SUSTAINED p-DIVISIBLE GROUPS AND LEAVES ON MODULI SPACES OF ABELIAN VARIETIES

CHING-LI CHAI

Abstract. We explain a concept of *sustained p-divisible groups*, discovered in collaboration with Frans Oort and motivated by the Hecke orbit problem. This concept leads to a scheme-theoretic definition of *central leaves* in moduli spaces of abelian varieties in characteristic $p > 0$, as well as central leaves in the space $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of sustained deformations of a p-divisible group over a field of characteristic p. The formal completion $\mathcal{C}^{/z_0}$ at any closed point z_0 of a central leaf $\mathcal{C}$ in a moduli space of abelian varieties and the sustained deformation space $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ both have natural *Tate-linear structure*, that they are "built up" from a family of torsors for p-divisible formal groups. We also formulate a notion of *strongly Tate-linear* formal subschemes of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$, and a local rigidity question on whether every reduced and irreducible closed formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ stable under a *strongly non-trivial* action of a p-adic Lie group is strongly Tate-linear.

1. Introduction

1.1. The goal of this article is to provides a short introduction to the notion of *sustained p-divisible groups*. A pleasant feature of the theory is that p-divisible groups appear naturally at various places, in stabilized Hom schemes, stabilized Isom schemes, and in sustained deformation spaces.

For a historical perspective, we recall in 1.2 and 1.3 Oort's idea on the foliation structures of a Siegel modular variety [25] in positive characteristic p. The generalization to PEL modular varieties is immediate.

The definition of a strongly sustained p-divisible group is given in 2.1. This notion is motivated by the search of a scheme-theoretic definition of central leaves on a PEL modular variety $\mathscr{M}$ over $\overline{\mathbb{F}}_p$. The sought-after answer is as follows:

Let $z_0 = [(A_0, \lambda_0, \mu_0)] \in \mathscr{M}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$-point of $\mathscr{M}$, corresponding to an abelian variety A_0 over $\overline{\mathbb{F}}_p$ with prescribed endomorphisms $\lambda_0 : \mathscr{O}_E \to \mathrm{End}(A_0)$ and a polarization μ_0 compatible with λ_0, where $\mathscr{O}_E$ is a maximal order of a central simple algebra of finite dimension over $\mathbb{Q}$.

> *The central leaf in $\mathscr{M}$ passing through z_0 is the largest locally closed subscheme $\mathcal{C}(z_0)$ of $\mathscr{M}$ such that the restriction to $\mathcal{C}(z_0)$ of the universal p-divisible group with PE structure is strongly $\overline{\mathbb{F}}_p$-sustained modeled on $(A_0[p^\infty], \mu_0[p^\infty], \lambda_0[p^\infty])$.*

Here $\lambda_0[p^\infty] : \mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(A_0[p^\infty])$ and $\mu_0[p^\infty] : A_0[p^\infty] \to A_0[p^\infty]^t$ are respectively the endomorphism structure and polarization on the p-divisible group $A_0[p^\infty]$ induced by λ_0 and μ_0.

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I have taken a shortcut and discuss only the notion of *strongly sustained p-divisible group*. The notion of *sustained p-divisible groups* is suppressed, to achieve a sharper focus. For the same reason, we consider only strongly sustained p-divisible groups and strongly sustained polarized p-divisible group. It is possible to extend our discussions to the case of p-divisible groups with prescribed endomorphism and polarization structure, and also the more general case of p-divisible groups with prescribed Tate-cycles. These generalizations are left to the ambitious readers.

Proofs are omitted as a rule. The only exception is a sketch of the smoothness of the sustained deformation functors. A complete documentation can be found in [11].

1.2. Foliation of $\mathcal{A}_g$ in characteristic $p > 0$.

The notion of *foliation* on the moduli space $\mathcal{A}_g$ of g-dimensional principally polarized abelian varieties in characteristic $p > 0$, due to Frans Oort, was announced in the conference *Moduli of Abelian Varieties, Texel '99*; see [25] for the published version. For any self-dual Newton polygon ξ of height $2g$, two foliations for the Newton stratum $\mathcal{W}_\xi(\mathcal{A}_g)$ in $\mathcal{A}_g$ associated to ξ are defined in [25], called the *central foliation* and the *isogeny foliation* of $\mathcal{W}_\xi(\mathcal{A}_g)$ respectively. Below is a list of their salient features.

(a) (foliation property) For every algebraically closed field $k \supseteq \mathbb{F}_p$ and every geometric point $x_0 \in \mathcal{W}_\xi(\mathcal{A}_g)(k)$, there is a unique central leaf $\mathcal{C}(x_0)$ passing through x_0 and a unique isogeny leaf $\mathcal{I}(x_0)$ passing through x_0. Both $\mathcal{C}(x_0)$ and $\mathcal{I}(x_0)$ are reduced closed subschemes of $\mathcal{W}_\xi(\mathcal{A}_g)\times_{\mathrm{Spec}(\overline{\mathbb{F}_p})}\mathrm{Spec}(k)$.

(b) Every central leave $\mathcal{C}(x_0)$ in $\mathcal{W}_\xi(\mathcal{A}_g)$ is equi-dimensional, and any two central leaves $\mathcal{C}(x_1), \mathcal{C}(x_2)$ in $\mathcal{W}_\xi(\mathcal{A}_g)$ in the same Newton polygon stratum $\mathcal{W}_\xi(\mathcal{A}_g)$ have the same dimension. Similarly every isogeny leave $\mathcal{I}(x_0)$ in $\mathcal{W}_\xi(\mathcal{A}_g)$ is equi-dimensional, and any two isogeny leaves $\mathcal{I}(x_1), \mathcal{I}(x_2)$ in $\mathcal{W}_\xi(\mathcal{A}_g)$ have the same dimension.

(c) Every central leaf $\mathcal{C}(x_0)$ as in (a) is smooth over k.

(d) (strong transversality) For every algebraically closed field $k \supseteq \mathbb{F}_p$ and every geometric point $x_0 \in \mathcal{W}_\xi(\mathcal{A}_g)(k)$, there exist k-schemes $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{I}}$ and k-morphisms

$$h_1 : \tilde{\mathcal{C}} \to \mathcal{C}(x_0), \quad h_2 : \tilde{\mathcal{I}} \to \mathcal{I}(x_0) \quad \text{and} \quad f : \tilde{\mathcal{C}} \times_{\mathrm{Spec}(k)} \tilde{\mathcal{I}} \to \mathcal{W}_\xi(\mathcal{A}_g)$$

such that h_1 is finite flat, while h_2 and f are finite surjective. In particular we have

$$\dim(\mathcal{C}(x_0)) + \dim(\mathcal{I}(x_0)) = \dim(\mathcal{W}_\xi(\mathcal{A}_g)) \quad \text{and} \quad \dim\big(\mathcal{C}(x_0) \cap \mathcal{I}(x_0)\big) = 0.$$

Except for the remark 1.4, our attention will be focused on the central foliation in the rest of this article.

1.3. Central leaves and gfc p-divisible groups.

We fix a prime number p in this paper. Let k be an algebraically closed field of characteristic p and let S be a scheme over k. Recall that a p-divisible group X over S, or respectively a polarized p-divisible group $(X, \lambda : X \to X^t)$ over S, is said to be *geometrically fiberwise constant* (abbreviated as *gfc*) relative to k if for any two (not necessarily closed) points $s_1, s_2 \in S$, there exist

- an algebraically closed field L containing k,
- k-morphisms $\iota_1 : \mathrm{Spec}(L) \to s_1$, $\iota_2 : \mathrm{Spec}(L) \to s_2$, and
- an L-isomorphism

$$\psi : X_{s_1} \times_{(s_1, \iota_1)} \mathrm{Spec}(L) \xrightarrow{\sim} X_{s_2} \times_{(s_2, \iota_2)} \mathrm{Spec}(L),$$

or respectively an L-isomorphism

$$\psi : (X_{s_1}, \lambda_{s_1}) \times_{(s_1, \iota_1)} \mathrm{Spec}(L) \xrightarrow{\sim} (X_{s_2}, \lambda_{s_2}) \times_{(s_2, \iota_2)} \mathrm{Spec}(L).$$

Let $n \geq 3$ be a positive integer prime to p. Denote by $\mathcal{A}_{g,1,n}$ the moduli scheme classifying all g-dimensional principally abelian schemes $(A \to S, \lambda : A \to A^t)$ over $\overline{\mathbb{F}_p}$, together with a symplectic level-n structure ζ.

For a geometric point $x_0 = [(A_0, \lambda_0, \zeta_0)] \in \mathcal{A}_{g,1,n}(k)$, the central leaf $\mathcal{C}(x_0)$ passing through x_0 is defined in [25] as the largest reduced locally closed subscheme of $\mathcal{A}_{g,1,n} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$ such that the principally polarized p-divisible group attached to the restriction to $\mathcal{C}(x_0)$ of the universal principally polarized abelian scheme is geometrically fiberwise constant. Equivalently, $\mathcal{C}(x_0)$ is the reduced locally closed subscheme of $\mathcal{A}_{g,1,n} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$ whose k-points are given by

$$\mathcal{C}(x_0)(k) = \big\{ y \in \mathcal{A}_{g,1,n}(k) \,\big|\, (A_y[p^\infty], \lambda_y[p^\infty]) \cong (A_{x_0}[p^\infty], \lambda_{x_0}[p^\infty]) \big\}.$$

Such a central leaf $\mathcal{C}(x_0)$ is a smooth locally closed subscheme of the moduli scheme $\mathcal{A}_{g,1,n} \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k)$, and is stable under all prime-to-p Hecke correspondences on $\mathcal{A}_{g,1,n}$. It seems quite appropriate to consider central leaves as "Shimura varieties in characteristic p". Among other things, we know from the Serre-Tate theorem that the deformation theory for any two closed points of a central leaf are isomorphic, so every central leaf is "homogeneous" in this weak sense.

The above definition of central leaves suffers from an obvious deficiency of the definition of $\mathcal{C}(x_0)$ in the previous paragraph because of the "point-wise" nature of the notion of gfc: it does *not* tell us how to characterise the set $\mathcal{C}(x_0)(S)$ of all S-points of a central leaf $\mathcal{C}(x_0)$, for non-reduced k-schemes S. This defect is rectified by the notion of *sustained p-divisible groups*, first discovered in December 2012 when F. Oort visited the author in Taipei. It retains the essence of geometrically fiberwise constant p-divisible groups and provides further insight into the local structure of central leaves.

1.4. Remark. The isogeny foliation can be understood via the Rapoport–Zink moduli space of quasi-isogenies for polarized p-divisible groups. Given a geometric point $x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g(k)$ corresponding to a g-dimensional principally polarized abelian variety (A_0, λ_0) over an algebraically closed field $k \supseteq \overline{\mathbb{F}_p}$, we have Rapoport–Zink moduli space

$$\mathcal{M}_{A_0[p^\infty], \lambda_0[p^\infty]} \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(k),$$

an adic formal scheme over k which classifies principally polarized p-divisible groups (X, λ) over k-schemes S, together with a quasi-isogeny

$$(A_0[p^\infty], \lambda_0[p^\infty]) \times_{\mathrm{Spec}(k)} S \dashrightarrow (X, \lambda)$$

which preserves the polarizations; see [28]. Moreover we have a natural morphism

$$f_{x_0} : \big(\mathcal{M}_{A_0[p^\infty],\lambda_0[p^\infty]} \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(k)\big)_{\mathrm{red}} \longrightarrow \mathcal{A}_g \times_{\mathrm{Spec}(\overline{\mathbb{F}_p})} \mathrm{Spec}(k),$$

which is locally quasi-finite. Here $\big(\mathcal{M}_{A_0[p^\infty],\lambda_0[p^\infty]} \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(k)\big)_{\mathrm{red}}$ is the Rapoport–Zink space over k with reduced structure; it is locally of finite type over k but has infinitely many irreducible components. The central leaf $\mathcal{I}(x_0)$ is the scheme-theoretic image of the above morphism f_{x_0}. One might think of $\mathcal{I}(x_0)$ as being "uniformized" by the reduced Rapoport–Zink space

$$\big(\mathcal{M}_{A_0[p^\infty],\lambda_0[p^\infty]} \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(k)\big)_{\mathrm{red}}$$

over k.

2. Strongly sustained p-divisible groups

2.1. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field, let Y_0 be a p-divisible group over κ, and let S be a κ-scheme.

(i) A p-divisible group X over S is *strongly κ-sustained modeled on* Y_0 if for every natural number n, the Isom scheme

$$\mathbf{Isom}_S\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^n]\big),$$

which represents the functor

$$T \;\mapsto\; \mathrm{Isom}_T\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} T,\, X[p^n] \times_S T\big) \qquad \forall\, S\text{-scheme } T$$

on the category of all S-schemes, is faithfully flat over S.

(ii) Let $\mu_0 : Y_0 \to Y_0^t$ be a polarization on Y_0. A polarized p-divisible group $(X, \lambda : X \to X^t)$ over S is *strongly κ-sustained modeled on* (Y_0, μ_0) if for every natural number n, the Isom scheme

$$\mathbf{Isom}_S\big((Y_0[p^n], \mu_0[p^n]) \times_{\mathrm{Spec}(\kappa)} S,\, (X[p^n], \lambda[p^n])\big),$$

is faithfully flat over S.

2.2. Remark. (a) The above definition articulates the basic idea that locally for the fppf topology, the $[p^n]$-kernel $X[p^n]$ of a strongly κ-sustained p-divisible group X modeled on Y_0 is isomorphic to the $[p^n]$-kernel $Y[p^n]$ of the "constant" p-divisible group Y_0.

(b) Being "constant" is fundamentally a relative concept, which explains the appearance of the base field κ in 2.1. Note that the defintion of geometrically fiberwise constant p-divisible groups recalled in 1.3 also depends on the algebraically closed base field k.

(c) It will be interesting to find a good generalization of defintion 2.1, in which the base scheme does not have to be the spectrum of a field $\kappa \supseteq \mathbb{F}_p$, and develop a satisfactory notion of a *family* of sustained p-divisible groups that is applicable to the family central leaves in a Newton polygon stratum of a Siegel modular variety $\mathcal{A}_{g,1,n}$.

(d) There is a related (and slightly weaker) notion of a *κ-sustained p-divisible group* over a κ-scheme S, which does not require the existence of a κ-model. More information can be found in [11]; see also [9].

2.3. Remark. Suppose that $X \to S$ is a strongly κ-sustained p-divisible group modeled on a p-divisible group Y_0 over κ as in 2.1.

(i) For each $n \in \mathbb{N}$, there exists a *finite locally free* morphism $T_n \to S$ and a T_n-isomorphism
$$Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} T_n \xrightarrow{\sim} X[p^n] \times_S T_n.$$

(ii) There exists a faithfully flat quasi-compact morphism $T \to S$ and a T-isomorphism
$$Y_0 \times_{\mathrm{Spec}(\kappa)} T \xrightarrow{\sim} X \times_S T.$$

(iii) If X is isoclinic, then for each $n \in \mathbb{N}$, there exists a *finite etale* morphism $T_n \to S$ and a T_n-isomorphism
$$Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} T_n \xrightarrow{\sim} X[p^n] \times_S T_n.$$

As the readers likely will expect, the obvious analogs of statements (i)–(iii) for a strongly κ-sustained *polarized* p-divisible group $(X \to S, \lambda : X \to X^t)$ hold.

2.4. Proposition (Slope filtration on sustained p-divisible groups)**.** *Let S be a scheme over a field $\kappa \supseteq \mathbb{F}_p$. Let $X \to S$ be a strongly κ-sustained p-divisible group modeled on a p-divisible group Y_0 over κ. Let*
$$(0) = \mathrm{Fil}_0(Y_0) \subsetneqq \mathrm{Fil}_1(Y_0) \subsetneqq \cdots \subsetneqq \mathrm{Fil}_m(Y_0) = Y_0$$
be the slope filtration of Y_0 by p-divisible subgroups $\mathrm{Fil}_i(Y_0)$ over κ, $i = 0, 1, \ldots, m$, such that $\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$ is isoclinic for $i = 1, \ldots, m$, and the slopes s_i of $\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$ satisfy
$$1 \geq s_1 > s_2 > \cdots > s_m \geq 0.$$
There exists a unique slope filtration
$$(0) = \mathrm{Fil}_0 X \subsetneqq \mathrm{Fil}_1 X \subsetneqq \cdots \subsetneqq \mathrm{Fil}_m X = X$$
of X by p-divisible subgroups $\mathrm{Fil}_i(X)$ such that

(a) *$\mathrm{Fil}_i X/\mathrm{Fil}_{i-1} X$ is a κ-sustained p-divisible group, isoclinic of slope s_i for $i = 1, \ldots, m$.*

(b) *The p-divisible group $\mathrm{Fil}_i(X)$ over S is strongly κ-sustained modeled on $\mathrm{Fil}_i(Y_0)$ for $i = 1, \ldots, m$.*

(c) *For any $i = 1, \ldots, m$, the isoclinic p-divisible group $\mathrm{Fil}_i(X)/\mathrm{Fil}_{i-1}(X)$ is strongly κ-sustained modeled on $\mathrm{Fil}_i(Y_0)/\mathrm{Fil}_{i-1}(Y_0)$.*

2.5. Remark. (i) The existence of the slope filtration on a p-divisible group over a field was proved in [30, Cor. 13].

(ii) The proof 2.4 is an exercise of flat descent, transferring the slope filtration for the model Y_0 to X via the flat covers
$$\mathbf{Isom}_S\big((Y_0[p^n], \mu_0[p^n]) \times_{\mathrm{Spec}(\kappa)} S,\ (X[p^n], \lambda[p^n])\big)$$
of S.

(iii) In general a p-divisible group Y over a base scheme S in characteristic p with constant Newton polygons may not admit a slope filtration. All one can say is that if the base scheme S is noetherian and normal, then Y is isogenous to a *completely slope divisible* p-divisible group Z over S; see [27, Thm. 2.1]. We refer to [27, (1.2)] for the definition of completely slope divisible p-divisible groups, and to [11] for relations between completely slope divisible and sustained p-divisible groups.

(iv) Assume that the base scheme S in 2.4 is the spectrum of an Artinian local κ-algebra R with residue field κ. Then $X_0 := X \times_S \mathrm{Spec}(\kappa)$ is a model of the strongly κ-sustained p-divisible group $X \to S$ in 2.4, and $\mathrm{Fil}_i(X)/\mathrm{Fil}_{i-1}(X)$ is canonically isomorphic to $\big(\mathrm{Fil}_i(X_0)/\mathrm{Fil}_{i-1}(X_0)\big) \times_{\mathrm{Spec}(\kappa)} S$ for $i = 1, \ldots, m$.

2.6. Remark. (i) Sometimes it is convenient to reindex the slope filtration by the slopes themselves. In the context of 2.4, define a decreasing filtration $\mathrm{Fil}^{\bullet}_{\mathrm{can}}X$ on X by

$$\mathrm{Fil}^t_{\mathrm{can}}X := \mathrm{Fil}_{m(t)}X, \quad \text{where} \quad m(t) = \begin{cases} 0 & \text{if } \ t > s_1 \\ i & \text{if } \ s_{i+1} < t \le s_i, \ i \in \{1, \ldots, m-1\} \\ m & \text{if } \ t \le s_m \end{cases}$$

for every $t \in \mathbb{R}$.

(ii) Suppose that $X \to S$ and $Y \to S$ are strongly κ-sustained p-divisible groups, and $\phi : X \to Y$ is an S-homomorphism of p-divisible groups. Then

$$\phi(\mathrm{Fil}^t_{\mathrm{can}}X) \subseteq \mathrm{Fil}^t_{\mathrm{can}}Y \quad \forall\, t \in \mathbb{R}.$$

2.7. Proposition (Backward compatibility with gfc)**.** *Let S be a* reduced *scheme over a field $\kappa \supseteq \mathbb{F}_p$. Let $X \to S$ be a p-divisible group over S and let Y_0 be a p-divisible group over κ.*

(a) *If X_s is strongly κ-sustained modeled on Y_0 for every $s \in S$, then $X \to S$ is strongly κ-sustained modeled on Y_0.*
(b) *Let λ be a polarization on X and let μ_0 be a polarization on Y_0. If (X_s, λ_s) is strongly κ-sustained modeled on (Y_0, μ_0) for every $s \in S$, then (X, λ) is strongly κ-sustained modeled on (Y_0, μ_0).*

2.8. Proposition. *Let n be a positive integer such that* $\gcd(n,p) = 1$*, $n \ge 3$. Let $d > 0$ be a positive integer. Denote by $\mathcal{A}_{g,d,n}$ the fine moduli scheme over $\overline{\mathbb{F}_p}$ which classifies all polarized abelian schemes $(A \to S, \lambda : A \to A^t)$ of relative dimension g with* $\deg(\lambda) = d^2$*, plus a symplectic level-n structure ζ, where S is an $\overline{\mathbb{F}_p}$-scheme. Let $(\boldsymbol{A}, \boldsymbol{\lambda})$ be the universal polarized abelian scheme over $\mathcal{A}_{g,d,n}$. Let $x_0 = [(A_0, \lambda_0, \zeta_0)]$ be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$.*

There exists a unique locally closed subscheme $\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0)$ of $\mathcal{A}_{g,d,n}$ with the following property:

> *For every polarized abelian scheme with level-n-structure (B, μ, ψ) over an $\overline{\mathbb{F}_p}$-scheme T, the modular morphism $T \to \mathcal{A}_{g,d,n}$ factors through the inclusion $\mathcal{C}(x_0) \hookrightarrow \mathcal{A}_{g,d,n}$ if and only if the polarized p-divisible group $(B[p^\infty], \mu[p^\infty])$ over T is strongly κ-sustained modeled on $(A_0[p^\infty], \lambda_0[p^\infty])$*

2.9. Definition (Updated definition of central leaves)**.** The locally closed subscheme

$$\mathcal{C}(x_0) = \mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0) \ \subseteq\ \mathcal{A}_{g,d,n}$$

in 2.8 is called the *central leaf* in $\mathcal{A}_{g,d,n}$ passing through x_0.

3. Stabilized Hom schemes for p-divisible groups

3.1. Definition (Stabilized Hom schemes for p-divisible groups)**.** Let Y, Z be p-divisible groups over a field $\kappa \supset \mathbb{F}_p$. For each $n \in \mathbb{N}$, we have a commutative group scheme

$$H_n := \mathbf{Hom}(Y[p^n],\, Z[p^n])$$

of finite type over κ, which satisfies the university property that

$$\mathbf{Hom}(Y[p^n], Z[p^n])(S) = \mathrm{Hom}_S(Y[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, Z[p^n] \times_{\mathrm{Spec}(\kappa)} S)$$

for every κ-scheme S. In addition we have

- restriction homomorphisms $r_{n,n+i}\colon H_{n+i} \to H_n$, and
- closed embeddings $\iota_{n+i,n}\colon H_n \hookrightarrow H_{n+i}$ given by (a) the epimorphism $Y_{n+i} \twoheadrightarrow Y_n$ induced by $[p^i]_{Y[p^{n+i}]}$, and (b) the inclusion $Z[p^n] \hookrightarrow Z[p^{n+i}]$.

Note that

$$\iota_{n+i,n} \circ r_{n,n+i} = [p^i]_{H_{n+i}} \quad \forall\, n, i \in \mathbb{N}.$$

Define closed subgroup scheme $\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n$ of H_n over κ by

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n := \mathrm{Im}(r_{i,n+i}\colon H_{n+i} \to H_n) \qquad \text{for} i \gg 0$$

It is clear from the above definition that the arrows

$$H_{n+i} \underset{\iota_{n+i,n}}{\overset{r_{n,n+i}}{\rightleftarrows}} H_n$$

induce arrows

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z)_{n+i} \underset{j_{n+i,n}}{\overset{\pi_{n,n+i}}{\rightleftarrows}} \mathbf{Hom}^{\mathrm{st}}(Y,Z)_n\ ,$$

and

$$j_{n+i,n} \circ \pi_{n,n+i} = [p^i]_{\mathbf{Hom}^{\mathrm{st}}(Y,Z)_{n+i}} \qquad \forall\, n, i \in \mathbb{N}.$$

3.2. Theorem. *Let Y, Z be p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$ as in* 3.1*.*

(a) *For each $n \in \mathbb{N}$, the commutative group scheme $\mathbf{Hom}^{\mathrm{stab}}(Y,Z)_n$ is* finite *over κ.*

(b) *The family*

$$\mathbf{Hom}^{\mathrm{st}}(Y,Z) := \big(\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n, j_{n+i,n}, \pi_{n,n+i}\big)_{n\in\mathbb{N}}$$

of commutative groups schemes $\mathbf{Hom}^{\mathrm{st}}(Y,Z)_n$ together with the homomorphisms $j_{n+i,n}, \pi_{n,n+i}$ is a p-divisible group over κ.

(c) *Suppose that the field κ is perfect, and let $\mathbb{D}_*(Y), \mathbb{D}_*(Z)$ be the covariant Dieudonné modules of Y and Z respectively. The covariant Dieudonné module of $\mathbf{Hom}^{\mathrm{st}}(Y,Z)$ is the largest $W(\kappa)$*-submodule *of $\mathrm{Hom}_{W(\kappa)}\big(\mathbb{D}_*(Y), \mathbb{D}_*(Z)\big) =: \boldsymbol{H}$ which is stable under the semi-linear operators F and V.*

(d) *Suppose that Y, Z are* isoclinic *over κ, with slopes s_Y and s_Z respectively.*
- *If $s_Y > s_Z$, then $\mathbf{Hom}^{\mathrm{st}}(Y,Z) = (0)$.*
- *If $s_Y \le s_Z$, then $\mathbf{Hom}^{\mathrm{st}}(Y,Z)$ is isoclinic of slope $s_Z - s_Y$ and height* $\mathrm{ht}(Z)\cdot \mathrm{ht}(Y)$.

Remark. (i) In 3.2 (c), the semi-linear operators F, V on $\boldsymbol{H}_{\mathbb{Q}}$ are defined as follows: for every $h \in \boldsymbol{H}$, we have

$$F(h)(y) = F(h(Vy)) \in \boldsymbol{H}$$

and

$$V(h)(y) = V(h(V^{-1}y)) = p^{-1}\cdot V(h(F(y))) \in \boldsymbol{H}_{\mathbb{Q}}$$

for all $y \in \mathbb{D}_*(Y)$. So $F(\boldsymbol{H}) \subseteq \boldsymbol{H}$, while $V(\boldsymbol{H}) \subseteq \boldsymbol{H}_{\mathbb{Q}}$.

(ii) In covariant Dieudonné theory, the operator V on the Dieudonné module $\mathbb{D}_*(X)$ of a p-divisible group over a perfect field $\kappa \supseteq \mathbb{F}_p$ corresponds to the geometric Frobenius operator on X.

In the situation of 3.2 (d), the assumptions on Y and Z means that asymptotically, $V^n_{\mathbb{D}_*(Y)}$ is roughly $p^{n\cdot s_Y}$ times an isomorphism and $V^n_{\mathbb{D}_*(Z)}$ is roughly $p^{n\cdot s_Z}$ times an isomorphism for $n \gg 0$. Therefore the recipe in 3.2 (c) for the Dieudonné module of $\mathbf{Hom}^{\mathrm{st}}(Y,Z)$ implies 3.2 (d).

3.3. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field and let Y be a p-divisible group over κ.

(i) Define a projective system $\mathbf{End}^{\mathrm{st}}(Y)$ of finite ring schemes over κ by

$$\mathbf{End}^{\mathrm{st}}(Y) := \big(\mathbf{End}^{\mathrm{st}}(Y)_n\big)_{n\ge 1}, \quad \mathbf{End}^{\mathrm{st}}(Y)_n := \mathbf{Hom}^{\mathrm{st}}(Y,Y)_n.$$

The group of units

$$\mathbf{Aut}^{\mathrm{st}}(Y) := \big(\mathbf{Aut}^{\mathrm{st}}(Y)_n\big)_{n\ge 1}, \quad \mathbf{Aut}^{\mathrm{st}}(Y)_n := \big(\mathbf{End}^{\mathrm{st}}(Y)_n\big)^{\times},$$

in $\mathbf{End}^{\mathrm{st}}(Y)$ is a projective system of finite group schemes over κ.

(ii) Let μ be a polarization of Y. For each $n \ge 1$, denote by $\mathbf{Aut}^{\mathrm{prest}}(Y)_n$ the closed subgroup scheme of $\mathbf{Aut}(Y,\mu)_n$ consisting of automorphisms of $Y[p^n]$ which respect the homomorphism

$$\mu[p^n] = \mu|_{Y[p^n]} : Y[p^n] \to Y^t[p^n] = Y[p^n]^D,$$

where $Y[p^n]^D$ is the Cartier dual of $Y[p^n]$. Define a closed subgroup scheme $\mathbf{Aut}^{\mathrm{st}}(Y,\mu)_n$ of $\mathbf{Aut}^{\mathrm{st}}(Y)_n$ by

$$\mathbf{Aut}^{\mathrm{st}}(Y,\mu)_n := \mathrm{Im}\big(\mathbf{Aut}^{\mathrm{prest}}(Y,\mu)_{n+i} \longrightarrow \mathbf{Aut}^{\mathrm{prest}}(Y,\mu)_n\big), \quad i \gg 0.$$

Denote by $\mathbf{Aut}^{\mathrm{st}}(Y,\mu)$ the projective system of finite group schemes

$$\mathbf{Aut}^{\mathrm{st}}(Y,\mu) := \Big(\mathbf{Aut}^{\mathrm{st}}(Y,\mu)_n\Big)_{n\ge 1}.$$

3.4. Definition. Let (Y,μ) be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$.

(i) Pick an isogeny $\nu : Y^t \to Y$ such that $\nu\circ\mu = [p^m]_Y$ and $\mu\circ\nu = [p^m]_{Y^t}$ for a natural number $m \in \mathbb{N}$. Define a quasi-isogeny ι_μ on the p-divisible group $\mathbf{End}^{\mathrm{st}}(Y)$ by

$$\iota(h) = p^{-m}\cdot \nu\circ h^t \circ \mu \qquad h \in \mathbf{End}^{\mathrm{st}}(Y).$$

This quasi-isogeny depends only on μ and is independent of the choice of ν and m, and satisfies

$$\iota_\mu^2 = \mathrm{id}.$$

We call ι_μ the *Rosati involution* on $\mathbf{End}^{\mathrm{st}}(Y)$

(ii) Denote by $\mathbf{End}^{\mathrm{st}}(Y)^{\iota_\mu=-1}$ the largest p-divisible subgroup of $\mathbf{End}^{\mathrm{st}}(Y)$ on which the Rosati involution ι_μ operates as $-\mathrm{id}$.

3.5. Lemma. *Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$.*

(i) *Let $X \to S$ be a strongly κ-sustained p-divisible group modeled on Y_0. For each $n \in \mathbb{N}$, there exists a positive integer i_0 such that the schematic image*

$$\mathrm{Im}\Big(\mathbf{Isom}_S\big(Y_0[p^{n+i}] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^{n+i}]\big) \longrightarrow \mathbf{Isom}_S\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^n]\big)\Big)$$

under the restriction homomorphism is independent of i for all $i \geq i_0$.

(ii) *The stabilized image*

$$\mathbf{Isom}_S^{\mathrm{st}}(Y_0, X)_n := \mathrm{Im}\Big(\mathbf{Isom}_S\big(Y_0[p^{n+i}] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^{n+i}]\big) \longrightarrow \mathbf{Isom}_S\big(Y_0[p^n] \times_{\mathrm{Spec}(\kappa)} S,\, X[p^n]\big)\Big), \quad i \gg 0$$

has a natural structure as a right torsor for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$. Moreover the natural projections maps

$$\mathbf{Isom}_S^{\mathrm{st}}(Y_0, X)_{n+i} \longrightarrow \mathbf{Isom}_S^{\mathrm{st}}(Y_0, X)_n$$

are faithfully flat and compatible with the projection maps

$$\mathbf{Aut}^{\mathrm{st}}(Y_0)_{n+i} \times_{\mathrm{Spec}(\kappa)} S \longrightarrow \mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S.$$

(iii) *Suppose that μ_0 is a polarization on Y_0 and (X, λ) is a strongly κ-sustained polarized p-divisible group modeled on (Y_0, μ_0). The obvious analogs of* (i) *and* (ii) *hold, and we have a projective family*

$$\Big(\mathbf{Isom}_S^{\mathrm{st}}((Y_0, \mu_0), (X, \lambda))_n\Big)_{n \geq 1}$$

of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_n \times_{\mathrm{Spec}(\kappa)} S$, compatible with the projections

$$\mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_{n+i} \longrightarrow \mathbf{Aut}^{\mathrm{st}}(Y_0, \mu_0)_n.$$

3.6. Lemma. *Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$, and let μ_0 be a polarization on Y_0. Let S be a scheme over κ.*

(i) *Let $(T_n)_{n \geq 1}$ be a compatible family of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$. For each $n \geq 1$, let*

$$X_n := T_n \wedge^{\mathbf{Aut}^{\mathrm{st}}(Y_0)_n} Y_0[p^n]$$

be the contraction product of T_n with $Y_0[p^n]$ with respect to the natural action of $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ on $Y_0[p^n]$. Then X_n is a BT_n group over S, and the family $\big(X_n\big)_{n \geq 1}$ together with the natural maps $X_n \hookrightarrow X_{n+1}$ and $X_{n+1} \twoheadrightarrow X_n$ is a p-divisible group over S.

(ii) *The constructions in* 3.5 (i)–(ii) *and* 3.6 (i) *define an equivalence between*

(a) *the category of strongly κ-sustained p-divisible groups over S modeled on Y_0, and*

(b) *the category of projective systems of right torsors $\big(T_n\big)_{n\geq 1}$ for the group schems $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\times_{\mathrm{Spec}(\kappa)}S$ over S which are compatible with the projective system of group schemes $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)\big)_{n\geq 1}$, in the sense that the projection map $T_{n+1}\to T_n$ is equivariant with respect to the projection $\mathbf{Aut}^{\mathrm{st}}(Y_0)_{n+1}\to \mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ for every n.*

(iii) *Let μ_0 be a polarization on Y_0. The obvious analog of* (i) *holds for strongly κ-sustained polarized p-divisible groups over S modeled on (Y_0,μ_0). This construction and* 3.5 (iii) *define an equivalence of categories between category of strongly κ-sustained polarized p-divisible groups over S modeled on (Y_0,μ_0) and the category of projective systems of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0,\mu_0)_n\times_{\mathrm{Spec}(\kappa)}S$ which are compatible with the projective system of group schemes $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0,\mu_0)\big)_{n\geq 1}$.*

4. Smoothness of sustained deformations

Let $\kappa\supseteq\mathbb{F}_p$ be a field. Let $\mathfrak{Art}_\kappa$ be the category whose objects are triples

$$(R,\, i:\kappa\to R,\, \epsilon: R\twoheadrightarrow \kappa),$$

where (R,i) is an Artinian local κ-algebra, and ϵ is a κ-linear surjective ring homomorphism whose kernel is the maximal ideal of R. A morphism in $\mathfrak{Art}_\kappa$ from (R_1,i_1,ϵ_1) to (R_2,i_2,ϵ_2) is a κ-linear ring homomorphism $h:R_1\to R_2$ such that $\epsilon_2\circ h_2=\epsilon_1$.

4.1. Definition. Let $\kappa\supseteq\mathbb{F}_p$ be a field and let Y_0 be a p-divisible group over κ.

(i) The functor $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of sustained deformations of Y_0 is the functor from $\mathfrak{Art}_\kappa$ to the category of sets which sends every object (R,i,ϵ) to the set of isomorphism classes of pairs

$$\big(X\to\mathrm{Spec}(R),\psi: Y_0\xrightarrow{\sim} X\times_{\mathrm{Spec}(R)}\mathrm{Spec}(\kappa)\big),$$

where $X\to\mathrm{Spec}(R)$ is a strongly κ-sustained p-divisible group modeled on Y_0 and ψ is a κ-isomorphism. Two pairs $(X_1\to\mathrm{Spec}(R),\,\psi_1)$, $(X_2\to\mathrm{Spec}(R),\,\psi_2)$ are isomorphic if there exists an isomorphism $\alpha: X_1\to X_2$ of p-divisible groups over R such that $\big(\alpha\times_S\mathrm{Spec}(\kappa)\big)\circ\psi_1=\psi_2$.

(ii) Let μ_0 be a polarization on Y_0. The functor $\mathbf{Def}^{\mathrm{sus}}(Y_0,\mu_0)$ of sustained deformations of Y_0 is the functor from $\mathfrak{Art}_\kappa$ to the category of sets which sends every object (R,i,ϵ) to the set of isomorphism classes of triples

$$\big(X\to\mathrm{Spec}(R),\lambda:X\to X^t,\psi:(Y_0,\mu_0)\xrightarrow{\sim}(X,\lambda)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(\kappa)\big),$$

where (X,λ) is a strongly κ-sustained p-divisible group over R modeled on (Y_0,μ_0), and ψ is an isomorphism from Y_0 to $X\times_S\mathrm{Spec}(\kappa)$ such that $\psi^*(\lambda\times_S\mathrm{Spec}(\kappa))=\mu_0$. Two triples $(X_1\to\mathrm{Spec}(R),\lambda_1,\psi_1)$, $(X_2\to\mathrm{Spec}(R),\lambda_2,\psi_2)$ are isomorphic if there exists an isomorphism $\alpha:X_1\to X_2$ over R such that $\alpha^*(\lambda_2)=\lambda_1$ and

$$\big(\alpha\times_{\mathrm{Spec}(R)}\mathrm{Spec}(\kappa)\big)\circ\psi_1=\psi_2.$$

Remark. Theorem 4.3 shows that the sustained deformation functors $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0,\mu_0)$ are representable. Each is isomorphic to the formal spectrum of a formal power series over κ in a finite number of variables, and $\mathbf{Def}^{\mathrm{sus}}(Y_0,\mu_0)$ is a closed formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$.

4.2. Lemma. *Let $x_0 = [(A_0,\lambda_0,\psi_0)]$ be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$ and let $\mathcal{C}(x_0)$ be the central leaf in $\mathcal{A}_{g,d,n}$ passing through x_0 as in* 2.9*. Denote by $\mathcal{C}(x_0)^{/x_0}$ the formal completion of $\mathcal{C}(x_0)$ at x_0. The natural morphism*

$$\mathcal{C}(x_0)^{/x_0} \longrightarrow \mathbf{Def}^{\mathrm{sus}}(A_0[p^\infty],\mu_0[p^\infty])$$

is an isomorphism.

4.3. Theorem. *Let $\kappa \supseteq \mathbb{F}_p$ be a field. Let Y_0 be a p-divisible group over κ and let μ_0 be a polarization of Y_0.*

(i) *The deformation functors $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0,\mu_0)$ are formally smooth over κ.*

(ii) *The dimension of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is given by*

$$\dim\big(\mathbf{Def}^{\mathrm{sus}}(Y_0)\big) = \dim\big(\mathbf{End}^{\mathrm{st}}(Y_0)\big),$$

the dimension of the p-divisible group $\mathbf{End}^{\mathrm{st}}(Y_0)$.

(iii) *We have*

$$\dim\big(\mathbf{Def}^{\mathrm{sus}}(Y_0,\mu_0)\big) = \dim\big(\mathbf{End}^{\mathrm{st}}(Y_0)^{\iota_{\mu_0}=-\mathrm{id}}\big),$$

where ι_{μ_0} is the Rosati involution on the p-divisible group $\mathbf{End}^{\mathrm{st}}(Y_0)$ defined in 3.4.

4.4. We sketch a proof of the smoothness of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and the statement 4.3 (ii). The proofs of the smoothness of the deformation functor $\mathbf{Def}^{\mathrm{sus}}(Y_0,\mu_0)$ and 4.3 (iii) are similar.

Let $h : (R',i',\epsilon') \to (R,i,\epsilon)$ be a morphism in such that $h : R' \to R$ is a surjection and $J := \mathrm{Ker}(h)$ is killed by the maximal ideal $\mathfrak{m}'$ of R'. In other words R' is a small extension of R. Let $S = \mathrm{Spec}(R)$, let $S' = \mathrm{Spec}(R')$, and let $S_0 := \mathrm{Spec}(\kappa)$. We need to show that, given a strongly κ-sustained p-divisible group X over R modeled on Y_0 plus a rigidification $\psi : Y_0 \xrightarrow{\sim} X \times_S S_0$, there exists a lifting of the pair (X,ψ) in $\mathbf{Def}^{\mathrm{sus}}(Y_0)(R)$ to a pair (X',ψ') in $\mathbf{Def}^{\mathrm{sus}}(Y_0)(R')$.

Let $(T_n)_{n\geq 1}$ be the projective family of right torsors for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S$ associated to the strongly κ-sustained p-divisible group over S as in 3.5, together with the trivializations

$$\phi_n : \mathbf{Aut}^{\mathrm{st}}(Y_0)_n \xrightarrow{\sim} T_n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$$

associated to ψ. According to 3.6 (ii), it suffices to show that the compatible family $\big(T_n,\phi_n\big)_{n\geq 1}$ of rigidified right torsors for $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S\big)_{n\geq 1}$ lifts to a compatible family $\big(T'_n,\phi'_n\big)_{n\geq 1}$ of rigidified right torsors for $\big(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'\big)_{n\geq 1}$.

For each $n \geq 1$, we have a perfect complex $\ell_{T_n/S}$ of $\mathscr{O}_S$-modules of amplitude $\subseteq [-1,0]$, called the co-Lie complex of T_n/S; see [13, Ch. 7, §2.4]. By [13, Ch. 7, Thm. 2.4.4], there is an obstruction element

$$o(T_n, S \hookrightarrow S') \in H^2(S, \ell^\vee_{T_n/S} \otimes^{\mathbb{L}}_R J)) \cong H^2(S_0, \ell^\vee_{T_n \times_S S_0/S_0} \otimes_\kappa J)$$

whose vanishing is the necessary and sufficient condition for the existence of a right torsor for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'$ which extends T_n. Since $\ell^\vee_{T_n \times_S S_0/S_0}$ is perfect of amplitude $\subseteq [0,1]$ and S_0 is affine,

$$H^2(S_0, \ell^\vee_{T_n \times_S S_0/S_0} \otimes_\kappa J) = (0).$$

Therefore T_n can be extended to a right torsor for $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'$, for every n.

The theorem [13, Ch. 7, Thm. 2.4.4] also tells us that the set of all liftings to S' of T_n has a natural structure as a torsor for

$$H^1(S_0, \ell^\vee_{T_n \times_S S_0/S_0} \otimes_\kappa J) = \nu_{T_n \times_S S_0/S_0} \otimes_\kappa J,$$

where

$$\nu_{T_n \times_S S_0/S_0} := H^1(\ell^\vee_{T_n \times_S S_0/S_0}) = H^1(\ell^\vee_{\mathbf{Aut}^{\mathrm{st}}(Y_0)_n/S_0}).$$

The last equality holds because the torsor $T_n \times_S S_0$ over $S_0 = \mathrm{Spec}(\kappa)$ is trivial.

Claim. The natural map

$$\nu_{T_{n+1} \times_S S_0/S_0} \otimes_\kappa J \longrightarrow \nu_{T_n \times_S S_0/S_0} \otimes_\kappa J$$

is an isomorphism for every $n \geq 1$.

Clearly the claim implies the existence of a projective system $\left(T'_n\right)_{n\geq 1}$ of right torsors for $\left(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \times_{\mathrm{Spec}(\kappa)} S'\right)_{n\geq 1}$ which extends $\left(T_n\right)_{n\geq 1}$. This finishes the proof of the smoothness of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$. The above claim also shows that the tangent space of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is naturally isomorphic to the κ-vector space $\nu_{T_n \times_S S_0/S_0}$, for any $n \geq 1$.

The key fact for the claim is the existence of a projective system of decreasing "slope filtration"

$$\left(\mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n,\ t \in [0,1]\right)_{n\geq 1}$$

on the projective system $\left(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\right)_{n\geq 1}$, where each $\mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ is a normal subgroup scheme of $\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$ for each $t \in [0,1]$ and every $n \geq 1$. This filtration has the following properties.

(a) The subquotient

$$\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n := \mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n / \mathrm{Fil}^{>t}_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n$$

is a commutative finite group scheme for every $t > 0$ and ever $n \geq 1$.

(b) $\nu_{\mathrm{gr}^0_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n/S_0} = (0)$ for every $n \geq 1$.

(c) For every $t \in (0,1]$, the projective system

$$\left(\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\right)_{n\geq 1}$$

"is" a p-divisible group, in the sense that there exist homomorphisms

$$\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n \to \mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_{n+1}$$

which together with the projections make the family of commutative group schemes $\left(\mathrm{gr}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\right)_{n\geq 1}$ a p-divisible group over κ.

Recall the fact that for any p-divisible group Z over κ, the natural map

$$\nu_{Z[p^{n+1}]/\kappa} \to \nu_{Z[p^n]/\kappa}$$

is an isomorphism for every $n \geq 1$; see [15, Prop. 2.2.1]. The Claim follows from this fact and dévissage, using the above filtration on $\left(\mathbf{Aut}^{\mathrm{st}}(Y_0)_n\right)_{n\geq 1}$ and the exactness properties of co-Lie complexes of group schemes.

Note that the above argument also shows that

$$\dim(\mathbf{Def}^{\mathrm{sus}}(Y_0)) = \dim(\nu_{\mathbf{End}^{\mathrm{st}}(Y_0)_n}) = \dim(\mathbf{End}^{\mathrm{st}}(Y_0)) \quad \forall\, n \geq 1.$$

The second equality is a general property of p-divisible groups over fields of characteristic $p > 0$. We have finished the sketch of the proofs the smoothness of the deformation functor $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and the statement (ii).

4.5. Corollary. *Let g, d, n be positive integers, with $n \geq 3$ and $\gcd(n,p) = 1$. For every $x_0 \in \mathcal{A}_{g,d,n}(\overline{\mathbb{F}}_p)$, the central leaf $\mathcal{C}_{\mathcal{A}_{g,d,n}}(x_0)$ as defined in 2.9 is smooth over $\overline{\mathbb{F}}_p$.*

PROOF. This is an immediate consequence of 4.2 and 4.3. □

4.6. Remark. When d is divisible by a high power of p, the moduli scheme $\mathcal{A}_{g,d,n}$ is known to be a local complete intersection, but otherwise it tends to exhibit many "unpleasant" phenomena. For instance it is non-reduced at every point, and the dimension of its supersingular locus is larger than the "expected dimension" $\lfloor g^2/4 \rfloor$. Thus the smoothness of central leaves in $\mathcal{A}_{g,d,n}$ might be a small surprise.

5. Local properties of central leaves with at most 3 slopes

5.1. In this section we illustrate two general phenomena of sustained deformation spaces $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$, where Y_0 is a p-divisible group over an algebraically closed field $k \supseteq \mathbb{F}_p$ and μ_0 is a polarization on Y_0:

(a) The formal schemes $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ are "built up" from p-divisible formal groups through a family of fibrations whose fibers are p-divisible formal groups. Formal schemes with such properties are said to be *Tate-linear*.

(b) (Local rigidity of Tate-linear formal schemes) Suppose that a p-adic Lie group G operates on a formal scheme $\mathbf{D}$ with a Tate-linear structure, and the action is *strongly non-trivial* in a strong sense. Then every irreducible reduced closed formal subscheme of $\mathbf{D}$ which is stable under the action of an open subgroup of G is a *Tate-linear formal subscheme* of $\mathbf{D}$.

For simplicity we will only illustrate phenomenon (a) for $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and phenomenon (b) for $\mathbf{D} = \mathbf{Def}^{\mathrm{sus}}(Y_0)$, both in the case when the p-divisible group Y_0 has at most three slopes. The formal schemes $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{D}$ we meet in these examples are either p-divisible formal groups over κ, or bi-extensions of p-divisible formal groups over κ. The precise definition of Tate-linear formal subschemes in either case is given in 5.6.

Note that the case when Y_0 is isoclinic is trivial, because $\mathbf{Def}^{\mathrm{sus}}(Y_0) = \mathrm{Spf}(k)$ if Y_0 is isoclinic.

5.2. Lemma. *Let Z_1, Z_2 be two isoclinic p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$, with slopes $s_1 > s_1$. Let $Y_0 := Z_1 \times Z_2$. Then $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is naturally isomorphic to the p-divisible formal group $\mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$.*

Note that $\mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$ is isoclinic of slope $s_1 - s_2$ and its height is $\mathrm{ht}(Z_1) \cdot \mathrm{ht}(Z_2)$, according to 3.2 (d).

5.3. Remark. (i) The proof of 5.2 is an easy consequence of 3.2 plus some diagram chasing with suitable Kummer sequences.

(ii) Usually a p-divisible group $Y \to S$ is considered as the *inductive* limit of its truncations $Y[p^n]$ in the category of sheaves on the flat site S_{fl} of S. If we adhere to this point of view, then the sheaf $\mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$ is *not* the "internal Hom" from Z_2 to Z_1 in the category of sheaves on S_{fl}. Rather it is essentially the "internal Ext^1" from Z_2 to Z_1.

5.4. Lemma. *Let Y_1, Y_2, Y_3 be isoclinic p-divisible groups over a field $\kappa \supseteq \mathbb{F}_p$, with slopes $s_1 > s_2 > s_3$. Let $Y_0 = Y_1 \times Y_2 \times Y_3$. Let $Z_{21} := \mathbf{Hom}^{\mathrm{st}}(Z_2, Z_1)$, let $Z_{32} := \mathbf{Hom}^{\mathrm{st}}(Z_3, Z_2)$, and let $Z_{31} := \mathbf{Hom}^{\mathrm{st}}(Z_3, Z_1)$. The formal scheme $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ has a natural structure as a bi-extension of the p-divisible formal groups (Z_{21}, Z_{32}) by the p-divisible formal group Z_{31}.*

We refer to [19] for the definition and basic properties of biextensioins of formal groups.

5.5. Remark. Suppose that Y is a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$ with three slopes. Let

$$(0) \subsetneqq Y_1 \subsetneqq Y_2 \subsetneqq Y$$

be the slope filtration of Y, so that

$$Z_1 := Y_1, \quad Z_2 := Y_2/Y_1, \quad Z_3 := Y/Y_2$$

are isoclinic p-divisible groups, of slopes

$$s_1 > s_2 > s_3$$

respectively. Proposition 2.4 defines maps

$$\pi_1 : \mathbf{Def}^{\mathrm{sus}}(Y) \to \mathbf{Def}^{\mathrm{sus}}(Y/Y_1) \quad \text{and} \quad \pi_1 : \mathbf{Def}^{\mathrm{sus}}(Y) \to \mathbf{Def}^{\mathrm{sus}}(Y_2).$$

The maps π_1 and π_2 define a morphism

$$\pi : \mathbf{Def}^{\mathrm{sus}}(Y) \longrightarrow \mathbf{Def}^{\mathrm{sus}}(Y/Y_1) \times_{\mathrm{Spec}(\kappa)} \mathbf{Def}^{\mathrm{sus}}(Y_2).$$

So far things look quite similar to 5.4. But unlike the situation in 5.4, here the map π does not make $\mathbf{Def}^{\mathrm{sus}}(Y)$ a bi-extension of $\big(\mathbf{Def}^{\mathrm{sus}}(Y/Y_1), \mathbf{Def}^{\mathrm{sus}}(Y_2)\big)$. The troubles are two fold.

(i) To begin with, $\mathbf{Def}^{\mathrm{sus}}(Y/Y_1)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_2)$ are torsors, not groups.
(ii) A more serious problem is that in general, the maps π_1, π_2 and π are not (formally) smooth.

The reason for (ii) is that the natural epimorphisms

$$\mathbf{End}^{\mathrm{st}}(Y) \longrightarrow \mathbf{End}^{\mathrm{st}}(Y/Y_1)$$

and

$$\mathbf{End}^{\mathrm{st}}(Y) \longrightarrow \mathbf{End}^{\mathrm{st}}(Y_2)$$

of p-divisible groups over κ may not be smooth, i.e. their kernels may not be p-divisible groups.

The moral here is that one needs to be careful when formulating the Tate-linear structure on the deformation spaces $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ and $\mathbf{Def}^{\mathrm{sus}}(Y_0, \mu_0)$ when the p-divisible group Y_0 over κ is not isomorphic to a product of isoclinic p-divisible groups.

5.6. Definition. Let $\kappa \supseteq \mathbb{F}_p$ be a field. Let Z, Z_1, Z_2, Z_3 be p-divisible formal groups over κ, and let B be a bi-extension of (Z_1, Z_2) by Z_3.

(a) A closed formal subscheme V of Z is *Tate-linear* if V is a p-divisible subgroup of Z.
(b) A closed formal subscheme V of B is *Tate-linear* if there exists a p-divisible subgroup Z_3' of Z_3 and a p-divisible subgroup U of $Z_1 \times Z_2$, such that V is stable under the action of Z_3' and the projection map $B \to Z_1 \times Z_2$ induces an isomorphism $V/Z_3 \xrightarrow{\sim} U$.

5.7. Definition. (a) An action of a p-adic Lie group on a p-divisible formal group Y over a field $\kappa \supseteq \mathbb{F}_p$ is said to be *strongly non-trivial* if none of the Jordan–Hölder component the induced action of the Lie algebra $\mathrm{Lie}(G)$ of G on $\mathbb{D}_*\big(Y \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa^{\mathrm{alg}})\big)_{\mathbb{Q}}$ is the trivial representation of $\mathrm{Lie}(G)$. Here κ^{alg} denotes an algebraic closure of κ.

(b) Let Y_1, Y_2, Z be p-divisible formal groups over κ. An action of a p-adic Lie group G on a bi-extension of (Y_1, Y_2) by Z which respects the bi-extension structure is said to be *strongly non-trivial* if the induced actions of G on Y_1, Y_2 and Z are all strongly non-trivial.

5.8. Theorem (Local rigidity for p-divisible formal groups and their biextensions). *Let $\kappa \supseteq \mathbb{F}_p$ be an algebraically closed field and let G be a p-adic Lie group. Let Y_1, Y_2, Z be p-divisible formal groups over κ.*

(1) *If $V \subseteq Z$ is an irreducible closed formal subvariety of Z stable under a strongly non-trivial action of G on Z, then V is a formal subgroups of Z.*
(2) *Let B be a bi-extension of $Y_1 \times Y_2$ by Z. Suppose that $V \subset B$ is a reduced irreducible closed formal subscheme of B stable under a strongly non-trivial action of G on B.*
    (a) *The formal subscheme V of the bi-extension B is Tate-linear.*

(b) *Furthermore if Y_1, Y_2 do not have any slope in common, then V is a sub-biextension of B.*

5.9. Remark. (a) See [4] for a proof of (a) and [11] for a proof of (b).

(b) It is expected that the method for (b) is can be extended to give a general local rigidity result for the sustained deformation spaces $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of any p-divisible group Y_0 over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$. See 6.3 for the precise statement.

(c) Let x_0 be an $\overline{\mathbb{F}_p}$-point of $\mathcal{A}_{g,d,n}$ such that $A_{x_0}[p^\infty]$ is isomorphic to a product of at most three isoclinic p-divisible groups. Let V be a reduced closed subscheme of a central leaf $\mathcal{C} = \mathcal{C}(x_0)$ in $\mathcal{A}_{g,d,n}$ which is stable under all prime-to-p Hecke correspondences on $\mathcal{A}_{g,d,n}$. Then theorem 5.8 applies to the closed formal subscheme $V^{/z_0} \subseteq \mathcal{C}^{/z}$, for every $\overline{\mathbb{F}_p}$-point z_0 of V, with G being a compact open subgroup of the group of all $\mathbb{Q}_p$-points of the Frobenius torus associated to the closed point z_0. One concludes that the formal completion $V^{/z_0}$ of V at any closed point $z_0 \in V$ is a Tate-linear formal subscheme of the Tate-linear formal scheme $\mathcal{C}^{/z_0}$.

5.10. Global rigidity questions. Let $\mathcal{C} \subseteq \mathcal{A}_{g,1,n}$ be a leaf in $\mathcal{A}_{g,1,n}$ over $\overline{\mathbb{F}_p}$. Let $Z \subseteq \mathcal{C}$ be an irreducible closed subscheme of $\mathcal{C}$. Let $z \in Z(\overline{\mathbb{F}_p})$ be a closed point of Z. Suppose that $Z^{/z} \subseteq$ is stable under a strongly non-trivial action of a p-adic Lie group G which respects the Tate-linear structure on $\mathcal{C}^{/z}$. Assume that the abelian variety A_z has at most three distinct slopes, and $A_z[p^\infty]$ is isomorphic to a product of isoclinic p-divisible groups. We formulate an expectation and a question below.

5.10.1. Expectation. The formal subscheme $Z \subseteq \mathcal{C}$ is Tate-linear at every closed point of Z.

5.10.2. Question. Is Z (an irreducible component of) the reduction of a Shimura subvariety of the Siegel modular variety $\mathcal{A}_{g,1,n}$ over $\overline{\mathbb{Q}}$?

Remark. (a) In few case when Z is contained in the reduction of a "small" Shimura subvariety of the Siegel modular variety, the answer to 5.10.2 is affirmative.

(b) Most people seem to believe that the answer to 5.10.2 is "yes", but there is little evidence other than our inability to produce a counter-example. The case when $g = 2$ is already a challenge.

6. Strongly Tate-linear formal subschemes of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$

6.1. Definition. Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$. Let

$$\big(\mathbf{Aut}^{\mathrm{st}}(Y)_n\big)_{n\geq 1} =: \big(\Gamma_n\big)_{n\geq 1}$$

be the projective family of stabilized Aut group schemes of Y_0, and let

$$\Big(\mathrm{Fil}^t\Gamma_n,\ t\in[0,1]\Big)_{n\geq 1} := \Big(\mathrm{Fil}^t_{\mathrm{can}}\mathbf{Aut}^{\mathrm{st}}(Y_0)_n,\ t\in[0,1]\Big)_{n\geq 1}$$

be the slope filtration on $\big(\Gamma_n\big)_{n\geq 1}$. A projective family $\big(H_n\big)_{n\geq 1}$ of subgroup schemes $H_n \subseteq \Gamma_n$ is said to be *stable* if the vertical arrows in the diagram

$$\begin{array}{ccccc} \mathrm{Fil}^{>t}H_{n+1} & \longrightarrow & \mathrm{Fil}^{\geq t}H_{n+1} & \longrightarrow & \mathrm{Fil}^{\geq t}H_{n+1}/\mathrm{Fil}^{>t}H_{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fil}^{>t}H_n & \longrightarrow & \mathrm{Fil}^{\geq t}H_n & \longrightarrow & \mathrm{Fil}^{\geq t}H_n/\mathrm{Fil}^{>t}H_n \end{array}$$

are faithfully flat for all $n \geq 1$ and all $t \in [0,1]$, where

$$\mathrm{Fil}^{>t}H_n := H_n \cap \mathrm{Fil}^{>t}\Gamma_n, \quad \mathrm{Fil}^{\geq t}H_n := H_n \cap \mathrm{Fil}^{\geq t}\Gamma_n,$$

and the projective system

$$\Big(\mathrm{Fil}^{\geq t}H_n/\mathrm{Fil}^{>t}H_n\Big)_{n\geq 1}$$

comes from a p-divisible group over κ, for every $t \in (0,1]$.

6.1.1. Definition. Let Y_0 be a p-divisible group over a field $\kappa \supseteq \mathbb{F}_p$, and let $\tilde{H} = \big(H_n\big)_{n\geq 1}$ be a stable family of subgroup schemes of the stabilized Aut group schemes $\big(\Gamma_n\big)_{n\geq 1}$ of Y_0 as in 6.1. Let

$$\tilde{T} := (T_n, \psi_n)_{n\geq 1}$$

be a compatible projective family of right Γ_n-torsors, in the sense that T_n is a right Γ_n-torsor and

$$\psi_n : T_{n+1} \wedge^{H_{n+1}} H_n \xrightarrow{\sim} T_n$$

is a isomorphism of right H_n-torsors for each $n \geq 1$. Here $T_{n+1} \wedge^{H_{n+1}} H_n$ denote the contration product of T_{n+1} with H_n via the faithfully flat homomorphism $H_{n+1} \to H_n$, which has a natural structure as a right H_n-torsor.

The deformation functor $\mathbf{Def}(\tilde{T})$ of $\tilde{T}$ is the functor from $\mathfrak{Art}_\kappa$ to the category of sets, which sends every object (R, j, ϵ) in $\mathfrak{Art}_\kappa$ to the set of all isomorphism classes of projective families

$$\Big(\boldsymbol{T}_n, \boldsymbol{\psi}_n, \zeta_n\Big)_{n\geq 1},$$

where

- $\boldsymbol{T}_n$ is a right torsor for $H_{n,R} := H_n \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(R)$,
- $\boldsymbol{\psi}_n : \boldsymbol{T}_{n+1} \wedge^{H_{n+1,R}} H_{n,R} \xrightarrow{\sim} \boldsymbol{T}_n$ is an isomorphism of right $H_{n,R}$-torsors,
- $\zeta_n : T_n \xrightarrow{\sim} \boldsymbol{T}_n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$ is an isomorphism of right H_n-torsors,
- $\boldsymbol{\psi}_n \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa)$ is naturally identified with ψ_n via the isomorphisms ζ_{n+1} and ζ_n

for every $n \geq 1$.

The argument for 4.3 also shows the following lemma 6.1.2.

6.1.2. Lemma. *Let Y_0 be a p-divisible group over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$, and let $\tilde{H} = \big(H_n\big)_{n\geq 1}$ be a stable family of subgroup schemes of the stabilized Aut group schemes $\big(\Gamma_n\big)_{n\geq 1}$ of Y_0 as in 6.1. Let*

$$T := (T_n, \psi_n)_{n\geq 1}$$

be a compatible projective family of right Γ_n-torsors over κ and let

$$\Bigl(\alpha_n : T_n \wedge^{H_n} \Gamma_n \xrightarrow{\sim} \mathbf{Aut}^{\mathrm{st}}(Y_0)_n\Bigr)_{n\geq 1}$$

be a compatible family of isomorphisms of right Γ_n-torsors. The deformation functor $\mathbf{Def}(\tilde{H})$ is formally smooth over κ, and the compatible family of isomorphisms $(\alpha_n)_{n\geq 1}$ defines a closed embedding

$$\mathbf{Def}(\tilde{H}) \hookrightarrow \mathbf{Def}^{\mathrm{sus}}(Y_0)\,.$$

6.2. Definition. Let Y_0 be a p-divisible group over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$. A smooth closed formal subscheme Z of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ is said to be a *strongly Tate-linear* formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ if there exists

- a stable projective family $\tilde{H} = \bigl(H_n)_{n\geq 1}$ of subgroup schemes of the stabilized Aut group schemes $\bigl(\Gamma_n\bigr)_{n\geq 1}$ of Y_0 as in 6.1,
- a compatible family of right H_n-torsors

$$\tilde{T} = \bigl(T_n\bigr)_{n\geq 1}$$

  over κ, and
- a compatible family of isomorphisms

$$\Bigl(\alpha_n : T_n \wedge^{H_n} \Gamma_n \xrightarrow{\sim} \mathbf{Aut}^{\mathrm{st}}(Y_0)_n\Bigr)_{n\geq 1}$$

  of right Γ_n-torsors over κ

such that Z is equal to the image of the close embedding

$$\mathbf{Def}(\tilde{T}) \hookrightarrow \mathbf{Def}^{\mathrm{sus}}(Y_0)$$

associated to $\bigl(\alpha_n\bigr)_{n\geq 1}$ as in 6.1.2.

6.2.1. Remark. (i) Formal completions of central leaves in the reduction modulo p of a Shimura subvariety in a Siegel modular variety are strongly Tate-linear. In particular, every central leaf $\mathcal{C}$ in a PEL type modular variety $\mathscr{M}$ over an algebraically closed field κ and every κ-point $z_0 \in \mathcal{C}$, the formal completion $\mathcal{C}^{/z_0}$ of $\mathcal{C}$ at z_0 is a strongly Tate-linear formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(A_{z_0}[p^\infty])$, where A_{z_0} is the abelian variety with PEL structure corresponding to the point z_0 in the modular variety $\mathscr{M}$.

(ii) There are good reasons to view central leaves in the reduction of a Shimura subvariety of a Siegel modular variety as characteristic-p analogues of Shimura varieties. If one adopts this perspective, then a strongly Tate-linear formal subscheme Z of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ as in 6.2 can be regarded as a local version of "characteristic-p Shimura variety", and the stable projective family of subgroup schemes $\tilde{H}$ attached to Z as in 6.2 serves as an analog of the reductive $\mathbb{Q}$-group in the Shimura input datum of a Shimura variety.

6.2.2. Remark. Suppose that Z is a strongly Tate-linear formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of the form $Z = \mathrm{Im}\bigl(\mathbf{Def}(\tilde{T}) \hookrightarrow \mathbf{Def}^{\mathrm{sus}}(Y_0)\bigr)$ for a compatible family of right H_n-torsors $\tilde{T} = \big(T_n\big)_{n\geq 1}$ as in 6.2. Let $\mathbf{Y}$ be the universal p-divisible group over $\mathbf{Def}^{\mathrm{sus}}(Y_0)$. Then we have a compatible family of isomorphisms

$$\phi_n : \mathbf{T}_n \wedge^{(H_n\times_{\mathrm{Spec}(\kappa)}Z)} (\Gamma_n \times_{\mathrm{Spec}(\kappa)} Z) \xrightarrow{\sim} \mathbf{Isom}^{\mathrm{st}}(Y_0,\mathbf{Y})_n \times_{\mathbf{Def}^{\mathrm{sus}}(Y_0)} Z$$

of right torsors for $\Gamma_n \times_{\mathrm{Spec}(\kappa)} Z$, where $\big(\mathbf{T}_n\big)_{n\geq 1}$ is the universal family of right H_n-torsors over $Z = \mathbf{Def}(\tilde{T})$. In the parlance of differential geometry, the stable family $\big(H\big)_{n\geq 1}$ of subgroup schemes of Γ_n is uniquely determined by Z as the smallest stable family of subgroup schemes of Γ_n such that the compatible family of right Γ_n-torsors

$$\Big(\mathbf{Isom}^{\mathrm{st}}(Y_0,\mathbf{Y})_n \times_{\mathbf{Def}^{\mathrm{sus}}(Y_0)} Z\Big)_{n\geq 1}$$

admits a "reduction of structural group" to the subgroup schemes H_n in a way that is compatible with the transition maps.

6.3. A local rigidity question for sustained deformation spaces.
NOTATION.

- Let Y_0 be a p-divisible group over an algebraically closed field $\kappa \supseteq \mathbb{F}_p$.
- Let $\mathbf{End}^{\mathrm{st}}(Y_0)$ be the p-divisible group formed by the stabilized End group schemes of Y_0, and let $\mathrm{Fil}^{>0}\mathbf{End}^{\mathrm{st}}(Y_0)$ be the largest p-divisible formal subgroup of $\mathbf{End}^{\mathrm{st}}(Y_0)$.
- Let Z be a reduced and irreducible closed formal subscheme of the formal scheme $\mathbf{Def}^{\mathrm{sus}}(Y_0)$.
- Let G be a closed subgroup of the compact p-adic group $\mathrm{Aut}(Y_0)$.

Expectation. Suppose that Z is stable under the natural action of G on the sustained deformation space $\mathbf{Def}^{\mathrm{sus}}(Y_0)$ of Y_0, and the natural action of G on the p-divisible formal group $\mathrm{Fil}^{>0}\mathbf{End}^{\mathrm{st}}(Y_0)$ over κ is strongly non-trivial in the sense of 5.7 (a). Then Z is a strongly Tate-linear formal subscheme of $\mathbf{Def}^{\mathrm{sus}}(Y_0)$.

6.3.1. Remark. (i) Theorem 5.8 says that the statement 6.3 holds when Y_0 is a product of at most three isoclinic p-divisible groups over κ. So there is considerable evidence supporting this expectation.

(ii) As remarked in 5.9 (b), it seems likely that the method for proving 5.8 (b) will also deliver the more general statement 6.3.

(iii) In applications to the Hecke orbit problem, we are given a reduced closed subscheme V of a central leaf $\mathcal{C}$ over $\overline{\mathbb{F}_p}$ in a Shimura subvariety $\mathcal{S}$ of a Siegel modular variety, such that V is stable under all prime-to-p Hecke correspondences on $\mathcal{S}$, and we want to show that $\mathcal{C}$ is equal to $\mathcal{S}$. Let z_0 be an $\overline{\mathbb{F}_p}$-point of the smooth locus of V, corresponding to a polarized abelian variety A_0 over $\overline{\mathbb{F}_p}$ with extra symmetries. Let $V^{/z_0}$ (respectively $\mathcal{C}^{/z_0}$) be the formal completion at z_0 of V (respectively $\mathcal{C}$). We have inclusions $V^{/z_0} \subseteq \mathcal{C}^{/z_0} \subseteq \mathbf{Def}^{\mathrm{sus}}(A_{z_0}[p^\infty])$. The fact that V is stable under all prime-to-p Hecke correspondences on $\mathcal{S}$ implies that there exists a compact open subgroup of the group G of p-adic points of the Frobenius torus attached to z_0, such that the formal subscheme $V^{/z_0}$ of $\mathbf{Def}(A_{z_0}[p^\infty])$ stable under the natural action of G on ($\mathcal{C}^{/z_0}$ and) $\mathbf{Def}(A_{z_0}[p^\infty])$. So the rigidity statement 6.3 implies that $V^{/z_0}$ is a strongly Tate-linear formal subscheme of $\mathcal{C}^{/z_0}$

and $\mathbf{Def}(A_{z_0}[p^\infty])$. This conclusion is still some distance away from the desired conclusion, but it is a structural constraint on Hecke-stable subvarieties of $\mathcal{C}$.

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