f^n) > n \cdot o_m(f)$, then $o_m(f^{n'}) > n' \cdot o_m(f)$ for all integers $n' \geq n$. It is clear that $o_m(f^{p^i}) = p^i \cdot o_m(f)$ for every natural number $i$, because $\text{supp}(f^{p^i}) = p^i \cdot \text{supp}(f)$. The statement (iii) follow.

\textbf{Remark.} We do not know whether the function $o_m$ is a valuation on $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$, i.e. whether the equality $o_m(f \cdot g) = o_m(f) + o_m(g)$ holds for all $f, g \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$.

A related question is the following. Let $I$ be the ideal of $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$ consisting of all elements $f \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$ such that $o_m(f) > 0$. This ideal $I$ is equal to its own radical, but we do not know whether it is a prime ideal.

\textbf{(4.5.5) Proof of 4.5.} 1. Suppose that $f \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$ for some $d' > 0$. We show that $f$ is in the fraction field $L$ of $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$ and is integral over $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$.

Pick a natural number $N \geq d' - d$. Then $t_1^N \cdot f \in \kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$, hence $f$ is an element $L$. Consider a power $f^{p^n}$ of $f$, where $n$ is a natural number. The support subset of $f^{p^n}$ consists of all elements of the form $p^n \cdot J$, where $J$ is an exponent in the support subset of $f$. We have

$$|p^nJ|_p = p^{-n} |J|_p \leq p^{-n} \cdot C \cdot (|J|_\sigma + d')^E = p^{-n} \cdot C \cdot (p^{-n}|p^nJ|_\sigma + d')^E.$$

Let $\delta := \text{Min}\{|J|_\sigma : J \in \text{supp}(f), J \neq 0\}$. Clearly $\delta > 0$. Choose $n_1$ sufficiently large so that $(1 - p^{-n_1}) \cdot p^{n_1} \cdot \delta > d' - d$. Then

$$|J|_p \leq C \cdot (|J|_\sigma + d')^E$$

for all $I \in \text{supp}(f^{p^n})$, which implies that $f^{p^n}$ is an element of $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$. Therefore $f$ is integral over the ring $\kappa\langle\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}}\rangle\rangle_{C,d}$.
2. By lemma 4.5.2, we may and do assume that $\kappa$ is an infinite field. Let $f = \frac{G}{F}$ be an element of the fraction field $L$ which is integral over the ring $\kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$. If the constant term of $F$ is a non-zero element of $\kappa$, then $F$ is a unit of the ring $\kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$ and there is nothing to prove. So we may and do assume that the constant term of $F$ is zero.

Let $e := \text{Min}(\{ |I|_\sigma : I \in \text{supp}(F) \})$. By lemma 4.5.1, after making a suitable linear change of coordinates, we may and do assume that $F$ is $i_m$-regular of order $e$. By 4.4.2 (2), there exist elements $U, R \in \kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d+2e}^{E,b}$ such that $G = U \cdot F + R$ and every exponent $I = (i_1, \ldots, i_m)$ of the support subset $\text{supp}(R)$ of $R$ has the property that $i_m < e$. Because $f = F/G$ is integral over $\langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$, the element $R/F$ is also integral over $\kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$. In other words, there exist a positive integer $n_1 > 0$ and elements $H_1, \ldots, H_n \in \kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$ such that

\[ R^{n_1} + H_1 \cdot R^{n_1-1} \cdot F + \cdots + H_{n-1} \cdot R \cdot F^{n_1-1} + H_n \cdot F^{n_1} = 0 \]

We may and do assume that $R \neq 0$. Let $e' := \text{o}_m(R)$. Clearly $e' < e$. By 4.5.4, we know that $\text{o}_m H_1 \cdot R^{n_1-1} \cdot F + \cdots + H_{n-1} \cdot R \cdot F^{n_1-1} + H_n \cdot F^{n_1} \geq (n_1 - 1)\cdot e' + e$, while $\text{o}_m(R^{n_1}) = n_1 \cdot e' < (n_1 - 1) + e$. This contradiction shows that $R = 0$. In other words $f \in \kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d+2e}^{E,b}$. We have proved that the integral closure of $\kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$ in its fraction field $L$ is contained in $\bigcup_{d'} \kappa \langle \langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle \rangle_{C,d}^{E,b}$.

\[ \square \]

§5. Action of a one-parameter subgroup on a biextension

In this section $k \supset \mathbb{F}_p$ is a perfect field, $X, Y, Z$ are $p$-divisible formal groups over $k$, and $\pi : E \to X \times Y$ is a biextension of $X \times Y$ by $Z$.

(5.1) Suppose we have a one-dimensional $p$-adic Lie group $\Gamma$ acting on a biextension $E$ of $X \times Y$ by $Z$. We will extract from such an action a collection of congruence relations; see proposition 3.5.3. This collection of congruence relations comes from the “leading term” of the action of a sequence $(\gamma_m)$ in $\Gamma$ with $\lim_{m \to \infty} \gamma_m = 1$, and can be regarded as a substitute for the “derivative” of the action of $\Gamma$ on $E$.

We will need the following congruence estimate for the morphisms $\eta_n : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z$ attached to a biextension $\pi : E \to X \times Y$ of $p$-divisible formal groups $X \times Y$ by $Z$.

**Proposition.** Let $m = m_E$ be the maximal ideal of the coordinate ring $R = R_E$ of the smooth formal scheme $E$ over $k$. Let $(\eta_n : \pi^{-1}(X[p^n] \times Y[p^n]) \to Z)_{n \in \mathbb{N}}$ be the compatible family of morphisms defined in 2.7.1. Let $\mu = \mu_{Z,\text{min}}$ be the maximum among the slopes of $Z$. There exist positive integers $n_2, c_2$ such that

$$\eta_n \equiv 0 \pmod{m(p^{[n/\mu]-c_2})}$$

for all $n \geq n_2$.

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1 The challenge of finding a good notion of “derivative” can be seen in a simple example: the standard action of $\mathbb{Z}_p^\times$ on the formal completion $\mathcal{G}_m = \text{Spf}(\mathbb{F}_p[[t]])$ of $\mathbb{G}_m$ over $\mathbb{F}_p$. The action of an element $a \in \mathbb{Z}_p^\times$ on $\mathcal{G}_m$ sends the coordinate $t$ to $(1 + t)^a - 1$. 

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(5.2.1) **Lemma.** Let $R_1, R_2, S_1, S_2$ be Noetherian local rings with maximal ideals $m_1, m_2, n_1, n_2$ respectively. Let $h_1 : R_1 \to S_1$ and $h_2 : R_2 \to S_2$ be injective local homomorphisms such that $S_i$ is a finitely generated $R_i$-module via $h_i$ for $i = 1, 2$. There exist positive integers $C, d$ with the following property:

Let $f, g : R_1 \to R_2$ and $f', g' : S_1 \to S_2$ be local homomorphisms such that $h_2 \circ f = f' \circ h_1$ and $h_2 \circ g = g' \circ h_1$. If $n \in \mathbb{N}$ and $f'(y) - g'(y) \equiv 0 \pmod{n_2^{Cn+d}}$ for all $y \in S_1$, then $f(x) - g(x) \equiv 0 \pmod{n_1^n}$ for all $x \in R_1$.

**Proof.** There exists a positive integer $a > 0$ such that $n_2^C \subset n_1S_2$. By the Artin–Rees lemma, there exists a natural number $e$ such that

$$S_1 \cap n_1^mS_2 \subset n_1^m \quad \forall n \in \mathbb{N}.$$ 

Lemma 5.2.1 holds for $C = a$ and $d = ae$. \[\hfill \square\]

(5.2.2) **Reduction steps for the proof of proposition 5.2.**

0. We may and do assume that the base field $k \supset \mathbb{F}_p$ is algebraically closed.

1. Suppose that there exists a biextension $E'$ of $p$-divisible formal groups $X' \times Y'$ by $Z'$ and an homomorphism $(\psi, \alpha, \beta, \gamma)$ from $E$ to $E'$ such that $\alpha : X \to X'$, $\beta : Y \to Y'$ and $\gamma : Z \to Z'$ are isogenies of $p$-divisible groups. It is easy to see from 2.6.3 that there exist isogenies $\alpha' : X' \to X$, $\beta' : Y' \to Y$, $\gamma' : Z' \to Z$ and a homomorphism $(\psi', \alpha', \beta', \gamma')$ from $E'$ to $E$. By 5.2.1, proposition 5.2 holds for $E$ if and only if it holds for $E'$. It suffices to prove 5.2 for a push-forward of the given biextension $E$ by an isogeny $E \to E'$ of $p$-divisible groups. Therefore we may and do assume that the $p$-divisible group $X$ is isomorphic to a product $X_1 \times \cdots \times X_a$ of isoclinic $p$-divisibles with distinct slopes, $Y$ is isomorphic to a product $Y_1 \times \cdots \times Y_b$ of isoclinic $p$-divisibles with distinct slopes, and $Z$ is isomorphic to a product $Z_1 \times \cdots \times Z_c$ of isoclinic $p$-divisible groups with distinct slopes.

2. In the situation at the end of step 1, the biextension $E$ is decomposed into a product of biextensions $E_{lmn}$, $l = 1, \ldots, a$, $m = 1, \ldots, b$, $n = 1, \ldots, c$, where each $E_{lmn}$ is a biextension of $X_l \times Y_m$ by $Z_n$. If 5.2 holds for all biextensions $E_{lmn}$, then it holds for $E$. Therefore we may and do assume that $X, Y, Z$ are all isoclinic.

3. If $\text{slope}(X) + \text{slope}(Y) \neq \text{slope}(Z)$, then the bilinear pairing $\Theta_E : M_*(X) \times M(Y) \to M(Z)$ is zero and the biextension $E$ splits canonically. In this case 5.2 holds for trivial reason. Therefore we may and do assume that $\text{slope}(X) + \text{slope}(Y) = \text{slope}(Z)$.

4. Modifying $Z$ by an isogeny if necessary, we may and do assume that there exist positive integers $a, r, s, n_0$ such that

- $\text{slope}(X) = \frac{a}{r}$,
- $Z[p^r] = \text{Ker}(\text{Fr}_{Z/k}^r)$,
- $X[p^{ns}] \supset \text{Ker}(\text{Fr}_{X/k}^{ns})$ and $Y[p^{ns}] \supset \text{Ker}(\text{Fr}_{Y/k}^{ns})$ for all $n \geq n_0$. 

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5. Let \( u_1, \ldots, u_b \) be a regular system of parameters of the coordinate ring of \( Z \) such that

\[ [p^m]^*(u_j) = u_j^f, \quad j = 1, \ldots, b. \]

We have to show that there exist positive integers \( n_2, c_2 \) such that

\[ \eta_a^*(u_j) \in m^{(p^{n_2}/m)^{-c_2}}, \quad \forall n \geq n_2, \ j = 1, \ldots, b. \]

6. It suffices to show that there exist positive integers \( n_3, c_3 \) such that

\[ \eta_a^*(u_j) \in m^{(p^{n_3}/m)^{-c_3}}, \quad \forall n \geq n_2, \ j = 1, \ldots, b. \]

**Proof of 5.2.** We are in a position to apply results in 3.5. For each \( j = 1, \ldots, b \) and each \( n \geq n_0 \), define

\[ a_{j,n} = \eta_a^*(u_j) \mod m^{(p^{n_0})}. \]

The sequence \( (a_{j,n})_{n \geq n_0} \) is \( \phi^r \)-compatible.

Pick a regular system of parameters \( t_1, \ldots, t_m \) of the coordinate ring of \( E \). By 3.5.3, there exists a formal sequence

\[ B_j = \sum_{l \in \mathbb{N}[1/p]^m} b_{j,l} t^l \]

such that the inequality (ast) in 3.5.3 whenever \( b_{j,l} \neq 0 \), and

\[ \eta_a^*(u_j) \equiv B_j^{p^m} \left( \mod \left\{ \sum_{|l| \geq p^m} r_{l} t^l \right\} \right) \]

for all \( n \geq n_0 \). The above congruence is modulo terms of max-degree at least \( p^m \), i.e. modulo formal series of the form \( \sum_{|l| \geq p^m} r_{l} t^l \). It is easily seen from the definition of the morphisms \( \eta_n \) that none of the \( a_{j,n}'s \) is a unit in \( k[[t]]/m^{(p^{n_0})} \). So the constant term of \( B_j \) is zero for all \( j = 1, \ldots, b \). The desired estimate for the \( \eta_a^*(u_j) \)'s follows immediately. \( \square \)

**5.3** Recall from 2.6.4 that the Lie algebra of the compact \( p \)-adic Lie group \( \text{Aut}_{\text{biext}}(E) \) consists of all triples \( (A,B,C) \) which kill the bilinear form \( \Theta_E \) as in 2.6.4 (2).

**5.3.1 Lemma.** Let \( v = (A,B,C) \) be an element of the Lie algebra of \( \text{Aut}_{\text{biext}}(E) \). Suppose that \( A \in \text{End}(X) \), \( B \in \text{End}(Y) \) and \( C \in \text{End}(Z) \). Then \( \exp(p^t A) \in \text{Aut}(X) \), \( \exp(p^t B) \in \text{Aut}(Y) \), \( \exp(p^t C) \in \text{Aut}(Z) \) and \( \exp(p^t v) \in \text{Aut}_{\text{biext}}(E) \) for all \( t \in \mathbb{Z}_p \).

**Proof.** The Taylor series for \( \exp(p^t A) \in \text{Aut}(X) \) converges \( p \)-adically and defines an element of \( \text{Aut}(X) \). Similarly \( \exp(p^t B) \in \text{Aut}(Y) \) and \( \exp(p^t C) \in \text{Aut}(Z) \). That \( \exp(p^t v) \in \text{Aut}_{\text{biext}}(E) \) follows from 2.6.4. \( \square \)

**5.3.2 Proposition.** Let \( v = (A,B,C) \) be an element of the Lie algebra of \( \text{Aut}_{\text{biext}}(E) \) such that \( A \in \text{End}(X) \), \( B \in \text{End}(Y) \) and \( C \in \text{End}(Z) \).
(i) For every integer \( n \geq 2 \), the infinite series
\[
\sum_{j \geq 2} \frac{p^n(j-1)}{j!} C^j
\]
converges to an element of \( \text{End}(Z) \).

(ii) The restriction of the automorphism \( \exp(p^n t v) \) to \( E_n = \pi^{-1}(X[p^n] \times Y[p^n]) \) is equal to
\[
\left(C \circ \eta_n + \sum_{j \geq 2} \frac{p^n(j-1)}{j!} C^j \circ \eta_n \right) \ast \text{id}_{E_n}
\]

**PROOF.** The statement (i) follows from the easy estimate
\[
\text{ord}_p(j!) < \frac{j}{p-1} \leq j,
\]
which implies that
\[
\text{ord}_p \left( \frac{p^{2(j-1)}}{j!} \right) \geq (n-1)(j-1) - 1.
\]
Clearly \( (n-1)(j-1) - 1 \geq 0 \) for all \( j \geq 2 \) and \( (n-1)(j-1) - 1 \to 0 \) as \( j \to \infty \). The statement (i) follows.

For (ii), we note that the automorphism \( \exp(p^n C) \times \exp(p^n A) \times \exp(p^n B) \) of \( Z \times X[p^n] \times Y[p^{2n}] \) descends to the restriction to \( E_n \) of the automorphism \( \exp(p^n v) \) of \( E \), according to 2.5.1 (iv) and 2.5.3. The statement (ii) follows from the definition of \( \eta_n \) in 2.7.1 and the Taylor expansion of \( \exp(p^n C) \). \( \square \)

(5.4) We adopt the following assumptions and notation. They are compatible with the assumptions and notation in 2.7.

(i) Let \( v = (A,B,C) \) be an element of the Lie algebra of \( \text{Aut}_{\text{biext}}(E) \). Assume that \( A \in \text{End}(X) \), \( B \in \text{End}(Y) \) and \( C \in \text{End}(Z) \).

(ii) Assume that \( a,s,r \) are three positive integers such that
- \( 0 < r < s \), and \( \frac{a}{r} \) is the largest slope of \( Z \)
- \( \frac{a}{s} \) is strictly bigger than every slope of \( X \) and every slope of \( Y \).

From general properties of slopes of \( p \)-divisible groups we know that there exist natural numbers \( n_0, c_0 \in \mathbb{N} \) with \( n_0 \geq \min(2, c_0/r) \) such that
\[
X[p^{na}] \supset \text{Ker}(\text{Fr}_X^{ns}) \quad \text{and} \quad Y[p^{na}] \supset \text{Ker}(\text{Fr}_Y^{ns})
\]
and
\[
Z[p^{na}] \supset \text{Ker}(\text{Fr}_Z^{n-r-c_0})
\]
for all \( n \geq n_0 \), where \( \text{Fr}_X^{ns} : X \to X(p^{ns}) \) (respectively \( \text{Fr}_Y^{ns} \)) is the \((ns)\)-th iterate of the relative Frobenius for \( X \) (respectively \( Y \)). Similarly for \( \text{Fr}_Z^{n-r-c_0} \).

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(iii) Let \( R = R_E \) be the affine coordinate ring of the smooth formal scheme \( E \), so that \( E = \text{Spf}(R) \) and \( R \) is non-canonically isomorphic to a formal power series ring in \( d \) variables, where \( d = \dim(E) \). Let \( m = m_E \) be the maximal ideal of \( R \). Let \( \phi = \phi_R \) be the absolute Frobenius endomorphism of \( R \), which sends every element \( x \in R \) to \( x^p \).

For every natural number \( j \), define an ideal of \( R \) by
\[
m^{(p^j)} := \phi^j(m)R.
\]

Note that
\[
m^{[p^j/d]} \subseteq m^{(p^j)} \subseteq m^{p^j}.
\]

Denote by \( E \mod m^{(j)} \) the Artinian scheme
\[
E \mod m^{(j)} := \text{Spec}(R/m^{(j)})
\]

(5.5) **Proposition.** We use the notation and assumption in 5.4 and 5.3.2. There exist positive integers \( n_3, c_3 \) such that the congruence
\[
\psi(\exp(p^{na}v)) \equiv (C \circ \eta_{na}) * \text{id}_{E_{na}} \pmod{m^{(p^{\min(na,2n-2)})}}
\]
for the action \( \psi(\exp(p^{na}v)) \) of the element \( \exp(p^{na}v) \in G \) on \( E \) holds for all integers \( n \geq n_3 \). In other words, the restrictions to the Artinian scheme \( E \mod m^{(p^j)} \) of the two automorphisms \( \psi(\exp(p^{na}v)) \) and \( (C \circ \eta_{na}) * \text{id}_{E_{na}} \) of the formal scheme \( E \) coincide. Here \( E_{na} = \pi^{-1}(X[p^{na}] \times Y[p^{na}]) \) as before.

**Proof.** This proposition is a straight-forward consequence of 5.2 and 5.3.2.

1. The assumption 5.4 (ii) tells us that \( E_{na} \supset \text{Spec}(R/m^{(p^j)}) \) for all \( n \geq n_0 \).

2. We know from 5.3.2 that the restriction of \( \psi(\exp(p^{na}v)) \) to \( E_{na} \) is equal to
\[
(C \circ \eta_{na} + \sum_{j \geq 2} \frac{p^{na(j-1)}}{j!} C^j \circ \eta_{na}) * \text{id}_{E_{na}},
\]
for all \( n \geq n_2 \).

3. We know from 5.2 that there exist positive integers \( n_2, c_2 \) such that
\[
\eta_{na} \equiv 0 \pmod{m^{(p^{na-2})}}
\]
for all \( n \geq n_2 \).

4. An elementary calculation shows that
\[
\text{ord}_p \frac{p^{na(j-1)}}{j!} > na(j-1) - \frac{j}{p-1} \geq na - 2 \quad \forall j \geq 2
\]
Let \( n_3 := \text{Min}(n_0, \lceil n_2/a \rceil) \). Combining 3 and 4 above we get an estimate of the typical “error term”
\[
\frac{p^{na(j-1)}}{j!} C^j \circ \eta_{na} \equiv 0 \pmod{m^{(p^{2n-2})}}
\]
where \( c_3 := 2c_0 + c_2 \), for all \( n \geq n_3 \) and all \( j \geq 2 \).
The following corollary 5.7 is a variant of 5.5 and will be convenient for our purpose. We will follow the general notation scheme in 5.4 and 5.5: $X, Y, Z$ are $p$-divisible groups over a perfect field $k \supset \mathbb{F}_p$, $\pi : E \to X \times Y$ is a biextension of $X \times Y$ by $Z$. Let $(R, m) = (R_E, m_E)$ be the coordinate ring of $E$.

(i) Assume that $X, Y, Z$ are $p$-divisible formal groups, i.e. every slope of $X, Y, Z$ is strictly positive.

(ii) Let $\nu = (A, B, C)$ be an element of the Lie algebra of $\text{Aut}_{\text{biext}}(E \to X \times Y)$, $A \in \text{End}(X)$, $B \in \text{End}(Y)$ and $C \in \text{End}(Z)$.

(iii) Assume that $Z$ is a product of isoclinic $p$-divisible groups; write $Z$ as a product of isoclinic $p$-divisible subgroups with distinct slopes: $Z = \prod Z_l$, where each $Z_l$ is isoclinic, the slopes of the $Z_l$ are mutually distinct, and the slope of $Z_l$ is the biggest among slopes of $Z$. Assume that the slope of $Z_1$ is strictly bigger than every slope of $X \times Y$.

(iv) Choose positive integers $a, r, s, n_3$ with $r < s$ such that the following conditions hold.

- $\text{slope}(Z_1) = \frac{a}{s}$
- $X[p^na] \supset \text{Ker}(\text{Fr}_X^{pns})$ and for all $n \geq n_3$.
- $Z_l[p^na] \supset \text{Ker}(\text{Fr}_Z^{pns})$ for all $l \neq 1$ and all $n \geq n_3$.

(v) For every $n \geq n_3$, define a morphism

$$\rho_{na} : \pi^{-1}(\text{Ker}(\text{Fr}_X^{pns}) \times \text{Ker}(\text{Fr}_Y^{pns})) \longrightarrow Z_l$$


\[ \rho_{na} = (\text{pr}_{Z_l} \circ \eta_{na}) \big|_{\pi^{-1}(\text{Ker}(\text{Fr}_X^{pns} \times \text{Ker}(\text{Fr}_Y^{pns}))}. \]

(5.7) **Corollary.** Notation and assumptions as in 5.6. In particular $a, r, s$ are positive integers, $r < s$, $\frac{a}{s}$ is the largest slope of $Z$, $\frac{a}{s}$ is strictly bigger than any slope of $X \times Y$ and any other slope of $Z$, $Z_1$ is the maximal $p$-divisible subgroup of $Z$ with slope $\frac{a}{s}$, and $Z_1[p^r] = \text{Ker}(\text{Fr}_{Z_1/k}^{p^r})$. There exist positive integers $n_4, c_4$ such that

$$\psi(\exp(p^r\nu)) \equiv (C|_{Z_1} \circ \rho_{na}) \ast \text{id}_{E \text{ mod } m} \mod m^{(\min(nr, 2nr - c_4))}$$

for all $n \geq n_4$, where $C|_{Z_1} \in \text{End}(Z_1)$ is the restriction to the factor $Z_1$ of $Z$ of the endomorphism $C \in \text{End}(Z)$.

Corollary 5.7 is an easy consequence of 5.5.

§6. **How to prove identities using powers of Frobenius**

(6.1) The main technical tool for proving local rigidity for $p$-divisible formal groups with respect to non-trivial actions of compact $p$-adic Lie groups is the statement [3, Prop. 3.1], about power series, which looks like a mess at first sight. A slightly different form, stated in [3, 3.1.1], is reproduced in 6.1.1 below for the convenience of the readers. A weak rigidity statement 6.2 for a section of a biextension stable under the action of a $p$-adic Lie group is presented as an application of 6.1.1.
(6.1.1) **Proposition.** Let \( k \supset \mathbb{F}_p \) be a field. Let \( \mathbf{u} = (u_1, \ldots, u_a), \mathbf{v} = (v_1, \ldots, v_b) \) be two tuples of variables. Let \( f(\mathbf{u}, \mathbf{v}) \in k[[\mathbf{u}, \mathbf{v}]] \) be a formal power series in the variables \( u_1, \ldots, u_a, v_1, \ldots, v_b \) with coefficients in \( k \). Let \( \mathbf{x} = (x_1, \ldots, x_m) \), \( \mathbf{y} = (y_1, \ldots, y_n) \) be two new sets of variables. Let \( (g_1(\mathbf{x}), \ldots, g_a(\mathbf{x})) \) be an \( a \)-tuple of power series such that \( g_i(\mathbf{x}) \in (\mathbf{x}) k[[\mathbf{x}]] \) for \( i = 1, \ldots, a \). Let \( (h_1(\mathbf{y}), \ldots, h_b(\mathbf{y})) \) be a \( b \)-tuple of power series with \( h_j(\mathbf{y}) \in (\mathbf{y}) k[[\mathbf{y}]] \) for \( j = 1, \ldots, b \). Let \( q = p^r \) be a power of \( p \) for some positive integer \( r \). Let \( n_0 \in \mathbb{N} \) be a natural number. Let \( (d_n)_{n \in \mathbb{N}, n \geq n_0} \) be a sequence of natural numbers such that \( \lim_{n \to \infty} \frac{d_n}{q^n} = 0 \). Suppose we are given power series \( \phi_{j,n}(\mathbf{v}) \in k[[\mathbf{y}]] \) for all \( j = 1, \ldots, b \) and all \( n \geq n_0 \) such that

\[
R_{j,n}(\mathbf{v}) := \phi_{j,n} - v_j^{d_n} \equiv 0 \pmod{(v)^{d_n}} \quad \forall j = 1, \ldots, b, \forall n \geq n_0.
\]

and

\[
f(\mathbf{g}(1, \ldots, a, \phi_{1,n}(h(\mathbf{x})), \ldots, \phi_{b,n}(h(\mathbf{x})))) \equiv 0 \pmod{(x)^{d_n}}
\]

in \( k[[\mathbf{x}]] \), for all \( n \geq n_0 \). Then

\[
f(\mathbf{g}(1, \ldots, a, \phi_{1,n}(h(\mathbf{x})), \ldots, \phi_{b,n}(h(\mathbf{x}))) = 0 \pmod{k[[\mathbf{x}]].}
\]

(6.1.2) **Remark.** As noted in [3, 3.1.1], the proof of [3, 3.1] also proves 6.1.1. We will reproduce the proof of 6.1.1 here. However we would like to remark here that the argument in [3] also works if the power series rings in the statement of 6.1.1 are replaced by complete restricted p-perfections of power series rings. This “easy exercise” will be carried out in 6.4.

(6.1.3) **Remark.** Readers who looked up [3, 3.1.1] may find it somewhat different from the statement of 6.1.1. There it is assumed that there exists an natural number \( b' \) with \( 0 \leq b' \leq b \) such that \( \phi_{j,n} - v_j^{d_n} \equiv 0 \pmod{(v)^{d_n}} \quad \forall j = 1, \ldots, b', \forall n \geq n_0 \), and \( \phi_{j,n} \equiv 0 \pmod{(v)^{d_n}} \quad \forall j = b' + 1, \ldots, \forall n \geq n_0 \). So the condition in [3, 3.1.1] that

\[
f(\mathbf{g}(1, \ldots, a, \phi_{1,n}(h(\mathbf{x})), \ldots, \phi_{b,n}(h(\mathbf{x})))) \equiv 0 \pmod{(x)^{d_n}} \quad \forall n \geq n_0
\]

is equivalent to the assumption that

\[
f'(\mathbf{g}(1, \ldots, a, \phi_{1,n}(h(\mathbf{x})), \ldots, \phi_{b,n}(h(\mathbf{x})))) \equiv 0 \pmod{(x)^{d_n}} \quad \forall n \geq n_0,
\]

where \( f'(u_1, \ldots, u_a, v_1, \ldots, v_{b'}) := f(u_1, \ldots, u_a, v_1, \ldots, v_{b'}, 0, \ldots, 0) \). The conclusion of [3, 3.1.1] is that \( f'(\mathbf{g}(1, \ldots, a, \phi_{1,n}(h(\mathbf{y})), \ldots, h_{b'}(\mathbf{y}))) = 0 \), which is the conclusion of 6.1.1 (with \( b \) replaced by \( b' \) and \( f \) replaced by \( f' \)). So the statements of 6.1.1 is really the same as that of [3, 3.1].

Proposition 6.2 can be reformulated as follows. We will use the following notation.

- Let \( k \supset \mathbb{F}_p \) be a field.
- Let \( U, V \) and \( X \) be local formal schemes over \( k \) which are isomorphic to the formal spectrum of a power series ring over \( k \) in a finite number of variables.
- For each positive integer \( m \), let \( U_m, V_m, X_m \) be the \( m \)-th infinitesimal neighborhood of the local formal schemes \( U, V, X \) respectively.
- Let \( U \times V \) be the fiber product of \( U \) and \( V \) over \( k \) in the category of formal schemes over \( k \).
• Let \( q = p^n \) be a positive power of \( p \).

• Let \( \text{Fr}^q_{Y/k} : V \to V^{(q)} \) be the \( r \)-th iterate of the relative Frobenius for \( V/k \).

• Let \( \delta : V^{(q)} \to V \) be an isomorphism over \( k \) and let \( \Phi = \delta \circ \text{Fr}^q_{V/k} \).

• For each \( n \in \mathbb{N} \), let

\[
(g, \Phi^n \circ h) : X \to U \times V
\]

be the morphism from \( X \) to the fiber product \( U \times V \) with components \( g \) and \( \Phi^n \circ h \). For each \( m \in \mathbb{N} \), let

\[
(g, \Phi^n \circ h)_m : X_m \to (U \times V)_m
\]

be the morphism from the \( m \)-th infinitesimal neighborhood of \( X \) to the \( m \)-th infinitesimal neighborhood of \( U \times V \) induced by \( (g, \Phi^n \circ h) \).

(6.1.4) **Proposition.** We use notations in the preceding paragraph. Let \( (d_n)_{n \geq n_0} \) be a sequence of positive integers such that

\[
\lim_{n \to \infty} \frac{q^n}{d_n} = 0.
\]

Let \( f \in \Gamma(U \times V, \mathcal{O}_{U \times V}) \) be a regular function on the formal scheme \( U \times V \). Suppose for each \( n \geq n_0 \), the function \( f \circ (g, \Phi^n \circ h)_{X_{d_n}} \) on the \( d_n \)-th infinitesimal neighborhood \( X_{d_n} \) of the closed point of \( X \) induced by the morphism \( (g, \Phi^n \circ h)_{d_n} : X_{d_n} \to (U \times V)_{d_n} \) and \( f \) is zero. Then the composition of \( f \) with \( g \circ h : X \times X \to U \times V \) is zero. In other words \( f \) vanishes on the schematic closure of the morphism \( g \circ h : X \times X \to U \times V \).

(6.1.5) **Remark.** Suppose that the base field \( k \supseteq \mathbb{F}_p \) in is algebraically closed field. Then the schematic closure of \( g \circ h : X \times X \to U \times V \) is equal to the product of the schematic closures of \( g : X \to U \) and \( h : X \to V \). So 6.1.4 simplifies to: the schematic closure of (the union of) the family of morphisms

\[
(g, \Phi^n \circ h)_{d_n} : X_{d_n} \to (U \times V)_{d_n} \to U \times V
\]

contains the product of the schematic closure of \( g : X \to U \) with the schematic closure of \( h : X \to V \).

The universal case of the last statement is when \( U = V = X \) and \( g, h \) are both equal to \( \text{id}_X \), and the family of morphisms in the statement is induced by the family of Frobenius-twisted diagonal maps

\[
(\text{id}_{X_{d_n}}, \Phi^n \circ \text{id}_{X_{d_n}}) : X_{d_n} \to X_{d_n} \times X_{d_n}.
\]

Note that \( (X \times X)_{d_n} \) is naturally identified with \( (X_{d_n} \times X_{d_n})_{d_n} \), the \( d_n \)-th infinitesimal neighborhood of the closed point of \( X_{d_n} \times X_{d_n} \).

(6.2) **Proposition.** Let \( k \supseteq \mathbb{F}_p \) be a field. Let \( X, Y, Z \) be \( p \)-divisible formal groups over \( k \). Let \( \pi : E \to X \times_{\text{Spec}(k)} Y \) be a biextension of \( X \times_{\text{Spec}(k)} Y \) by \( Z \). Let \( s : X \times_{\text{Spec}(k)} Y \to E \) be a section of \( \pi \), i.e. \( \pi \circ s = \text{id}_{X \times_{\text{Spec}(k)} Y} \). Let \( G \) be a compact \( p \)-adic Lie group action on the biextension \( E \to X \times Y \).

(i) Suppose that \( Z \) is isoclinic, and the slope of \( Z \) is strictly bigger than every slope of \( X \) and every slope of \( Y \).

(ii) Assume that the induced action of \( G \) on \( Z \) is strongly non-trivial.
(iii) Assume moreover that the graph of the section $s$ is stable under the action of $G$ on $E$.

Then the biextension $\pi : E \to X \times_{\text{Spec}(k)} Y$ is trivial, and the section $s$ is its canonical splitting.

**Proof.** There are two preliminary reduction steps before the main argument

**Reduction Step 1.** According to the last paragraph of 2.4.1, it suffice to verify the assertion in the last sentence of 6.2 after extending the base field from $k$ to an algebraic closure of $k$. So we may and do assume that $k$ is algebraically closed.

**Reduction Step 2.** Let $\nu$ be the slope of $Z$. We know that there exists an isogeny $\xi : Z \to Z_1$ such that $Z_1$ is completely slope divisible in the sense that there exist positive integers $a_1, r_1$ with $a_1 = r_1 \cdot \nu$ such that the kernel of the $N$-th iterate

$$\text{Fr}^{(p^a)}_{Z/k} : Z \to Z^{(p^N)}$$

of the relative Frobenius is equal to $Z[p^{a_1}]$. Let $E_1 \to X \times_{\text{Spec}(k)} Y$ be the push forward by $\xi$ of the biextension $E$, and let $\tilde{\xi} : E \to E_1$ be the canonical biextension homomorphism from $E$ to $E_1$. For each element $g \in G$, let $\alpha(g), \beta(g)$ and $\gamma(g)$ be the automorphism of $X, Y$ and $Z$ induced by the action of $g$ on the biextension $E$. There exists an open subgroup $G_1$ of $G$ such that the isogeny $\xi \circ \alpha(g) \circ \xi^{-1}$ is an automorphism of $Z$. One verifies without difficulty that there exists a continuous homomorphism $\rho_1 : G_1 \to \text{Aut}_{\text{biext}}(E_1)$ such that $\rho_1(g) \circ \tilde{\xi} = \tilde{\xi} \circ \rho(g)$ for every $g \in G_1$. From the last paragraph of 2.4.4, to show that the biextension $E$ of $X \times_{\text{Spec}(k)} Y$ by $Z$ is trivial, it suffices to show that the biextension $E_1$ of $X \times_{\text{Spec}(k)} Y$ by $Z_1$ is trivial. In this case the composition of the canonical splitting $E_1$ is the unique section of $E \to X \times_{\text{Spec}(k)} Y$ whose composition with $\tilde{\xi}$ is equal to the canonical splitting of the biextension $E_1 \to X \times_{\text{Spec}(k)} Y$. So we may and do assume that $Z[p^{a_1}] = \text{Ker}(\text{Fr}^p_{Z})$.

By [3, Thm. 4.3], the graph of the restriction of $s$ to $X \times 0_Y$ is a $p$-divisible subgroup of $\pi^{-1}(X \times 0_Y) \cong X \times_{\text{Spec}(k)} Z$, meaning that the restriction of $s$ to $X \times 0_Y$ is a group homomorphism from $X \to \pi^{-1}(X \times 0_Y)$. Because $X$ and $Z$ does not have common slope, $s$ coincides with $\varepsilon_2$ on $X \times 0_Y$. Similarly $s$ coincides with $\varepsilon_1$ on $0_X \times Y$. Let $\tau : (X \times X) \times Y \to Z$ and $\sigma : X \times (Y \times Y) \to Z$ be defined by formulas (a), (b) in 2.2.1. To prove 6.2, it suffices to show that both $\tau$ and $\sigma$ are zero, in the sense that each is equal to the composition of the zero section $0_Z$ of $Z$ with the projection of its source to $\text{Spec}(k)$.

The assumption that the graph of the section $s$ is stable under the action of $G$ means that the map $\mu_{\rho(g)} : X \times Y \to Z$ in 2.3.4 is identically 0 for each $g \in G$. Therefore

$$\gamma(g)(\tau(x_1, x_2; y)) - \tau(\alpha(g)(x_1), \alpha(g)(x_2); \beta(g)(y)) = 0$$

$$\gamma(g)(\sigma(x; y_1, y_2)) - \sigma(\alpha(g)(x); \beta(g)(y_1), \beta(g)(y_2)) = 0$$

for all $x, x_1, x_2 \in X$ and all $y, y_1, y_2 \in Y$.

The key observation here is that equations (1), (2) above for elements $g \in G$ close to the identity element of $G$, under the assumption that the slope of $Z$ is strictly bigger than all slopes of $X$ and $Y$, produces a large number of identities which are increasingly close to equations of the form

$$d\gamma(C) \cdot \tau = 0_Z = d\gamma(C) \cdot \sigma$$

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in the sense specified in 6.1.1, so that the identity principle with Frobenius powers 6.1.1 is applicable. Here $C$ is any element of Lie$(G)$ such that $d\gamma(C)$, a priori an element of $\text{End}(Z) \otimes \mathbb{Z} \mathbb{Q}$, is actually an endomorphism of $Z$. The limit equations resulting from the identity principle is the following.

\[(*) \text{ If } \nu \text{ is an element of Lie}(G) \text{ such that the image of } \nu \text{ under the representation} \]

\[ (d\alpha, d\beta, d\gamma) : \text{Lie}(G) \to \text{End}(Z) \otimes \mathbb{Z} \mathbb{Q} \]

\[ \text{lies in } \text{End}(X) \oplus \text{End}(Y) \oplus \text{End}(Z), \text{ then } d\gamma(\nu) \text{ kills both } Z\text{-valued functions } \tau \text{ and } \sigma \]

\[ \text{from } X \times Y \text{ to } Z. \]

Since the action of $G$ on $Z$ is assumed to be strongly non-trivial, the statement $(*)$ above implies that the maps $\tau : (X \times X) \times Y \to Z$ and $\sigma : X \times (Y \times Y) \to Z$ are both identically zero and 6.2 follows.

It remains to prove $(*)$. Let $\mu = \text{slope}(Z)$. After extending the base field $k$, we may and do assume that $k$ is algebraically closed. Changing $Z$ by an isogeny, we may and do assume that there exist positive integers $a_1, r_1$ such that $Z[p^{a_1}] = \text{Ker}(\text{Fr}_{Z/k})$.

Let $\nu$ be an element of Lie$(G)$ such that $A = d\alpha(\nu)$ is an endomorphism of $Z$, $B = d\beta(\nu)$ is an endomorphism of $Y$ and $C = d\gamma(\nu)$ is an endomorphism of $Z$. There exists positive integers $a, r, s, c, n_0$ such that (i)--(iv) below hold for all integers $n \geq n_0$.

(i) $a, r$ are multiples of $a_1, r_1$ respectively, $r < s$, and $a/s$ is strictly bigger than every slope of $X$ and every slope of $Y$.

(ii) $\gamma(\exp(p^{na}C)) \equiv \text{id}_Z + p^{na}C \pmod{\text{Ker}(\text{Fr}_{Z/k}^{na-c})}$,

(iii) $\alpha(\exp(p^{na}A)) \in \text{End}(X)$, $\beta(\exp(p^{na}B)) \in \text{End}(Y)$,

(iv) $\alpha(\exp(p^{na}A)) \equiv \text{id}_X \pmod{\text{Ker}(\text{Fr}_{X/k}^{na-c})}$, and $\beta(\exp(p^{na}B)) \equiv \text{id}_Y \pmod{\text{Ker}(\text{Fr}_{Y/k}^{na-c})}$.

The equations (6.2.1), (6.2.2 with $g = \exp(p^{na}A$ and the congruences (ii) and (iv) above implies that

\[(p^{na}C) \cdot \tau(x_1, x_2; y) \equiv 0 \pmod{\text{Ker}(\text{Fr}_{Y/k}^{na-c})} \]

and

\[(p^{na}C) \cdot \sigma(x; y_1, y_2) \equiv 0 \pmod{\text{Ker}(\text{Fr}_{Y/k}^{na-c})} \]

for all $n \geq n_0$. Applying proposition 6.1.1, we conclude that

\[ C \cdot \tau(x_1, x_2; y) = 0 \text{ and } C \cdot \sigma(x; y_1, y_2) = 0. \]

We have proved the statement $(*)$ and proposition 6.2. $\square$
We make some preparations before stating proposition 6.4. The latter is a generalization of 6.1.1 to the context of complete restricted perfections of power series rings.

**Definition.** Let $E > 0$, $C, d \geq 1$ be real numbers. Define the support subset

$$\text{supp}(m : E; C, d) \subset \mathbb{N}[1/p]^m$$

with parameters $(E; C, d)$ by

$$\text{supp}(m : E; C, d) = \{ I \in \mathbb{N}[1/p]^m : |I|_p \leq C \cdot (|I|_\sigma + d)^E \}$$

**Definition.** Let $x = (x_1, \ldots, x_m)$ be a tuple of variables,

(i) The total degree of monomials in $x$ gives rise to a decreasing filtration

$$\text{Fil}_{\text{deg}} \geq 0$$

on $k\langle x_1, \ldots, x_m \rangle_{E, \sigma}^{C, d}$, indexed by real numbers:

$$\text{Fil}_{\text{deg}}^u(\langle x_1, \ldots, x_m \rangle_{C, d}^{E, \sigma}) := \left\{ \sum_{I \in \text{supp}(m : E; C, d)} a_I \cdot x_I^I \mid a_I \in k \forall I, \ a_I = 0 \text{ if } |I|_\sigma < u \right\}$$

for every $u \in \mathbb{R}$.

(ii) For every real number $u$, define $\text{Fil}_{\text{deg}}^u$ by

$$\text{Fil}_{\text{deg}}^u(\langle x_1, \ldots, x_m \rangle_{C, d}^{E, \sigma}) := \left\{ \sum_{I \in \text{supp}(m : E; C, d)} a_I \cdot x_I^I \mid a_I \in k \forall I, \ a_I = 0 \text{ if } |I|_\sigma \leq u \right\}$$

The following lemma deals with the perfection $k[x_1^{\sigma \rightarrow \infty}, \ldots, x_m^{\sigma \rightarrow \infty}] = \cup_{n \in \mathbb{N}} k[x_1^{\sigma \rightarrow n}, \ldots, x_m^{\sigma \rightarrow n}]$ of the polynomial ring $k[x_1, \ldots, x_m]$ over the perfect base field $k$. Notice that one can evaluate any element of $k[x_1^{\sigma \rightarrow \infty}, \ldots, x_m^{\sigma \rightarrow \infty}]$ at any $m$-tuple $(c_1, \ldots, c_m) \in k^m$. Lemma 6.3.3 provides a dichotomy when an element $F(x_1, \ldots, x_m) \in k[x_1^{\sigma \rightarrow \infty}, \ldots, x_m^{\sigma \rightarrow \infty}]$ is evaluated at all $\text{Fr}_q$-powers

$$\{(c_1^q, \ldots, c_m^q) : n \in \mathbb{N}\}$$

of a given $m$-tuple $(c_1, \ldots, c_m)$, where $q = p^r$ is a power of $p$, $r \in \mathbb{N}_{>0}$:

- either $F(c_1^q, \ldots, c_m^q) = 0$ for infinitely many natural numbers,
- or $F(c_1^q, \ldots, c_m^q) = 0$ for all $n \in \mathbb{Z}$.

**Lemma.** Let $r$ be a positive integer, and let $q = p^r$. Let $F(x_1, \ldots, x_m)$ be an element of $k[x_1^{\sigma \rightarrow \infty}, \ldots, x_m^{\sigma \rightarrow \infty}]$. Suppose that $(c_1, \ldots, c_m) \in k^m$ is an element of $k^m$ and $n_0$ is a natural number such that

$$F(c_1^q, \ldots, c_m^q) = 0$$

for all integers $n \geq n_0$. Then $F(c_1^q, \ldots, c_m^q) = 0$ for all $n \in \mathbb{Z}$. In particular $F(c_1, \ldots, c_m) = 0$.

**Proof.** When $F(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$, this statement was proved in [3, 2.2]. The general case follows because there exists a positive integer $i$ such that $F(x_1, \ldots, x_n)^{p^i} \in k[x_1, \ldots, x_n]$. □
(6.4) **Proposition.** Let \( x = (x_1, \ldots, x_m) \), \( y = (y_1, \ldots, y_m) \), \( u = (u_1, \ldots, u_r) \) and \( v = (v_1, \ldots, v_b) \) be four tuples of variables. Let \((E_1; C_1, d_1)\) and \((E_2; C_2, d_2)\) be two triples of real parameters with \( E_1, E_2 > 0 \) and \( C_1, C_2, d_1, d_2 \geq 1 \). Let

\[
f(u, v) \in k \langle \langle u_1^{p^{-\infty}}, \ldots, u_r^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_b^{p^{-\infty}} \rangle \rangle_{E_1}^{E_1, b} \]

be an element of \( k \langle \langle u_1^{p^{-\infty}}, \ldots, u_r^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_b^{p^{-\infty}} \rangle \rangle_{E_1}^{E_1, b} \) such that the support set \( \text{supp}(f) \) of \( f \) is contained in the product \( \text{supp}(a : E_1; C_1, d_1) \times \text{supp}(b : E_1; C_1, d_1) \):

\[
(6.4.1) \quad \text{supp}(f) \subseteq \text{supp}(a : E_1; C_1, d_1) \times \text{supp}(b : E_1; C_1, d_1).
\]

In other words \( f \) lies in the closure in \( k \langle \langle u_1^{p^{-\infty}}, \ldots, u_r^{p^{-\infty}}, v_1^{p^{-\infty}}, \ldots, v_b^{p^{-\infty}} \rangle \rangle_{E_1}^{E_1, b} \) of the subring

\[
k \langle \langle u_1^{p^{-\infty}}, \ldots, u_r^{p^{-\infty}} \rangle \rangle_{E_1}^{E_1, b} \otimes_k k \langle \langle v_1^{p^{-\infty}}, \ldots, v_b^{p^{-\infty}} \rangle \rangle_{E_1}^{E_1, b}.
\]

Let

\[
(g_1(x), \ldots, g_a(x)) \in (\text{Fil}_{t, \text{deg}}^{>0} k \langle \langle x_1^{p^{-\infty}}, \ldots, x_m^{p^{-\infty}} \rangle \rangle_{C_2; d_2}^{E_2, b})^a
\]

be an \( a \)-tuple of elements in \( k \langle \langle x_1^{p^{-\infty}}, \ldots, x_m^{p^{-\infty}} \rangle \rangle_{C_2; d_2}^{E_2, b} \) whose constant terms are 0. Let

\[
(h_1(y), \ldots, h_b(y)) \in (\text{Fil}_{t, \text{deg}}^{>0} k \langle \langle y_1^{p^{-\infty}}, \ldots, y_m^{p^{-\infty}} \rangle \rangle_{C_2; d_2}^{E_2, b})^b
\]

be a \( b \)-tuple of elements in \( k \langle \langle y_1^{p^{-\infty}}, \ldots, y_m^{p^{-\infty}} \rangle \rangle_{C_2; d_2}^{E_2, b} \) whose constant terms are 0. Let \( q = p^r \) be a power of \( p \), where \( r > 0 \) is a positive integer. Let \( n_0 \) be a natural number. Suppose that there exists a sequence \( (d_n)_{n \geq n_0} \) of natural numbers such that

\[
(6.4.2) \quad \lim_{n \to \infty} \frac{q^n}{d_n} = 0
\]

and

\[
(6.4.3) \quad f(g_1(x), \ldots, g_a(x), h_1(y), \ldots, h_b(y)) q^n \equiv 0 \pmod{\text{Fil}_{t, \text{deg}}^{d_n}} \quad \forall n \geq n_0.
\]

Then

\[
(6.4.4) \quad f(g_1(x), \ldots, g_a(x), h_1(y), \ldots, h_b(y)) = 0.
\]

In the above the congruence relation \( 6.4.3 \) takes place in \( k \langle \langle x_1^{p^{-\infty}}, \ldots, x_m^{p^{-\infty}} \rangle \rangle_{C_3; d_3}^{E_3, b} \), and the equation \( 6.4.4 \) holds in the ring \( k \langle \langle x_1^{p^{-\infty}}, \ldots, x_m^{p^{-\infty}}, y_1^{p^{-\infty}}, \ldots, y_m^{p^{-\infty}} \rangle \rangle_{C_3; d_3}^{E_3, b} \), where

- \( E_3 = E_1 + E_2 + E_1 E_2 \),
- \( C_3 = C_1^1 + E_2 + C_2^1 + E_1 + E_2 + (1 + d) E_1 E_2 (1 + E_2) \), and
- \( d_3 \) is a sufficiently large constant depending on \((E_1; C_1, d_1)\) and \((E_2; C_2, d_2)\).

See 4.3.1 and the trivial lower bound for \( e_2 \) there.

**Remark.** For application to rigidity of biextensions of \( p \)-divisible formal groups, we will need only the special case of 6.4 when \( f(u, v) \in k[[u_1, \ldots, u_r, v_1, \ldots, v_b]] \), i.e. \( f(u, v) \) is a usual power series. It is interesting to know whether 6.4 holds without the assumption 6.4.1 on \( f(u, v) \). We do not know the answer.
(6.5) Proof of Proposition 6.4. Let

\[
L = \{ (i,j) | (i,j) \in \{1,\ldots,b \} \times (\text{supp}(m : E_2; C_2, d_2) \setminus \emptyset) \}
\]

be an infinite array of variables indexed by \( \{1,\ldots,b \} \times (\text{supp}(m : E_2; C_2, d_2) \setminus \emptyset) \), where \( \emptyset \) is the zero element of the support subset \( \text{supp}(m : E_2; C_2, d_2) \subset \mathbb{N}[1/p] \) defined in 6.3.1. For each \( i = 1,\ldots,b \),

\[
h_i(y) = \sum_{0 \neq K \in \text{supp}(m : E_2; C_2, d_2)} c_{i,K} y^K
\]

with \( c_{i,K} \in k \) for all \( J \in S(m : E_2; C_2, d_2) \setminus \{ \emptyset \} \). Let

\[
H_i(t;y) := \sum_{0 \neq K \in \text{supp}(m : E_2; C_2, d_2)} L_{i,K} y^K
\]

The assumption 6.4.1 implies that the composition

\[
f(g_1(x),\ldots,g_n(x),H_1(t;y),\ldots,H_1(t;y))
\]

is a well-defined formal series \( k((x_1^{p^{-\infty}},\ldots,x_m^{p^{-\infty}},y_1^{p^{-\infty}},\ldots,y_m^{p^{-\infty}}))_{E_3,b} \) whose support is contained in the product \( \text{supp}(m : E_3; C_3, d_3) \times \text{supp}(m : E_3; C_3, d_3) \):

\[
f(g(x),H(t;y)) = \sum_{(I,J) \in \text{supp}(m : E_3; C_3, d_3) \times \text{supp}(m : E_3; C_3, d_3)} A_{I,J}(t) x^I y^J
\]

Moreover each coefficient \( A_{I,J}(t) \) is an element in the perfection

\[
k\left[ t^{p^{-\infty}} \right] = k\left[ t_{i,K}^{p^{-\infty}} \right]_{i \in \{1,\ldots,b \}, K \in \text{supp}(m : E_2; C_2, d_2) \setminus \emptyset}
\]

of the polynomial ring

\[
k\left[ t^{p^{-\infty}} \right] = k\left[ t_{i,K} \right]_{i \in \{1,\ldots,b \}, K \in \text{supp}(m : E_2; C_2, d_2) \setminus \emptyset}
\]

in infinitely many variables \( t_{i,K} \). Clearly For every \( n \in \mathbb{N} \), we have

\[
f(g_1(x),\ldots,g_n(x),h_1(x)^q^n,\ldots,h_b(x)^q^n) = \sum_{I,J} A_{I,J}(c^{q^n}) x^I q^n y^J.
\]

In particular

\[
f(g_1(x),\ldots,g_n(x),h_1(x),\ldots,h_b(x)) = \sum_{I,J} A_{I,J}(c) x^I y^J.
\]

By assumption 6.4.2, we get

\[
\sum_{(I,J) \text{ s.t. } y^J \text{ s.t. } y^J < d_n} A_{I,J}(c^{q^n}) x^I q^n y^J = 0 \quad \forall n \geq n_0.
\]

We want to show that \( A_{I,J}(c) = 0 \) for all \((I,J) \in \text{supp}(m : E_3; C_3, d_3) \times \text{supp}(m : E_3; C_3, d_3) \). Suppose to the contrary that \( A_{I_0,J_0}(c) \neq 0 \) for some \((I_0,J_0) \in \text{supp}(m : E_3; C_3, d_3) \times \text{supp}(m : E_3; C_3, d_3) \).
By lemma 6.3.3, there exists infinitely many natural numbers \( n \) such that \( A_{I_0J_0}(\xi^n) \neq 0 \). Define a subset \( T \subseteq \text{supp}(m : E_3; C_3, d_3) \times \text{supp}(m : E_3; C_3, d_3) \) by

\[
T := \left\{ (I, J) : I, J \in \text{supp}(m : E_3; C_3, d_3), A_{I,J}(\xi^n) \neq 0 \text{ for infinitely many } n \in \mathbb{N} \right\}.
\]

This set \( T \) is non-empty because it contains \((I_0, J_0)\). Again by lemma 6.3.3 we know that

\[
A_{I,J}(\xi^n) = 0 \quad \forall n \in \mathbb{Z} \text{ if } (I, J) \notin T,
\]

and equation 6.5.5 becomes

\[
\sum_{(I,J) \in T \text{ s.t. } |I + q^n J|_\sigma < d_n} A_{I,J}(\xi^n) \xi^{I + q^n J} = 0 \quad \forall n \geq n_0.
\]

Let

\[
M_2 := \min \{|J|_\sigma : (I, J) \in T\}
\]

and let

\[
M_1 := \min \{|I|_\sigma : (I, J) \in T \text{ and } |J|_\sigma = M_2\}.
\]

The minimum defining \( M_2 \) (respectively \( M_1 \)) exists because every subset \( \text{supp}(m : E_3; C_3, d_3) \) whose archimedean norm is bounded above is a finite set. This finiteness property for \( \text{supp}(m : E_3; C_3, d_3) \) also implies that there exists a positive number \( \varepsilon_2 > 0 \) such that

\[
J \in \text{supp}(m : E_3; C_3, d_3) \quad \text{and} \quad |J|_\sigma > M_2 \implies |J|_\sigma > M_2 + \varepsilon_2.
\]

The subset

\[
T_1 := \{ (I, J) \in T : |J|_\sigma = M_2, |I|_\sigma = M_1 \}
\]

is a non-empty finite set. There exists a natural number \( n_1 \geq n_0 \) such that properties 6.5.7–6.5.9 below hold.

\[
M_1 + q^n M_2 < d_n - 2 \quad \forall n \geq n_1, n \in \mathbb{N}
\]

\[
q^n \cdot \varepsilon_2 > M_1 \quad \forall n \geq n_1, n \in \mathbb{N}
\]

\[
(I_1, J_1), (I_2, J_2) \in T_1, \quad I_1 + q^n J_1 = I_2 + q^n J_2 \quad \text{and} \quad n \geq n_1 \implies (I_1, J_1) = (I_2, J_2)
\]

Consider the set

\[
S_n := \{ (I, J) \in T : |I + q^n J|_\sigma = M_1 + q^n M_2 \}.
\]

The property 6.5.8 and the inequality 6.5.6 implies that \( S_n = T_1 \) for all \( n \geq n_1 \). Because \( S_n = T_1 \), when we examine terms of total degree \( M_1 + q^n M_2 \) in equation 6.5.5, we find that

\[
\sum_{(I,J) \in T_1} A_{I,J}(\xi^n) \xi^{I + q^n J} = 0 \quad \forall n \geq n_1.
\]

By property 6.5.9 and equation 6.5.9, we see that

\[
A_{I,J}(\xi^n) = 0
\]

for all \((I, J) \in T_1\) and all \( n \geq n_1 \), therefore \( T_1 \) is the empty set. This is a contradiction. We have proved proposition 6.4. \( \square \)
Remark. (a) The assumption 6.4.1 on the support of \( f(u, v) \) implies the uniform bound 6.5.1 on the support of the composition \( f(g_1(x), \ldots, g_{n}(x), H_1(t; y), \ldots, H_{p}(t; y)) \). This observation allows us to take advantage of the finiteness property of the support set \( \text{supp}(m; E_3; C_3, d_3) \). The rest of the argument in the proof of 6.4 is identical with the proof of [3, 3.1].

(b) Our proof is not strong enough to show that 6.4 holds for every element \( f \) in \( k\langle u^{p^{-\infty}}, u^{p^{-m}} \rangle^{E_1, b}_{C_1; d_1} \). But we don’t have a counter-example either. It will be interesting if one can find a larger class of formal series \( f(u, v) \) in \( k\langle u^{p^{-\infty}}, u^{p^{-m}} \rangle^{E_1, b}_{C_1; d_1} \) for which the statement 6.4 holds.

§7. Rigidity results for biextensions

(7.1) Notation and basic setup. In this section \( k \) is a perfect field of characteristic \( p > 0 \).

(i) Let \( X, Y, Z \) be \( p \)-divisible formal groups, and \( \pi : E \rightarrow X \times_{\text{Spec}(k)} Y \) is a biextension of \( X \times_{\text{Spec}(k)} Y \) by \( Z \).

(ii) Let \( G \) be a compact \( p \)-adic Lie group. Let \( (\rho, \alpha, \beta, \gamma) \) be an action of \( G \) on the biextension \( E \rightarrow X \times Y \), where \( \rho : G \rightarrow \text{Aut}_{\text{biext}}(E \times X \times Y) \) is a continuous homomorphism, and \( \alpha : G \rightarrow \text{Aut}(X) \) (respectively \( \beta : G \rightarrow \text{Aut}(Y) \), \( \gamma : G \rightarrow \text{Aut}(Z) \)) is the action of \( G \) on \( X \) (respectively \( Y, Z \)) underlying \( \rho \). We know from 2.6.4 that the group homomorphism

\[
(\alpha, \beta, \gamma) : G \longrightarrow \text{Aut}(X) \times \text{Aut}(Y) \times \text{Aut}(Z)
\]

is a closed embedding of compact \( p \)-adic Lie groups, and the induced map

\[
(d\alpha, d\beta, d\gamma) : \longrightarrow \text{Lie}(G) \otimes \text{End}(X)_{\mathbb{Q}} \oplus \text{End}(Y)_{\mathbb{Q}} \oplus \text{End}(Z)_{\mathbb{Q}}
\]

is an injection of finite dimensional vector spaces over \( \mathbb{Q}_p \). We often use the map \( (\alpha, \beta, \gamma) \) to \( G \) with a subgroup of \( \text{Aut}(X) \times \text{Aut}(Y) \times \text{Aut}(Z) \), and regard \( \text{Lie}(G) \) as a \( \mathbb{Q}_p \)-vector subspace of \( \text{Lie}(G) \otimes \text{End}(X)_{\mathbb{Q}} \oplus \text{End}(Y)_{\mathbb{Q}} \oplus \text{End}(Z)_{\mathbb{Q}} \).

(iii) Let \( W \subset E \) be a formal subvariety of \( E \), in the sense that there exists a prime ideal \( I_W \) of the coordinate ring \( R_E \) of \( E \) such that \( W = \text{Spf}(R_E/I_W) \). Assume that \( W \) is stable under the action of \( G \).

(iv) The formal subvariety \( V = \text{Spf}(R_{X \times \text{Spec}(k) Y / (I_W \cap R_X \times Y)}) \subseteq X \times_{\text{Spec}(k)} Y \) will be said to be the image of \( W \) in \( X \times_{\text{Spec}(k)} Y \).

(7.2) Theorem. Let \( W \) be a formal subvariety of \( E \) stable under the action of \( G \). Let \( \mu_1 \) be the largest slope of \( Z \). Let \( Z_1 \) be the maximal \( p \)-divisible subgroup of \( Z \) which is isoclinic of slope \( \mu_1 \). Let \( Z_2 \) be the largest among all isoclinic \( p \)-divisible subgroups of \( Z \) with slope \( \mu_1 \) which are contained in \( W \). Let \( \Upsilon_{Z_2} : Z_2 \times E \rightarrow E \) be the morphism

\[
\Upsilon : Z_2 \times E \rightarrow E \quad (z_2, e) \mapsto z_2 \ast e,
\]

corresponding to the restriction to \( Z_2 \) of the action of \( Z \) on \( E \). Let \( v = (A, B, C) \in \text{Lie}(G) \) be an element of the Lie algebra of \( G \) such that \( A \in \text{End}(X), B \in \text{End}(Y) \) and \( C \in \text{End}(Z) \).
(1) Assume that $\mu_1$ is strictly bigger than every slope of $X$ and every slope of $Y$. Then

$$\{Y \circ (C|_{Z_2} \times \text{id}_W)\}(Z_2 \times W) \subseteq W.$$

In other words the formal subvariety $W \subseteq E$ is stable under translation by the $p$-divisible subgroup $C(Z_2)$ of $Z$.

(2) Assume in addition that the action of $G$ on $Z_2$ is strongly non-trivial. Then

$$Y(Z_2 \times W) \subseteq W.$$

(7.2.1) $7.2(1) \implies 7.2(2)$.

The assumption that the action of $G$ on $Z_2$ is strongly non-trivial implies that there exists elements $v_{ij} = (A_{ij}, B_{ij}, C_{ij}) \in \text{Lie}(G)$, indexed by a finite subset

$$\{(i, j) \in \mathbb{N}^2 : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i\}\},$$

where $n_i \in \mathbb{N}_{\geq 1}$ for each $i = 1, \ldots, m$, such that

$$\sum_{1 \leq i \leq m} C_{i,1}|_{Z_2} \circ \cdots \circ C_{i,n_i}|_{Z_2} \in \text{End}(Z_2)_\mathbb{Q}^\times.$$

Here $C_{i,j} \in \text{End}(Z_2)_\mathbb{Q}$ stands for the restriction to $Z_2$ of the element $C_{i,j} \in \text{End}(Z)_\mathbb{Q} = \text{End}(Z) \otimes \mathbb{Q}$. See [3, 4.1.1] for this lemma on representation theory. The statement (2) follows from statement (1) and the above linear algebra consequence of the assumption that $G$ operates strongly non-trivially on $Z_2$. The statement (1) will be proved in 7.2.3.

(7.2.2) The main ingredients in the proof of the statement (1) have been developed in previous sections. Here is a brief summary of how these ingredients come into the proof

(i) The analysis of the action on $E$ of $\psi(\exp(p^{na}v))$ for elements $\exp(p^{na}v) \in G$ close to the identity element of $G$ in 5.7, where $v = (A, B, C) = (d\alpha(v), d\beta(v), d\gamma(v))$ is an element of the Lie algebra $\text{Lie}(G)$ of $G$. This analysis says that the difference between $\psi(\exp(p^{na}v))$ is represented modulo $m_E^{p^{ns}}$ by a linearized “main term”

$$C|_{Z_1} \circ \rho_{na} \quad (\text{mod } m_E^{p^{ns}})$$

for natural numbers $n \gg 0$. The map

$$\rho_{na} : E \mod m_E^{p^{ns}} \to Z_1 \mod m_E^{p^{ns}}$$

comes from the composition of the map

$$\eta_{na} \mod m_E^{p^{ns}} : E \mod m_E^{p^{ns}} \to Z \mod m_Z^{p^{ns}}$$

declared in 2.7 with the projection $\text{pr}_{Z_1} : Z \to Z_1$ from $Z$ to its factor $Z_1$. The phenomenon is that the main term shows up already at the level of $m_E^{c,p^{ns}}$, where $\frac{a}{r} = \mu_1$ and $s > r$, while the error term lies in $m_E^{p^{ns}}$. Thus the error term goes to 0 at a rate much faster than the rate at which the main term does. It is tempting to replace “much faster” by “doubly exponential” in the preceding sentence in order to better convey the comparison. At any rate, this family of congruences relations for $\psi(\exp(p^{na}v))$ can be regarded as a reasonable substitute in the attempt to differentiate the function $t \mapsto \psi(\exp(tv))$ for $t$ in a small neighborhood of 0 in $\mathbb{Z}_p$.  

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(ii) In order to analyse the compatible family of maps $\rho_n$ defined in 3.3.2, one is naturally led to the notion of complete restricted perfection of equi-characteristic-$p$ complete Noetherian local domains in §3. When applied to the coordinate ring $(R_E, \mathfrak{m}_E)$ of a biextension $E$, this procedure produces many of mutually related of rings, depending on parameters; each one is a completion of some suitable subring of the perfection of $R_E$. After picking a good regular system of parameters $(z_1, \ldots, z_d)$ for the coordinate ring of $Z_1$, the compatible family of maps $\rho_n$’s is controlled by a $b$-tuple $h_1, \ldots, h_b$ of elements of a suitable complete restricted perfection $\tilde{R}_E$ of $R_E$, so that $\rho_{na}$ is represented by $(h_1^{pa}, \ldots, h_b^{pa})$ for all large $n$’s. Restricting to a formal subvariety $W$ of $E$, one gets a $d$-tuple of elements of $\tilde{R}_W$, where $\tilde{R}_W$ is a complete restricted perfection of the coordinate ring $R_W$ of $W$.

(iii) Let $\Delta_2 : R_E \to R_{Z_2} \hat{\otimes} R_E$ be the ring homomorphism corresponding to the translation action $Z_2 \times E \to E$ of $Z_2$ on $E$, and let $C|_{Z_2} : R_{Z_2} \to R_{Z_2}$ be the ring endomorphism corresponding to the restriction to $Z_2$ of $C \in \text{End}(Z)$.

To show that a $G$-invariant formal subvariety $W$ is stable under translation by $C|_{Z_2}(Z_2)$, one needs to show that every element $f$ in the prime ideal $I_W$ for $W$ is sent to an element of $R_{Z_2} \cdot I_W$ under $(C|_{Z_2} \otimes 1_{R_E}) \circ \Delta_2$, or equivalently, the element $(C|_{Z_2} \otimes 1_{R_E})(\Delta_2(f))$ is mapped to 0 under the natural surjection $R_{Z_2} \hat{\otimes} R_E \to R_{Z_1} \hat{\otimes} R_E$. Following [3], our strategy is to try to show that a suitable element $\tilde{f} \in \tilde{R}_W \hat{\otimes} \tilde{R}_W$ attached to $\Delta(f_1)$ is zero.

(iv) To simplify the algebra involved, one deploys an easy form of local uniformization and embed $R_W$ in a formal power series ring $S$. The situation then is that we have an element $f(u_1, \ldots, u_a, v_1, \ldots, v_b) \in \hat{S} \hat{\otimes} \hat{S}$ attached to $f_1$, and all we have going for us is a family of congruences of the form

$$f(g_1(x), \ldots, g_a(x), h_1(x)^{pa}, \ldots, h_b(x)^{pa}) \equiv 0 \pmod{(x)^{p^{(s-t)}-c}}$$

for all $n \gg 0$, with integer constants $s > r > 0, c > 0$, where $x = (x_1, \ldots, x_m)$ are variables for the power series ring $S$. These congruences are consequences of the analysis of $\psi(\exp(p^{na}v))$ in (i) above.

At this point the identity principle 6.4 comes to the rescue. It says that the above family of congruences implies that the function $f(g_1(x), \ldots, g_a(x), h_1(y), \ldots, h_b(y)$ in two sets of variables $x_1, \ldots, x_m, y_1, \ldots, y_m$ is identically equal to 0.

(7.2.3) **Proof of Theorem 7.2.**

1. Preliminary reduction steps.

(a) It suffices to verify the statement of 7.2 after extending the base field $k$ to an algebraic closure of $k$. So we may and do assume that $k$ is algebraically closed.

(b) If $E \to E'$ is an isogeny of biextensions, the statement of 7.2 holds for $E$ if and only if it holds for $E'$. Modifying $E$ by suitable isogenies, we may and do assume that $X, Y, Z$ are product of isoclinic $p$-divisible groups. Moreover may and do assume that there exist positive integers $a, r$ such that $a = \frac{a}{r}$ and $Z_1[p^a] = \text{Fr}_{Z_1}$.
(c) Choose a suitable regular system of parameters \((u_1, \ldots, u_b)\) for the coordinate ring \(Z_1\) such that \(Z_1 = \text{Spf}(k[[u_1, \ldots, u_b]])\) and 
\[
[p^r]_i(u_i) = u_i^{c_1}
\]
for \(i = 1, \ldots, b\).

(d) The largest slope \(\mu_1\) of \(Z\) is assumed to be bigger than every slope appearing in \(X \times Y\). Multiplying \(a, r\) by a suitable positive integer, we may and do assume that there exists positive integers \(s, n_0\) such that \(s > r\) and \(\frac{s}{r}\) is strictly bigger than every slope of \(X \times Y\), and 
\[
X[p^{na}] \supset \ker(\text{Fr}^{ns}_{X/k}) \quad \text{and} \quad Y[p^{na}] \supset \ker(\text{Fr}^{ns}_{Y/k})
\]
for all \(n \geq n_0\).

(e) Let \(R_W = R_E/I_W\) be the coordinate ring of \(W\), where \(I_W\) is the prime ideal of \(R_E\) corresponding to \(W\). By [3, 2.1], there exists a \(k\)-linear injective local homomorphism 
\[
t : R_W = R_E/I_W \leftrightarrow k[[t_1, \ldots, t_m]]
\]
from \(R_W\) to a formal power series ring in \(m\) variables, where \(m = \dim(R_W)\).

2. By 5.7, there exist positive integers \(n_4 \geq n_0\) and \(c_4\) such that

\[\psi(\exp(p^{na}v)) \equiv (C \mid Z_1 \circ \rho_{na} \ast \text{id}_{E \mod m} \pmod{p^{\min(s, 2nr - c_4)}})\]

for all \(n \geq n_4\), where 
\[
\rho_{na} = (\text{pr}_{Z_1} \circ \eta_{na}) \bigg|_{\pi^{-1}(\ker(\text{Fr}^{ns}_{X} \times \ker(Fr^{ns}_{Y})))} : \pi^{-1}(\ker(\text{Fr}^{ns}_{X} \times \ker(Fr^{ns}_{Y})) \rightarrow Z_1
\]
is the restriction to \(\pi^{-1}(\ker(\text{Fr}^{ns}_{X} \times \ker(Fr^{ns}_{Y})) of the composition of \(\eta_{na}\) with the projection 
\[
\text{pr}_{Z_1} : Z \rightarrow Z_1.
\]

For each \(j = 1, \ldots, b\), defined a \(\phi^r\)-compatible sequence \((a_{j,n})_{n \geq n_4}\) by 
\[
a_{j,n} := \rho_{na}^*(u_j) \in R_E/m^{(p^{na})}_E
\]
for all \(n \geq n_4\). Let \(i_1 := \max(s - r, \lceil \frac{n_4}{r} \rceil)\). For each \(j = 1, \ldots, b\), let 
\[
\tilde{a}_j \in (R_E, m^\text{perf.}_E)_{s: \phi^r; [i_1]}
\]
be the formal series corresponding to the \(\phi^r\)-compatible sequence \((a_{j,n})_{n \geq n_4}\).

Although \((R_E, m^\text{perf.}_E)_{s: \phi^r; [i_1]}\) is more tightly related to \(\phi^r\)-compatible sequences through the construction in 3.5, we will pass to the larger ring \((R_E, m^\text{perf.}_E)_{s: \phi^r; [i_1]}\), and consider the \(\tilde{a}_j\)'s as elements of \((R_E, m^\text{perf.}_E)_{s: \phi^r; [i_1]}\) in the rest of the proof.
3. The elements \( \tilde{a}_1, \ldots, \tilde{a}_b \in (R_E, m_E)_{s; \phi'; [1]}^{\text{perf}, \#} \) defines a ring homomorphism

\[
\tilde{\eta} : R_{Z_1} = k[[u_1, \ldots, u_b]] \longrightarrow (R_E, m_E)_{s; \phi'; [1]}^{\text{perf}, \#}.
\]

Let

\[
\omega_1 : (R_E, m_E)_{s; \phi'; [1]}^{\text{perf}, b} \longrightarrow (R_{Z_1}, m_{Z_1})_{s; \phi'; [1]}^{\text{perf}, b}
\]

be the ring homomorphism induced by the inclusion \( Z_1 \hookrightarrow E \). Because the restriction to \( Z \) of the morphism \( \eta : \pi^{-1}(X[p^n] \times Y[p^n]) \rightarrow Z \) is equal to \([p^n]_{Z}\) for every \( n \in \mathbb{N} \), we see that

(7.2.3.2) \[
\omega_1 \circ \tilde{\eta} = j_{R_{Z_1}}
\]

where \( j_{R_{Z_1}} : R_{Z_1} \hookrightarrow (R_{Z_1}, m_{Z_1})_{s; \phi'; [1]}^{\text{perf}, b} \) is the natural injection from \( R_{Z_1} \) to its complete restricted perfection \( (R_{Z_1}, m_{Z_1})_{s; \phi'; [1]}^{\text{perf}, b} \).

5. We also have the following ring homomorphisms.

(a) The canonical homomorphism \( R_E \rightarrow R_E/I_W = R_W \) gives rise to a homomorphism

\[
\tau : (R_E, m_E)_{s; \phi'; [1]}^{\text{perf}, b} \longrightarrow (R_W, m_W)_{s; \phi'; [1]}^{\text{perf}, b}.
\]

(b) The injective local homomorphism \( \iota : R_W \rightarrow k[[t_1, \ldots, t_m]] \) induces a injective continuous homomorphism

\[
\tilde{\iota} : (R_W, m_W)_{s; \phi'; [1]}^{\text{perf}, b} \longrightarrow k\langle \langle t_1^{-\infty}, \ldots, t_m^{-\infty} \rangle \rangle_{s; \phi'; [1]}^{\text{perf}, b}.
\]

(c) Continuous ring homomorphisms

\[
\Delta_1 : R_E \rightarrow R_{Z_1} \hat{\otimes} R_E \quad \text{and} \quad \Delta_2 : R_E \rightarrow R_{Z_2} \hat{\otimes} R_E
\]

corresponding to the actions \( Z_1 \times E \rightarrow E \) and \( Z_2 \times E \rightarrow E \) of \( Z_1 \) and \( Z_2 \) on \( E \).

(d) The ring homomorphism

\[
\omega_2 : (R_W, m_W)_{s; \phi'; [1]}^{\text{perf}, b} \longrightarrow (R_{Z_2}, m_{Z_2})_{s; \phi'; [1]}^{\text{perf}, b}
\]

induced by the surjective ring homomorphism \( R_W \rightarrow R_{Z_2} \) which corresponds to the inclusion \( Z_2 \hookrightarrow W \).

(e) The ring endomorphisms \( C|_{Z_1}^2 : R_{Z_1} \rightarrow R_{Z_1} \) and \( C|_{Z_2}^2 : R_{Z_2} \rightarrow R_{Z_2} \) corresponding to the endomorphisms \( C|_{Z_1} \) (respectively \( C|_{Z_2} \)) of the \( p \)-divisible group \( Z_1 \) (respectively \( Z_2 \)).

(f) The ring homomorphism

\[
\tilde{q} : (R_{Z_1}, m_{Z_1})_{s; \phi'; [1]}^{\text{perf}, b} \longrightarrow (R_{Z_2}, m_{Z_2})_{s; \phi'; [1]}^{\text{perf}, b}
\]

induced by the canonical surjection \( q : R_{Z_1} \twoheadrightarrow R_{Z_2} \).
Clearly we have

\[(7.2.3.3) \quad \omega_2 \circ \tau = \tilde{q} \circ \omega_1 \quad \text{and} \quad C|Z_2^* \circ q = q \circ C|Z_1^*
\]

The following diagram

\[(7.2.3.4) \quad \begin{array}{ccc}
R_{Z_1} & \xrightarrow{\tilde{\eta}} & (R_E, m_E)_{x: \phi^r_i; [1]} \xrightarrow{\tau} (R_W, m_W)_{x: \phi^r_i; [1]} \\
\downarrow \omega_1 & = & \downarrow \omega_2 \\
R_{Z_1} & \xrightarrow{j_{RZ_1}} & (R_{Z_1}, m_{Z_1})_{x: \phi^r_i; [1]} \xrightarrow{q} (R_{Z_2}, m_{Z_2})_{x: \phi^r_i; [1]}
\end{array}
\]

commutes by 7.2.3.2. It follows that the diagram

\[(7.2.3.5) \quad \begin{array}{ccc}
R_E & \xrightarrow{=} & R_E \\
\Delta_1 & & \Delta_2 \\
R_{Z_1} \otimes R_E & \xrightarrow{q \otimes 1} & R_{Z_1} \otimes R_W \\
\downarrow C|Z_1^* \otimes 1 & & \downarrow C|Z_1^* \otimes 1 \\
R_{Z_1} \otimes R_E & \xrightarrow{1 \otimes 1} & R_{Z_1} \otimes R_W \\
\downarrow \tilde{\eta} \otimes 1 & & \downarrow \tilde{\eta} \otimes 1 \\
(R_E, m_E)^{\text{perf}, b}_{x: \phi^r_i; [1]} \otimes (R_E, m_E)^{\text{perf}, b}_{x: \phi^r_i; [1]} & \xrightarrow{\omega_1 \otimes \tau} & (R_{Z_1}, m_{Z_1})^{\text{perf}, b}_{x: \phi^r_i; [1]} \otimes (R_{Z_2}, m_{Z_2})^{\text{perf}, b}_{x: \phi^r_i; [1]} \\
\uparrow \tau \otimes 1 & & \uparrow \tilde{q} \otimes 1 \\
(R_W, m_W)^{\text{perf}, b}_{x: \phi^r_i; [1]} \otimes (R_W, m_W)^{\text{perf}, b}_{x: \phi^r_i; [1]} & \xrightarrow{\omega_2 \otimes 1} & (R_{Z_2}, m_{Z_2})^{\text{perf}, b}_{x: \phi^r_i; [1]} \otimes (R_{Z_2}, m_{Z_2})^{\text{perf}, b}_{x: \phi^r_i; [1]}
\end{array}
\]

also commutes.

6. Recall that \(I_W\) is the prime ideal of the coordinate ring of \(E\) consisting of all functions on \(E\) which vanishes on the \(G\)-invariant formal subvariety \(W \subset E\). We want to show that

(A) \(\left(C|Z_2^* \times 1_{R_E}\right) \circ \Delta_2(f) = 0 \quad \forall f \in I_W\)

We know from 7.2.3.5 that

\[
(C|Z_2^* \times 1_{R_E}) \circ \Delta_2(f) = \left((q \otimes 1) \circ (C|Z_1^* \times 1_{R_E}) \circ \Delta_1\right)(f).
\]

Because \(j_{RZ_2}\) and \(j_{RW}\) are both injective, our goal 6A is to equivalent to

(B) \(\left((j_{RZ_2} \otimes j_{R_W}) \circ (q \otimes 1) \circ (C|Z_1^* \times 1_{R_E}) \circ \Delta_1\right)(f) = 0 \quad \forall f \in I_W\)

What we will show is a stronger statement

(C) \(\left((\tau \otimes \tilde{\eta}) \circ (C|Z_1^* \times 1_{R_E}) \circ \Delta_1\right)(f) = 0 \quad \forall f \in I_W\)
In other words, the composition of the four vertical arrows at the left edge of the diagram 7.2.3.5 kills every element of the prime ideal $I_W$. It follows immediately from the commutative diagram 7.2.3.5 that

$$(C) \implies (B) \iff (A).$$

It remains to prove (C).

7. Suppose that $f$ is an element of $I_W$. Define an element $	ilde{f} \in (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \otimes (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]}$ by

$$
\tilde{f} := ((j_{R_Z} \otimes j_{R_{Z_1}}) \circ (q \otimes 1) \circ (C|_{Z_1} \times 1_{R_E}) \circ \Delta_1)(f).
$$

Let $\phi$ be the Frobenius endomorphism $x \mapsto x^p$ on $(R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]}$. Let

$$
\nu_W: (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \otimes (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \longrightarrow (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]}
$$

be map which defines multiplication for the ring $(R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]}$. Geometrically $\nu_W$ corresponds to the diagonal morphism from $\text{Spec}(R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]}$ to its self-product.

Because the formal subvariety $W \subset E$ is assumed to be stable under $G$, therefore stable under $\psi(\exp p^{n_r}v)$ for all $n \geq n_4$. Hence the congruence property 7.2.3.1 implies that under the homomorphism

$$
\phi^{n_r} \otimes 1: (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \otimes (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \longrightarrow (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \otimes (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]},
$$

we have

$$
(\phi^{n_r} \otimes 1)(\tilde{f}) \equiv 0 \pmod{\text{Fil}^{n_s-i_1}} \quad \forall n \geq n_4.
$$

8. We claim that the family of congruence conditions 7.2.3.6 satisfied by $\tilde{f}$ implies that $\tilde{f} = 0$. By 4.2.4, the homomorphism $\bar{i}: (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \rightarrow k\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{x, \varphi^r; [i_1]}$ induced by $i: R_W \hookrightarrow k[[t_1, \ldots, t_m]]$ is also an injection. Therefore it suffices to show that the injective map

$$
\bar{i} \otimes \bar{i}: (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \otimes (R_W, m_W)^{\text{perf}, b}_{x, \varphi^r; [i_1]} \longrightarrow k[[t_1, \ldots, t_m]] \otimes k[[t_1, \ldots, t_m]]
$$

sends $\tilde{f}$ to $0$. Under the ring homomorphism $\bar{i} \otimes \bar{i}$ the family of congruence conditions 7.2.3.6 for $\tilde{f}$ is transformed into a similar family of congruence conditions for the element $(\bar{i} \otimes \bar{i})(\tilde{f}) \in k[[t_1, \ldots, t_m]] \otimes k[[t_1, \ldots, t_m]]$. Let $E = \frac{t}{s-r}$, choose suitable parameters $C > 1$, $d \geq 0$ so that the ring $k\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{x, \varphi^r; [i_1]}^{\text{perf}, b}$ is contained in $k\langle t_1^{p^{-\infty}}, \ldots, t_m^{p^{-\infty}} \rangle_{C, d}^{\text{perf}, b}$. Apply the identity principle 6.4, we conclude that $(\bar{i} \otimes \bar{i})(\tilde{f}) = 0$. Hence $\tilde{f} = 0$. We have proved the claim and concludes the proof of the statement (C). We have seen the (C) implies (A), which is exactly the statement of 7.2.4.

(7.2.4) **Corollary.** Let $W$ be a formal subvariety of $E$ stable under the action of $G$. Let $Z_1$ be the largest $p$-divisible subgroup of $Z$ contained in $W$. Let

$$
\nu_{Z_1}: Z_1 \times \text{Spec}(k) \rightarrow E
$$

be the translation action of $Z_1$ on $E$. Assume that every slope of $Z_1$ is strictly larger then every slope of $X \times Y$ and the action of $G$ on $Z_1$ is strongly non-trivial. Then

$$
\nu_{Z_1}(Z \times \text{Spec}(k)) \subset W.
$$
PROOF. We may and do assume that $X_1 = X$ and $Y_1 = Y$. We prove 7.2.4 by induction on dim$(Z_1)$. If $Z_1$ is isoclinic, then $Z_2 = Z_1$ and 7.2.4 follows immediate from 7.2. Let $\zeta : Z \to Z/Z_2$ be the canonical map from $Z$ to the quotient $p$-divisible group $Z/Z_2$. Let $\zeta : E \to \zeta, E =: \overline{E}$ be the canonical map from $E$ to its push-forward by $\zeta$, which is naturally identified with the map “quotient by the $p$-divisible group $Z_2$”. Let $W$ be the image of $W$ in $E$ under $\overline{\zeta}$. By induction, the formal subvariety $\overline{W}$ in $\overline{E}$ is stable under the natural action of $Z_1/Z_2$. It follows that $W$ is stable under the action of $Z_1$. 

(7.3) PROPOSITION. Let $W$ be a formal subvariety of $E$ stable under the action of $G$. Let $Z_1$ be the largest $p$-divisible subgroup of $Z$ contained in $W$. We make the following assumptions.

(i) The $p$-divisible groups $X$ and $Y$ have no slope in common.

(ii) Every slope of the $p$-divisible group $Z_1$ is strictly bigger every slope of $X \times Y$.

(iii) The action of $G$ on $E$ is strictly non-trivial.

Then $W$ is a sub-biextension of $E$. In other words there exists a $p$-divisible subgroup $X_1$ of $X$, a $p$-divisible subgroup $Y_1$ of $Y$, a $p$-divisible subgroup $Z_1$ of $Z$, such that

- $\pi(W) \subset X_1 \times Y_1$,
- $W$ is stable under the two relative group laws $+_1$ and $+_2$,
- $W$ is stable under translation by $Z_1$,
- the morphism $\pi|_W : W \to X_1 \times Y_1$ is formally smooth,
- $\pi|_W : W \to X_1 \times Y_1$ gives $W$ a natural structure as a $Z_1$-torsor over $X_1 \times Y_1$.

PROOF. The image $V$ of $W$ in $X \times_{\text{Spec}(k)} Y$ is a $p$-divisible subgroup of $X \times_{\text{Spec}(k)} Y$, by local rigidity for $p$-divisible groups [3, Thm. 4.3]. Because the $p$-divisible formal groups $X$ and $Y$ do not have common slopes, there exists $p$-divisible subgroups $X_1 \subset X$ and $Y_1 \subset Y$ such that $V = X_1 \times_{\text{Spec}(k)} Y_1$. Again by local rigidity for $p$-divisible groups, the intersection $(W \cap Z)$ with reduced structure is a $p$-divisible subgroup $Z_3$ of $Z$. We also know that

$$\dim(Z_3) \geq \dim(W) - \dim(X_1) - \dim(Y_1)$$

from basic commutative algebra.

By 7.2.4, $W$ is stable under the action of $Z_3$. The quotient $\overline{E} := E/Z_3$ has a natural structure as a biextension of $X \times_{\text{Spec}(k)} Y$ by $\overline{Z} := Z/Z_3$. The quotient $\overline{W} := W/Z_3$ is a formal subvariety of $\overline{E}$, stable under the action of $G$. Its dimension $\dim(\overline{W})$ is equal to $\dim(W) - \dim(Z_3)$. The image over $\overline{W}$ in $X \times_{\text{Spec}(k)} Y$ is $X_1 \times_{\text{Spec}(k)} Y_1$, hence $\dim(\overline{W}) \geq \dim(X_1) + \dim(Y_1)$. Combined with the displayed inequality in the previous paragraph, we see that $\dim(\overline{W}) = \dim(X_1 \times Y_1)$. On the other hand the closed fiber of the morphism $\overline{q} : \overline{W} \to X_1 \times Y_1$ is zero-dimensional with exactly one point. Hence the morphism $\overline{q}$ of formal schemes, which corresponds to a local homomorphism between complete noetherian $k$-algebras, is finite and purely inseparable.

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Let \( \pi_1 : \overline{E}_1 \to X_1 \times Y_1 \) be the restriction to \( X_1 \times Y_1 \) of the biextension \( \overline{E} \to X \times Y \). Because \( X_1 \) and \( Y_1 \) are both \( p \)-divisible formal groups, the purely inseparable map \( \overline{q} \) is dominated by suitable isogenies: There exists an isogeny \( u : X_2 \to X_1 \), an isogeny \( v : Y_2 \to Y_1 \) and a morphism \( \xi : X_2 \times Y_2 \to \overline{W} \) such that \( \overline{q} \circ \xi = u \times v \).

Consider the pull-back \( E_2 := (u \times v)^* \overline{E}_1 \to X_2 \times Y_2 \) of the biextension \( \pi_1 : \overline{E}_1 \to X_1 \times Y_1 \). We know that the compact \( p \)-adic Lie group \( G \) operates on the biextension \( \overline{E}_1 \), and \( \overline{W} \) is stable under the action of \( G \). There exists a compact open subgroup \( G_2 \subset G \) which operates on the biextension \( \overline{E}_2 \), and the natural map \( h : \overline{E}_2 \to \overline{E}_1 \) is equivariant with respect to the inclusion \( G_2 \hookrightarrow G \). The morphism \( \xi : X_2 \times Y_2 \to \overline{W} \) defines a morphism \( \xi_2 : X_2 \times Y_2 \to \overline{E}_2 \) such that \( h \circ \xi_2 = \xi_1 \). It follows that

\[
(u \times v) \circ \xi_2 = \pi_1 \circ h \circ \xi_2 = \pi_1 \circ \xi_1 = u \times v.
\]

Therefore

\[
\pi_2 ^* \xi_2 = \text{id}_{X_2 \times Y_2}.
\]

In other words \( \xi_2 \) is a section of the biextension \( \overline{E}_2 \) of \( X_2 \times Y_2 \) by \( \overline{Z} \). Moreover \( \xi_2 \) is equivariant with respect to the action of \( G_2 \) on \( \overline{E}_2 \).

Because every slope of \( \overline{Z} \) is strictly bigger than every slope of \( X_2 \times Y_2 \), we know by 6.2 that the biextension \( \overline{E}_2 \) splits, and \( \xi_2 \) is its canonical splitting. So the biextension \( \overline{E}_1 \) also splits. The inverse image of the canonical splitting of \( \overline{E}_1 \) in \( \pi^{-1}(X_1 \times Y_1) \) is a biextension \( E' \) of \( X_1 \times Y_1 \) by \( Z_1 \), which is a sub-biextension of \( E \). This sub-biextension \( E' \) is contained in \( W \) and has the same dimension as \( W \), hence \( W \) is equal to \( E' \).

\[\square\]

**7.4** Proposition. Let \( W \) be a formal subvariety of \( E \) stable under the action of \( G \). Assume that the slope of \( X, Y \) are mutually distinct, \( Z \) is isoclinic, and the slope of \( Z \) is strictly bigger than every slope of \( X \times Y \). Assume also that the action of \( G \) on \( E \) is strongly non-trivial in the sense of 2.6.5. Then \( W \) is a sub-biextension of \( E \).

**Proof.** We will first perform some reduction steps.

**Reduction step 1.** Being a sub-biextension means that \( W \) is stable under the two relative group laws \(+_1 \) and \(+_2 \), which can be verified after extending the base field \( k \) to an algebraic closure of \( k \). So we may and do assume that \( k \) is algebraically closed.

**Reduction step 2.** It is easy to see that the statement 7.4 does not change under isogeny: if \( E' \to E \) and \( E \to E'' \) are isogenies of biextensions, then the statement 7.4 for \( E \) is equivalent to the statement for \( E' \) and also equivalent to the statement for \( E'' \). Changing \( X, Y, Z \) by isogeny if necessary, we may and do assume that both \( X \) and \( Y \) are product of isoclinic \( p \)-divisible subgroups. Moreover if \( U \) is the maximal isoclinic \( p \)-divisible subgroup for a given slope in one of the three \( p \)-divisible groups \( X, Y, \) or \( Z \), then there exists positive integers \( c, d > 0 \) such that \( U[p^c] = \text{Ker}[\text{Fr}_U]/k] \).

**Reduction step 3.** We may and do assume that the image of \( W \) in \( X \times Y \) is equal to \( X \times Y \).

**Claim.** The statement 7.4 holds under the additional assumptions in reduction steps 1–3.
We have seen that 7.4 follows from the claim. We will prove the claim by induction on the number of slopes of $X \times Y$. Denote this number by $\#\text{slopes}(X \times Y)$.

**The case when $\#\text{slopes}(X \times Y) = 2$, namely both $X$ and $Y$ are isoclinic.** This is the initial step of the induction.

Suppose first that $\text{slope}(X) + \text{slope}(Y) \neq \text{slope}(Z)$, then the $W(k)$-bilinear pairing

$$\Theta_E : M_*(X) \times M_*(Y) \to M_*(Z)$$

is identically zero. Then the biextension $E \to X \times Y$ splits by 2.6.3, and the canonical splitting of $E$ induces a natural isomorphism $h$ from $E$ to the trivial biextension $X \times Y \times Z$. The trivial biextension $X \times Y \times Z$ has a natural structure as a $p$-divisible group, and every automorphism of this trivial biextension is also an automorphism for the $p$-divisible group structure. Apply the local rigidity for $p$-divisible formal groups [3, 4.3], we conclude that the $G$-stable formal subvariety $W$ of $E \cong X \times Y \times Z$ is of the form $Z_1 \times X \times Y$ for a $p$-divisible subgroup of $Z$. Therefore $W$ is a sub-biextension of $E$.

Assume now that $\text{slope}(X) + \text{slope}(Y) = \text{slope}(Z)$. Recall that both $X$ and $Y$ are $p$-divisible formal groups, hence $\text{slope}(X) > 0$ and $\text{slope} = 0$ and 7.2.4 applies. We have proved the claim when both $X$ and $Y$ are isoclinic.

**The induction step.** Suppose that $\#\text{slopes}(X \times Y) = m_0$, $m_0 \geq 3$, and that the claim holds whenever $\#\text{slopes}(X \times Y) \leq m_0 - 1$. By symmetry may assume that the largest slope of $X$, $\mu_1$, is bigger than the largest slope of $Y$. Let $X_1$ be the largest $p$-divisible subgroup of $X$ with slope $\mu_1$. According to the assumption in reduction step 2, the $p$-divisible group $X$ is isomorphic to a product $X_1 \times X_2$, where $X_2$ is a $p$-divisible subgroup such that every slope of $X_2$ is strictly smaller than $\mu_1$.

There are two cases to consider. If $\mu_1 < \text{slope}(Z)$, then the claim holds for the biextension $E$ by 7.2.4. It remains to treat the case when $\mu_1 > \text{slope}(Z)$.

Suppose now that $\mu_1 > \text{slope}(Z)$. Let $E_1 := \pi^{-1}(X_1 \times Y)$ and $E_2 := \pi^{-1}(X_2 \times Y)$. We have

$$E \cong (+Z : Z \times Z \to Z)_*(E_1 \times_Y E_2),$$

where $+Z : Z \times Z \to Z$ is the group law for $Z$, and the push-out by $+Z$ of the fiber product $E_1 \times_Y E_2$ is an analog of the familiar Baer sum construction for extensions of commutative groups; we will call it the $Y$-Baer sum of $E_1$ and $E_2$ Notice that the $Y$-Baer sum $(+Z)_*(E_1 \times_Y E_2)$ of two biextensions of $X \times Y$ by the same $p$-divisible group $Z$ has a natural structure as a biextension, and the above isomorphism is compatible with the biextension structures on both sides of the arrow. The biextension $E_1 \to X_1 \times Y$ of $X_1 \times Y_1$ by $Z$ splits, and we have a canonical $G$-equivariant isomorphism $\nu : E_1 \cong X_1 \times Y \times Z$. Let $\text{pr}_{X_1} : E \to X_1$ be the composition of $\pi : E \to X \times Y = X_1 \times X_2 \times Y$ with the projection $\text{pr}_{X_1} : X_1 \times X_2 \times Y \to X_1$. The splitting of the biextension $E_1$ and the partial group law $+1$ defines a natural translation action

$$T : X_1 \times \text{Spec}(k) E \to E \quad (x_1, e) \mapsto x_1 *_{X_1} e := T(x_1, e) \quad x_1 \in X_1, \ e \in E$$

of $X_1$ on $E$, which is compatible with the composition $\text{pr}_{X_2 \times Y} \circ \pi$, where $\pi : E \to X \times Y = X_1 \times X_2 \times Y$ is the structural map of the biextension $E$ and $\text{pr}_{X_2 \times Y}$ is the projection $\text{pr}_{X_2 \times Y} : X_1 \times X_2 \times Y \to X_2 \times Y$. By local rigidity of $p$-divisible formal groups, we know that $W \supset \nu^{-1}(X_1)$. 

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It is important to observe that because the slope of \( X_1 \) is strictly bigger than every slope appearing in \( Z, X_2 \) or \( Y \), for every element \( \nu = (A,B,C) \in \text{Lie}(G) \) such that \( A \in \text{End}(X), B \in \text{End}(Y), \) and \( C \in \text{End}(Z) \), the action of \( \exp(p^n \nu) \) on \( E \) is essentially translation by \( p^n C \circ \text{pr}_{X_1} \) for all sufficiently large natural numbers \( n \in \mathbb{N} \). More precisely, there exist positive integers \( r_1, s_2, a_1, n_1 \) such that

- \( 0 < r_1 < s_1, a_1 > 0, n_1 \geq 2, \)
- \( \text{slope}(X_1) = \frac{a_1}{r_1}, \) and \( X_1[p^{n_1}] = \ker(\text{Fr}_{X_1}^r), \)
- \( \psi(\exp(p^{n_1} \nu)) \equiv (p^{n_1} C \circ \text{pr}_{X_1}) \ast_{X_1} \text{id}_E \pmod{m^n_E} \) for all \( n \geq n_1. \)

The above properties allow us to apply the identity principle 6.1.1 as in the proof of 7.2 to conclude that for the morphism

\[
T_{X_1} \circ \left( (C \circ \text{pr}_{X_1}) \times \text{id}_E \right) : E \times E \rightarrow E, \quad (e_1, e_2) \mapsto T_{X_1} \left( C \circ \text{pr}_{X_1}(e_1), e_2 \right), \quad e_1, e_2 \in E
\]

we have

\[
(T_{X_1} \circ \left( (C \circ \text{pr}_{X_1}) \times \text{id}_E \right)) (W \times W) \subseteq W.
\]

Note that the above proof that the formal variety \( W \) is stable under translation by \( (C \circ \text{pr}_{X_1}) \) is the same as the argument in step 1 of the proof of [3, 4.3]; complete restricted perfection of power series rings are not used.

The rest of the proof is quite formal. Because the action of \( G \) on \( X \) is strongly non-trivial, we deduce that

\[
(T_{X_1} \circ \left( \text{pr}_{X_1} \times \text{id}_E \right)) (W \times W) \subseteq W.
\]

In particular \( W \) is stable under translation by \( X_1 \). The quotient of \( E \) by \( X_1 \) is canonically isomorphic to \( E_2 \). Under this canonical isomorphism \( W/X_1 \) becomes a formal subvariety of \( E_2 \). By induction \( W/X_1 \) is sub-biextension of \( E_2 \), which is a biextension of \( X_2 \times Y \) by the \( p \)-divisible subgroup \( W \cap Z \subseteq Z \) of \( Z \). It follows that \( W \) itself is a sub-biextension of \( E \).

**Theorem (7.5)** Let \( W \) be a formal subvariety of \( E \) stable under the action of \( G \). Assume that the slope of \( X, Y, Z \) are mutually distinct, i.e. no two of the \( p \)-divisible formal groups \( X, Y, Z \) share any common slope. Assume also that the action of \( G \) on \( E \) is strongly non-trivial in the sense of 2.6.5. Then \( W \) is a sub-biextension of \( E \).

**Proof.** As in the proof of 7.4, we may and do assume that the assumption in the reduction steps 1–3 in the proof of 7.4 hold. We will use induction on the pair

\[
\text{inv}_E := (\#\text{slopes}(Z), \#\text{slopes}(X \times Y))
\]

under the lexicographic ordering. The case when \( \#\text{slopes}(Z) = 1 \) has been treated in 7.4.

Suppose that \( \#\text{slopes}(Z) = m_2 \geq 2, \#\text{slopes}(X \times Y) = m_3 \geq 2, \) and assume that the statement of 7.5 holds whenever \( \text{inv}_E < (m_2, m_3) \) in the lexicographic order. Let \( \nu_2 \) be the largest slope of \( Z \), and let \( \nu_3 \) be the largest slope of \( X \times Y \). We may and do assume that \( \nu_3 \) is a slope of \( X \). Write \( X = X_3 \times X_4 \), where \( X_3 \) is isoclinic of slope \( \nu_3 \), and \( \nu_3 \) is not a slope of \( X_4 \). Similarly \( Z = Z_2 \times Z_4 \), where \( Z \) is isoclinic of slope \( \nu_2 \), and \( \nu_2 \) is not a slope of \( Z_4 \).
There are two cases to consider.

**Case 1.** Suppose first that $v_2 > v_3$.

Let $E_2 := (\text{pr}_{Z_2} : Z \to Z_2)_*E$, $E_4 := (\text{pr}_{Z_4} : Z \to Z_4)_*E$. We have a natural isomorphism

$$E \xrightarrow{\sim} E_2 \times_{(X \times Y)} E_4$$

over $X \times Y$. Let $W_2$ be the image of $W$ in $E_2$, and let $W_4$ be the image of $W$ in $E_4$. By induction, we know that $W_2$ is a sub-biextension $E'_2$ of $X \times Y$ by a $p$-divisible subgroup $Z'_2$ of $Z_2$. Clearly $Z'_2 = Z_2 \cap W$, where we have regarded $Z_2$ as a formal subvariety of $Z = \pi^{-1}(0,0) \subset E$. Again by induction, we know that $W_4$ is a sub-biextension $E'_4$ of $X \times Y$ by a $p$-divisible subgroup $Z'_4$ of $Z_4$.

Modifying $Z$ by an isogeny, we may and do assume that there exists a $p$-divisible subgroup $Z''$ of $Z$ such that $Z \cong Z'_2 \times Z'_4$. Let $E'_2$ and $E''_4$ be the push-out of the biextension $E_2$ by the projections from $E_2$ to $E'_2$ and $E''_4$ respectively. We have a natural isomorphism

$$E_2 \xrightarrow{\sim} E'_2 \times_{(X \times Y)} E''_4.$$ 

Clearly the formal subvariety $W$ of $E$ is contained in the sub-biextension $E'_2 \times_{(X \times Y)} E_4$ of $E$. If $Z'' \neq (0)$, we are done by induction. So we may and do assume that $Z'_2 = Z$.

By 7.2 we know that $W$ is stable under translation by $Z_2$. Under the natural isomorphism $E / Z_2 \xrightarrow{\sim} E_4$ the quotient $W / Z_2$ corresponds to a $G$-invariant formal subvariety of the biextension $E_4 \to X \times Y$. By induction, this $G$-invariant formal subvariety $W / Z_2$ of $E_4$ is a sub-biextension of $E_4 \cong E / Z_2$. It follows that $W$ itself is a sub-biextension of $E$. We have finished the case when the largest slope $v_2$ of $Z$ is bigger than every slope of $X \times Y$.

**Case 2.** $v_2 < v_3$. There exists a $p$-divisible subgroups $X_3, X_4$ of $X$ such that $X_3$ is isoclinic of slope $v_3$, and every slope of $X_4$ is strictly smaller than $v_3$. Let $E_3, E_4$ be the pull-back of $E$ to $X_3 \times Y$ and $X_4 \times Y$ respectively. The biextension $E$ is naturally isomorphic to the “$Y$-Baer sum” $(+Z : Z \times Z \to Z)_*(E_3 \times_Y E_4)$ of the biextensions $E_3$ and $E_4$:

$$E \xrightarrow{\sim} (+Z : Z \times Z \to Z)_*(E_3 \times_Y E_4).$$

Because the slope $v_3$ of $X_3$ is strictly bigger than every slope of $Z$, the biextension $E_3 \to X_3 \times Y$ of $X_3 \times Y$ by $Z$ splits:

$$E_3 \xrightarrow{\sim} X_3 \times Y \times Z.$$ 

As in the proof of 7.4, the condition that $v_3 > v_2$ implies that for every element $v = (A, B, C) \in \text{Lie}(G)$ such that $A \in \text{End}(X)$, $B \in \text{End}(Y)$ and $C \in \text{End}(Z)$, the action of $\psi(\exp(p^m)v)$ on $E$ has a “main term” corresponding to translation by the composition of the endomorphism $p^nC |_{X_3}$ of $X_3$ with the projection $\text{pr}_{X_3} : E \to X_3$. More precisely the automorphism $\psi(\exp(p^m)v)$ of $E$ satisfies a congruence relation

$$(p^mC \circ \text{pr}_{X_3})_*X_3 \not\equiv \text{id}_E \mod mp^{m/(v_2 + v_2)}$$

for a suitably small real number $\varepsilon_2 < v_3 - v_2$, and for all natural numbers $m \geq m_0$, where $m_0$ is a natural number which depends on the biextension $E \to X \times Y$ and $\varepsilon_2$. Applying the identity principle 6.1.1 for power series, we deduce that

$$(T_{X_3} \circ ((C \circ \text{pr}_{X_3}) \times \text{id}_E))(W \times W) \subseteq W,$$

where
• $\text{pr}_{X_3} : E \rightarrow X_3$ is the composition of $\pi : E \rightarrow X \times Y = X_3 \times X_4 \times Y$ with the projection $X_3 \times X_4 \times Y \rightarrow X_3$,

• $T_{X_3} : X_3 \times E \rightarrow E$ is the translation action of $X_3$ on $E$ induced by the action of $X_3$ on the trivial biextension $E_3 \cong X_3 \times Y \times Z$ and the trivial action of $X_3$ on $E_4$, through the natural isomorphism $E \cong (+_Z : Z \times Z \rightarrow Z) \times (X \times Y) E_4$.

Because the action of $G$ on $X$ is strictly non-trivial, it follows that $W$ is stable under translation by $X_3$ in the following sense:

$$(T_{X_3} \circ (\text{pr}_{X_3} \times \text{id}_E)) (W \times W) \subseteq W.$$ 

The quotient $W/X_3$ is a formal subvariety of $E/X_3 \cong E_4$. The induction hypothesis implies that $W/X_3$ is a sub-biextension of $E_4$, which in turn implies that $W$ is a sub-biextension of $E$. We have finished the induction step. □

References