Density of Hecke orbits for abelian varieties of p-corank one

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Abstract

Let k be a field of characteristic $p > 0$ and let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties over k of dimension g. Suppose that (A, λ) is a principally polarized abelian variety over k such that $\mathrm{Card}(A[p](k^{\mathrm{alg}})) = p^{g-1}$. Then the symplectic separable isogeny class of (A, λ) is Zariski dense in the zero locus of the Hasse invariant in $\mathcal{A}_g$.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. We denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties over k of dimension g. It was proved in [3] that for every prime number $\ell \neq p$, the ℓ-power Hecke orbit of every ordinary point (A, λ) of $\mathcal{A}_g$ is Zariski dense in $\mathcal{A}_g$. In this paper we study the Hecke orbits of certain non-ordinary points in $\mathcal{A}_g$ which are quite close to being ordinary.

Clearly the separable Heck correspondences preserve all stratifications of $\mathcal{A}_g$ defined using the p-power torsion points of the universal abelian variety. On the other hand, the equivalence relation given by Zariski closure

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of prime-to-p Hecke orbits is strictly finer than the above stratifications if $g > 1$, because the prime-to-p Hecke orbit of every super-singular principally polarized abelian variety is finite; see [3, Prop. 1]. This suggests that a complete understanding of the Zariski closure of all prime-to-p Hecke orbits in $\mathcal{A}_g$ may be difficult.

For an abelian variety A of dimension g over an overfield of k, the p-rank $f(A)$ of A is defined to be the height of the maximal étale quotient of the Barsotti-Tate group attached to A. The p-rank $f(A)$ of A is an integer between 0 and g, and we shall call $g - f(A)$ the *p-corank* of A. The main result of this paper says that for every prime number $\ell \neq p$, every ℓ-power Hecke orbit in the p-corank-one locus of $\mathcal{A}_g$ is Zariski dense. This of course is the strongest possible statement for the p-corank-one locus, just as the case of the ordinary locus. For higher p-corank, our method can be used to show that the codimension of the Zariski closure of the ℓ-power Hecke orbit of a point (A, λ) with p-corank r is at most $\frac{r(r+1)}{2}$. This is clearly not the definitive statement though. So we have chosen not to present the higher p-corank case, in favor of keeping the exposition shorter than it would be otherwise. This generalization to the case of higher p-corank is left to the interested readers "as an exercise"—a nontrivial one which the author has not properly done, and there may be potential pitfalls.

Here is an outline of the argument in four steps. Much of the ideas already appeared in [3]; the precise definition of the Hecke orbits can also be found there. Let Z be the Zariski closure of the ℓ-power Hecke orbit of a point $x_0 = [(A_0, \lambda_0)]$ in $\mathcal{A}_g(k)$. As in the beginning of §3 of [3], it suffices to deal with the case when (A_0, λ) is defined over a finite field $\mathbb{F}_q$. Therefore we may assume from the beginning that $k = \mathbb{F}$, the algebraic closure of $\mathbb{F}_p$.

1) Use embedded Hilbert-Blumenthal varieties, c.f. [3, §3].

   It is well-known that an abelian variety defined over a finite field has sufficiently many complex multiplications; see [18]. Especially $\mathrm{End}(A)^\circ := \mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a product F of totally real number fields, such that $[F : \mathbb{Q}] = g$ and F is fixed under the Rosati involution $*$ on $\mathrm{End}(A)^\circ$. This allows us to construct an algebraic correspondence f from a Hilbert-Blumenthal moduli space $\mathcal{M}_F$ associated to F to $\mathcal{A}_g$, given by a proper morphism $\pi : \widetilde{\mathcal{M}_F} \to \mathcal{M}_F$ which is finite over the locus with p-corank at most one and a morphism $\tilde{f} : \widetilde{\mathcal{M}_F} \to \mathcal{A}_g$. Moreover there is a point x_1 in $\mathcal{M}_F$, represented by an abelian variety A_1

with multiplication by $\mathcal{O}_F$ which is F-isogenous to A_0, such that x_1 corresponds to x_0 under f. The ℓ-power Hecke orbit of x_1 in $\mathcal{M}_F$ is "mapped into" the ℓ-power Hecke orbit of x_0 in $\mathcal{A}_g$ in the obvious sense.

2) Solve the problem for Hilbert-Blumenthal varieties.

   Let F be a product of totally real number fields with $[F:\mathbb{Q}] = g$. Let A_1 be an abelian variety of dimension g over $k = \mathbb{F}$ with multiplication by $\mathcal{O}_F$ such that the p-rank of A_1 is equal to $g-1$. So the Barsotti-Tate group $A_1[p^\infty]$ splits into a direct sum of two Barsotti-Tate groups, $A_1[p^\infty]_{\mathrm{ord}}$ and $A_1[p^\infty]_{\mathrm{ss}}$; $A_1[p^\infty]_{\mathrm{ord}}$ has height $2g-2$ and slopes 0 and 1, while $A_1[p^\infty]_{\mathrm{ss}}$ has height 2 and slope $\frac{1}{2}$. Moreover there exists a degree-one prime $\wp$ of $\mathcal{O}_F$ over p, such that $A_1[p^\infty]_{\mathrm{ss}} = A_1[\wp^\infty]$.

   Using the method of [3, §4, §5], one shows that the Zariski closure Z_1 of the ℓ-power Hecke orbit of A_1 in $\mathcal{M}_F$ is equal to the locus in $\mathcal{M}_F$ where the Barsotti-Tate group $A[\wp^\infty]$ of the $\wp$-power torsion points of the universal abelian variety over $\mathcal{M}_F$ has slope $\frac{1}{2}$. The dimension of Z_1 is therefore equal to $g-1$. We note that the theory of linear subvarieties as explained in [2] can be applied to simplify the proof of [3, §4], as well as the corresponding part in this step.

3) Explore the action of the stabilizer subgroup at a super-singular point.

   The main result of [8] implies that Z_1 contains a supersingular point z_1. This is used in the previous step, just as in [3, §5]. Let z_0 be an image of z_1 under f, and let (A_z, λ_z) be the principally polarized abelian variety corresponding to z_0. As explained in [3, §1], since Z is stable under the prime-to-ℓ Hecke operators, the formal completion $\mathcal{A}_g^{/z_0}$ of $\mathcal{A}_g$ at z_0 is stable under the natural action of an open subgroup K of the group of automorphisms of $(A_z[p^\infty], \lambda_z)$. The latter group is a compact open subgroup of the group of $\mathbb{Q}_p$-points of an inner form of $\mathrm{Sp}_{2g}(\mathbb{Q}_p)$.

   Our commutative semisimple algebra F is embedded in $\mathrm{End}(A_z)^\circ$, fixed by the Rosati involution $*$; the embedding is denoted by j. Also recall the F has a degree-one place $\wp$. Let K be an open subgroup of the $\mathbb{Q}_p$-points of the unitary group attached to $(\mathrm{End}(A_z)^\circ, *)$. One shows that there exists an element $u \in K$ such that $u \cdot j(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cdot u^{-1}$ is the $\mathbb{Q}_p$-span of a $\mathbb{Q}$-subalgebra F' of $\mathrm{End}(A_z)^\circ$ and $u \cdot j(F_\wp) \cdot u^{-1}$ is the $\mathbb{Q}_p$-span of a $\mathbb{Q}$-subalgebra F'' of F'. One deduces that Z contains a point x_2 such that the corresponding principally polarized abelian variety (A_2, λ_2) is such that A_2 is isogenous to a product of an ordinary abelian variety with a supersingular elliptic curve.

4) At this point one can finish the proof without much difficulty. First, using the part of the ℓ-power Hecke correspondences which preserves the splitting of A_2 up to isogeny, one sees that by "varying the ordinary factor of A_2 by symplectic isogenies" and then take the Zariski closure, one gets a subvariety W of $\mathcal{A}_g$ of dimension $\frac{g(g-1)}{2}$. It is known that the local completion D of the p-corank-one locus at x_2 has a canonical structure as a T-torsor over a formal group G, where T is a formal torus of dimension $\frac{g(g-1)}{2}$ and G is non-canonically isomorphic to $g-1$ copies of the formal completion of a supersingular elliptic curve; see [4, Thm. 4.4, Prop. 4.7]. The subvariety W corresponds to the fiber T over the origin of G. It is easy to see that W is a proper subset of Z. Changing the point x_2 to another point of Z if necessary, we may assume that the ordinary factor of A_2 is the $(g-1)$st self-product of an ordinary elliptic curve over $\mathbb{F}$. Since the formal completion $Z^{/x_2}$ of Z at x_2 is stable under the natural action of the group of automorphisms of $(A_2[p^\infty], \lambda_2)$, this forces $Z^{/x_2}$ to be equal to D. In other words $\dim(Z) = \dim(D) = \frac{g(g+1}{2} - 1$, and Z is equal to the zero-locus of the Hasse invariant on $\mathcal{A}_g$.

In the exposition in the rest of this paper, the quaternionic linear algebra involved in step 3) is explained in §2; see the statement of Theorem 1. Step 2) is carried in §3. The rest of the proof are in §4. Roughly, the first two paragraphs of §4 set up the scene as sketched in step 1). In the next two paragraphs the results of §2 is applied to complete step 3). The rest of §4 completes the proof as sketched in step 4). Actually our presentation deviates a little from the above outline. We found it convenient to use an isogeny correspondence between $\mathcal{A}_{g,d}$ and $\mathcal{A}_g$, and a morphism i from $\mathcal{M}_F$ to $\mathcal{A}_{g,d}$. Here $\mathcal{A}_{g,d}$ denotes the moduli stack of abelian varieties of dimension g with a polarization of degree d^2, where d is a suitable positive integer. So in §4 the morphism i plays the role of f in the steps above.

Although the situation in this paper is different from that in [3], the ideas for steps 1), 2), 4) already appeared there. Only the method used in step 3) is really new. We also formulated a slight generalization, which can be proved by the same method; see Prop. 4.

This paper owes its existence to questions of F. Oort on relations between Hecke orbits and stratifications of the moduli space $\mathcal{A}_g$, see [15, 16A?] He also went through a first draft, pointed out many errors and inaccuracies, and suggested improvements. It is a great pleasure to thank him for stimulating discussions and continuing encouragement.

2 Linear algebra for the stabilizer subgroup at supersingular abelian varieties

Let (A_z, λ_z) be a polarized supersingular abelian variety over $\mathbb{F}$ of dimension g. It is well-known that A_z is isogenous to a self-product of a supersingular elliptic curve over $\mathbb{F}$. Therefore $E = \mathrm{End}_{\mathbb{F}}(A_z)^\circ$ is isomorphic to a matrix algebra $\mathrm{M}_g(B)$, where B is the quaternion division algebra over $\mathbb{Q}$ ramified exactly at p and ∞. The Rosati involution attached to λ_z is a positive definite involution on E. Clearly E is a central simple algebra over $\mathbb{Q}$. We fix a simple left E-module N and let $D = \mathrm{End}_E(N)$ be the centralizer of E operating on N. If one chooses an isomorphism of E with $\mathrm{M}_g(B)$, then N is isomorphic to the set $B^{\oplus g}$ of all $g \times 1$ column vectors with entries in B, and D is isomorphic to $B^{\mathrm{opp}} \cong B$; the action of D on the left of N becomes the right multiplication of B on $B^{\oplus g}$. It is a general fact that every positive definite involution on E comes from a positive definite D-valued hermitian form on N; we denote by h the hermitian form on N corresponding to the Rosati involution $*$. Thus $h : N \times N \to D$ satisfies

$$h(d_1 \cdot n_1, d_2 \cdot n_2) = d_1 \cdot h(n_1, n_2) \cdot d_2^*$$

for all $n_1, n_2 \in N$ and all $d_1, d_2 \in D$, where d_2^* denotes the image of d_2 under the standard involution of D. Moreover

$$h(n_1, n_2) = h(n_2, n_1)^* \quad \forall n_1, n_2 \in N; \qquad h(n, n) > 0 \quad \forall 0 \neq n \in N\ .$$

We recall that the standard involution is the unique positive definite involution on $D \cong B$; abusing the notation, $*$ will also denote the standard involution on $D \cong B$. Let $D_p = D \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $N_p = N \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $E_p = E \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Let $\mathrm{GU}(E, *) = \mathrm{GU}(h)$ be the group of unitary similitudes defined by $(E, *)$ or equivalently (h, N). For any commutative $\mathbb{Q}$-algebra R, $\mathrm{GU}(E, *)(R)$ consists of all elements $u \in (E \otimes_{\mathbb{Q}} R)^\times$ such that $u \cdot u^* = u^* \cdot u \in R^\times \cdot 1$; it is also the group of D-linear automorphisms of N which preserve the hermitian form h up to multiples.

Suppose that F is a commutative semisimple algebra over $\mathbb{Q}$, and let $j : F \hookrightarrow E$ be an embedding of F into E such that $j(F)$ is fixed by the involution $*$ on E. Since the restriction to F of the trace form for E is a positive linear functional on F with respect to the involution $*$, F is isomorphic to a product of totally real number fields. Clearly every conjugate of $j(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ by an element of $\mathrm{GU}(E,*)(\mathbb{Q}_p)$ is a commutative semisimple subalgebra of E_p of dimension g over $\mathbb{Q}$ fixed by $*$. Our goal is to show that one can conjugate $j(F)$ by a suitable element u of $\mathrm{GU}(E,*)(\mathbb{Q}_p)$ such that $u \cdot j(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cdot u^{-1}$ is the $\mathbb{Q}_p$-span of a commutative semisimple subalgebra F' of E of dimension g over $\mathbb{Q}$, which is "as split as possible".

Theorem 1. *As above let $j : F \hookrightarrow E$ be an embedding of a commutative semisimple algebra F into E which is fixed under the involution $*$ on E. Suppose that $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = F_{\wp_1} \times \cdots \times F_{\wp_a}$, where each $F_{\wp_i}$ is a finite extension field of $\mathbb{Q}_p$. Then there exist commutative semisimple subalgebras $F'_1, \ldots, F'_a$ of E and an element $u \in \mathrm{GU}(E,*)(\mathbb{Q}_p)$ such that $u \cdot j(F_{\wp_i}) \cdot u^{-1}$ is the $\mathbb{Q}_p$-span of F'_i for each $i = 1, \ldots, a$.*

Remark. (a) Clearly each F'_i is a totally real number field.

(b) Since the reduced norm $\mathrm{Nm}^{\mathrm{red}}_{D_p/\mathbb{Q}_p} : D^\times \to \mathbb{Q}_p^\times$ is surjective, one can replace the group of unitary similitudes $\mathrm{GU}(E,*)(\mathbb{Q}_p)$ by the unitary group $\mathrm{U}(E,*)(\mathbb{Q}_p) = \mathrm{U}(h)(\mathbb{Q}_p)$. We formulated the theorem with $\mathrm{GU}(E,*)(\mathbb{Q}_p)$ since this is more natural, but the element u we construct in the proof actually belongs to $\mathrm{U}(E,*)(\mathbb{Q}_p)$.

(c) Of course the totally real fields $F'_1, \ldots, F'_a$ are not uniquely determined by $j(F)$; one only knows that $F'_i \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong F_{\wp_i}$ for each i.

Corollary 2.1. *Under the hypotheses of Theorem 1, let K be an open subgroup of $U(E,*)(\mathbb{Q}_p)$. Then there exist commutative semisimple subalgebras $F'_1, \ldots, F'_a$ of E and an element $u_1 \in K$ such that the conclusion of Theorem 1 holds for u_1.*

Proof of Corollary 2.1. The group of $\mathbb{Q}$-rational points $U(E,*)(\mathbb{Q})$ is dense in the p-adic group $U(E,*)(\mathbb{Q}_p)$ by the weak approximation theorem. By the remark above, there exists an element $u \in \mathrm{U}(E,*)(\mathbb{Q}_p)$ such that each $u \cdot j(F_{\wp_i}) \cdot u^{-1}$ is the $\mathbb{Q}_p$-span of a $\mathbb{Q}$-subalgebra F'_i of E. Pick an element $v \in U(E,*)(\mathbb{Q})$ such that $v^{-1} \cdot u \in K$. Then $u_1 = v^{-1} \cdot u$ does the job. $\square$

Proof of Theorem 1. The image under j of the identity element in each $F_{\wp_i}$ is an idempotent e_i in E_p. For each i, the D_p-submodule $e_i \cdot N_p$ of N_p is a free D_p module of rank

$$r_i = \dim_{\mathbb{Q}_p}\left(F_{\wp_i}\right) = \mathrm{Tr}^{\mathrm{red}}_{E_p/\mathbb{Q}_p}(e_i) \ .$$

This assertion is easily checked after base change to $\overline{\mathbb{Q}_p}$. Clearly the $e_i N_p$'s are orthogonal to each other with respect to the hermitian form h. One sees quickly that for each i, the subalgebra $e_i E_p e_i$ of E_p is stable under the involution $*$, $e_i N_p$ is a simple left $e_i E_p e_i$-module, and $e_i E_p e_i = \mathrm{End}_{D_p}(e_i N_p)$. The subalgebra $j(F_{\wp_i})$ of $e_i E_p e_i$ is fixed by the involution.

It is an easy fact that every nondegenerate D_p-valued hermitian form on the D_p-module N_p is diagonal. Since the reduced norm $\mathrm{Nm}^{\mathrm{red}}_{D_p/\mathbb{Q}_p} : D_p^\times \to \mathbb{Q}_p^\times$ is surjective, any two non-degenerate D_p-valued hermitian form on N_p are equivalent under $U(h,*)(\mathbb{Q}_p)$. This argument already appeared in the proof of [7, §1, Thm. 3.1]. It follows that there exists an element $u_2 \in \mathrm{U}(h,*)(\mathbb{Q}_p)$ such that each $u_2 \cdot e_i N_p$ is the $\mathbb{Q}_p$ span of a D-submodule N_i of N. In other words $e'_i := u_2 \cdot j(e_i) \cdot u_2^{-1} \in E$ for $i = 1, \ldots, a$.

For each $i = 1, \ldots, a$, let $E'_i = e'_i E e'_i = \mathrm{End}_D(N_i)$; it is stable under the involution $*$ on E. The subfield $F_i := u \cdot F_{\wp_i} \cdot u^{-1}$ of $E'_i \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is fixed under the involution. Its has dimension r_i, while E_i is non-canonically isomorphic to $\mathrm{M}_{r_i}(B)$. Let S_i denote the subspace of all symmetric elements of E'_i, that is elements of E_i fixed by the involution. Consider the algebraic map

$$C_i : U(h|N_i \otimes \mathbb{Q}_p) \times F_i \to S_i \otimes \mathbb{Q}_p$$

over $\mathbb{Q}_p$ defined by conjugating elements of F_i by elements of the unitary group. Pick a generator y_i of the field F_i. One easily checks that the differential of C_i at the point (e'_i, y_i) is an isomorphism. Hence the image of C_i contains an open neighborhood of y_i with respect to the p-adic topology of $S_i \otimes \mathbb{Q}_p$. Clearly S_i is dense in $S_i \otimes \mathbb{Q}_p$. So there exists an element $s_i \in U(h|N_i)(\mathbb{Q}_p)$ such that $s_i \cdot y_i \cdot s_i^{-1} \in E'_i$. The elements $s_1, \ldots, s_a$ together define an element $s \in \mathrm{U}(h,N)(\mathbb{Q}_p) = \mathrm{U}(E,*)(\mathbb{Q}_p)$. Take $u = s \cdot u_2$ and let F'_i be the subfield of E'_i generated by $s_i \cdot y_i \cdot s_i^{-1}$ for $i = 1, \ldots, a$. The theorem is proved. $\square$

3 The case for Hilbert-Blumenthal modular varieties

In this section we solve the Hecke orbit problem for the p-corank-one locus of a Hilbert-Blumenthal modular variety. We first restrict ourself to the essential case of one totally real number field. The more general case of a product of a finite number of totally real number fields can be easily deduced from the essential case.

Let F be a totally real number field, and let $\mathcal{O}_F$ be the ring of integers of F. We refer the readers to [16] and [6] for background material on the Hilbert-Blumenthal modular variety; generally we use the notation in [6]. For any invertible $\mathcal{O}_F$-module Λ with a notion of positivity, let $\mathcal{M}_F^\Lambda$ be the moduli stack of abelian varieties A of dimension $g = [F : \mathbb{Q}]$ with endomorphisms $\iota : \mathcal{O}_F \to \mathrm{End}(A)$ and a Λ-polarization $\phi : A \otimes_{\mathcal{O}_F} \Lambda \xrightarrow{\sim} A^t$, where A^t is the dual abelian variety of A; see [6, §2]. Let $\mathrm{GL}_2^{\mathrm{red}}$ be the reductive algebraic group over $\mathbb{Q}$ whose $\mathbb{Q}$-rational points consist of all elements of $\mathrm{GL}_2(F)$ with determinant in $\mathbb{Q}^\times$. Denote by $\mathbb{A}_f^{(p)}$ the prime-to-p part of the finite adèles for $\mathbb{Q}$. Elements of $\mathrm{GL}_2^{\mathrm{red}}(\mathbb{A}_f^{(p)})$ give rise to prime-to-p Hecke correspondences on $\mathcal{M}_F^\Lambda$; similarly elements of $\mathrm{GL}_2^{\mathrm{red}}(\mathbb{Q}_\ell)$ give ℓ-power Hecke correspondences for any prime number $\ell \neq p$. Geometrically, the prime-to-p Hecke orbit of a point (A_1, ι_1, ϕ_1) of $\mathcal{M}_F^\Lambda$ is the set consisting of all points (A, ι, ϕ) such that there exists a prime-to-p $\mathcal{O}_F$-linear isogeny $\alpha : A \to A_1$ such that the pull-back of ϕ_1 is a multiple of ϕ; similarly for the ℓ-power Hecke orbit. Sometimes we will use the term 'F-Hecke orbit' to emphasize that we are considering the Hecke orbits for $\mathcal{M}_F^\Lambda$.

Suppose that (A_1, ι_1, ϕ_1) is a point of $\mathcal{M}_F^\Lambda(\mathbb{F})$ such that its p-rank $f(A_1)$ is equal to $g - 1 = [F : \mathbb{Q}] - 1$. We assume that $g > 1$ to avoid the trivial case $F = \mathbb{Q}$. The Barsotti-Tate group $A_1[p^\infty]$ of A_1 decomposes canonically into a direct sum of its toric part $A_1[p^\infty]^{\mathrm{toric}}$, its étale part $A_1[p^\infty]^{\text{ét}}$, and a third factor isomorphic to the Barsotti-Tate group of a supersingular elliptic curve over $\mathbb{F}$; denote the third factor by $A_1[p^\infty]^{\mathrm{ss}}$. The $\mathcal{O}_F$-action on A_1 extends canonically to an action of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and the latter preserves the decomposition above. Since the action of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on the character group of $A_1[p^\infty]^{\mathrm{toric}}$ is dual to its action on the p-adic Tate module of $A_1[p^\infty]^{\text{ét}}$, it follows that the ideal of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which operates trivially on both $A_1[p^\infty]^{\mathrm{toric}}$ and $A_1[p^\infty]^{\text{ét}}$ is a direct summand of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ corresponding to a degree-one place $\wp_1$ of F above p. Let $\wp_2, \ldots, \wp_r$ be the remaining places of F above p. We have $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_{\wp_1} \times \cdots \times \mathcal{O}_{\wp_r}$. The character group of $A_1[p^\infty]^{\mathrm{toric}}$ and the the p-adic Tate module of $A_1[p^\infty]^{\text{ét}}$ are both free $\mathcal{O}_{\wp_2} \times \cdots \times \mathcal{O}_{\wp_r x}$-modules of rank one.

The Serre-Tate theorem says that deforming (A_1, ι_1, ϕ_1) is equivalent to deforming the corresponding Barsotti-Tate group $(A_1[p^\infty], \iota_1[p^\infty], \phi_1[p^\infty])$. The decomposition $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_{\wp_1} \times \cdots \times \mathcal{O}_{\wp_r}$ of $\mathcal{O}_F$ therefore induces a decomposition of the deformation space of (A_1, ι_1, ϕ_1) in $\mathcal{M}_F^\Lambda$ into a product $D_1 \times \cdots \times D_r$ of r factors, where each D_i is the deformation space of the triple $(A_1[\wp_i^\infty], \iota_1[\wp_i^\infty], \phi_1[\wp_i^\infty])$. Clearly D_1 is isomorphic to the deformation space of a supersingular elliptic curve, while for each $i = 2, \ldots, r$ D_i is a formal torus of dimension $g_i = [F_{\wp_i} : \mathbb{Q}_p]$ because $A_1[\wp_i^\infty]$ is ordinary.

Let $\ell \neq p$ be a prime number, and let Z_1 be the Zariski closure of the ℓ-power Hecke orbit of $x_1 = [(A_1, \iota_1, \phi_1)] \in \mathcal{M}_F^\Lambda(\mathbb{F})$ in M_F^Λ over $\mathbb{F}$. It is clear that the one-dimensional Barsotti-Tate group of height two attached to the $\wp_1$-power torsion point of any point $[(A, \iota, \phi)]$ in Z_1 is supersingular in the sense that $\frac{1}{2}$ is its only slope. Therefore the formal completion $Z_1^{/x_1}$ of Z_1 at x_1 is contained in (the base point of D_1) $\times D_2 \times \cdots D_r$. Since x_1 is defined over a finite field, it has sufficiently many complex multiplications. As in [3, §4], this implies that $Z_1^{/x_1}$ is stable under the action under an open subgroup of $\mathcal{O}_{\wp_2} \times \cdots \times \mathcal{O}_{\wp_r}$. By [3, Prop. 4], there exists a non-empty subset $I \subseteq \{2, \ldots, r\}$ such that $Z_1^{/x_1}$ is equal to the product

$$\prod_{i \in I} D_i \times \prod_{i \notin I} (\text{the base point of } D_i) \ .$$

As in [3, Prop. 5], this subset J of the set of all places of F above p is independent of the choice of points of p-corank one on of Z_1.

As in [3, Prop. 6], since Z_1 is stable under all ℓ-power F-Hecke correspondences, the main result of [8] implies that Z_1 contains a supersingular point z_1. This point $z_1 \in Z_1(\mathbb{F})$ corresponds to a triple $(A_{z_1}, \iota_{z_1}, \phi_{z_1})$, where A_{z_1} is a supersingular abelian variety. Again by the Serre-Tate theorem, the formal completion $(\mathcal{M}_F^\Lambda)^{/z_1}$ of $\mathcal{M}_F^\Lambda$ at z_1 is canonically isomorphic to a product $D_{\wp_1}^{z_1} \times \cdots \times D_{\wp_r}^{z_r}$. The formal completion $Z_1^{/z_1}$ of Z_1 at z_1 is a closed formal subscheme of $(\mathcal{M}_F^\Lambda)^{/z_1}$, and the first projection of the product structure sends $Z_1^{/z_1}$ to the base point of $D_{\wp_1}^{z_1}$. The argument of "inspecting at the supersingular point" in the proof of [3, Prop. 7] then shows that $J = \{2, \ldots, r\}$: The stabilizer subgroup at z_1 contains an open subgroup $\prod_{i=1}^r U_i$ of the product $\prod_{i=1}^r (B_{\wp_i})_1^\times$ of norm-one elements in quaternion algebras $B_{\wp_i}$'s over $F_{\wp_i}$. For each i, the action of elements of U_i changes the i-th coordinate with respect to the product structure of $(\mathcal{M}_F^\Lambda)^{/z_1}$. This implies that $i \in J$ unless the i-th projection of $Z_1^{/z_1}$ is equal to the base point of $D_1^{z_1}$. On the other hand, if for some $i \geq 2$ the i-th projection of $Z_1^{/z_1}$ is equal to the base point of $D_1^{z_1}$, then the the p-rank of every point of Z_1 is at most $g-2$. This contradiction proves that $\dim(Z_1) = g-1$.

The conclusion that $\dim(Z_1) = g-1$ can be reformulated in terms of Hasse-invariants: The decomposition $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Q}_p = \mathcal{O}_{\wp_1} \times \cdots \times \mathcal{O}_{\wp_r}$ gives a decomposition of the Barsotti-Tate group attached to the universal abelian variety A over $\mathcal{M}_F^\Lambda$ and also a decomposition of the locally free sheaf $\underline{\mathrm{H}}^1(A, \mathcal{O}_A)$ of rank g on $\mathcal{M}_F^\Lambda$. So one gets r Hasse invariants from the relative Frobenius action on the r factors of $\underline{\mathrm{H}}^1(A, \mathcal{O}_A)$. See [10] for documentation on Hasse invariants and stratification for $\mathcal{M}_F^\Lambda$. The Hasse invariant for the factor $\wp_1$ is a section of an invertible sheaf on $\mathcal{M}_F^\Lambda$, and $\wp_1$ is a degree-one place of F above p. Clearly the ℓ-power F-Hecke correspondences preserve the zero locus of each Hasse invariant. What we proved in this section is that for each degree-one place $\wp$ above p and each p-corank-one point x in the zero locus $Z_\wp$ of the Hasse invariant attached to $\wp$, the Zariski closure of the ℓ-power F-Hecke orbit of x is a union of irreducible components of the divisor $Z_\wp$ of $\mathcal{M}_F^\Lambda$.

We summarize our discussion so far as

Theorem 2. *Let $x_1 \in \mathcal{M}_F^\Lambda$ be a point of p-corank one over an algebraically closed field k of characteristic $p > 0$. Then there exists a unique degree-one place $\wp$ of F above p such that x_1 lies in the zero locus $Z_\wp \subset \mathcal{M}_F^\Lambda$ of the Hasse invariant attached to $\wp$. Moreover for any prime number $\ell \neq p$, the Zariski closure of the ℓ-power F-Hecke orbit of x_1 in $\mathcal{M}_F^\Lambda$ over k is a union of irreducible components of the divisor $Z_\wp$ on $\mathcal{M}_F^\Lambda$.*

Now we take up the more general situation. Let $F = F_1 \times \cdots \times F_s$, where each F_j is a totally real number field. Let $\mathcal{O}_F = \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_s}$. Let Λ be an invertible $\mathcal{O}_F$-module with a notion of positivity. Then Λ decomposes naturally into a direct sum $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_s$, where each Λ_j is an invertible $\mathcal{O}_{F_j}$-module with a notion of positivity. We define a moduli stack $\mathcal{M}_F^\Lambda$ just as before. Clearly every triple (A, ι, ϕ) in $\mathcal{M}_F^\Lambda$ decomposes as a product $(A_1 \times \cdots A_s, \iota_1 \times \cdots \iota_s, \phi_1 \times \cdots \phi_s)$, since $\mathcal{O}_F$ is a product. This shows that

$\mathcal{M}_F^\Lambda$ decomposes as $\mathcal{M}_{F_1}^{\Lambda_1} \times \cdots \times \mathcal{M}_{F_s}^{\Lambda_s}$. Similarly, the reductive group attached to $\mathcal{M}_F^\Lambda$ is equal to the product of the reductive groups attached to $\mathcal{M}_{F_j}^{\Lambda_j}$ up to center. So it will be more convenient to use the derived groups. In other words one uses $\mathrm{SL}_2(F) = \mathrm{SL}_2(F_1) \times \cdots \times \mathrm{SL}_2(F_s)$. The effect for Hecke orbits is that one replaces the ℓ-power Hecke orbit of a point $x = [(A, \iota, \phi)]$ by something slightly smaller: the set of all points $[(A', \iota', \phi')]$ such that there exists an $\mathcal{O}_A$-linear $\mathbb{Q}_\ell$-quasi-isogeny between A' and A which preserves the polarizations. We call this set the *reduced ℓ-power F-Hecke orbit* of x. One verifies easily that the statements of Theorem 2 and [3, Thm. 1] remain true if one replaces the ℓ-power F-Hecke orbit of x_1 by the reduced ℓ-power F-Hecke orbit of x_1; the same proofs work while the statements are strengthened a little bit. One advantage of using the reduced ℓ-power Hecke orbit is that the reduced ℓ-power F-Hecke orbit of a point $(A_1 \times \cdots A_s, \iota_1 \times \cdots \iota_s, \phi_1 \times \cdots \phi_s)$ in $\mathcal{M}_F^\Lambda$ is simply the product of the reduced ℓ-power F-Hecke orbits of the (A_j, ι_j, ϕ_j)'s. Combining Thm. 2 with [3, Thm. 1], we obtain the following generalization of Thm. 2 using the reduced F-Hecke orbits.

Theorem 3. *Let $F = F_1 \times \cdots \times F_s$ be a product of totally real number fields, $\mathcal{O}_F = \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_s}$, and let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_s$ be an invertible $\mathcal{O}_F$-module with a notion of positivity as above. Let $x_1 \in \mathcal{M}_F^\Lambda$ be a point of p-corank one over an algebraically closed field k of characteristic $p > 0$. Then there exists a unique degree-one place $\wp$ of F above p such that x_1 lies in the zero locus $Z_\wp \subset \mathcal{M}_F^\Lambda$ of the Hasse invariant attached to $\wp$. Moreover for any prime number $\ell \neq p$, the Zariski closure of the reduced ℓ-power F-Hecke orbit of x_1 in $\mathcal{M}_F^\Lambda$ over k is a union of irreducible components of the divisor $Z_\wp$ on $\mathcal{M}_F^\Lambda$.*

4 End of Proof

Let (A_0, λ_0) be a principally polarized abelian variety over $\mathbb{F}$ of dimension g such that the p-rank $f(A_0)$ of A_0 is equal to $g-1$. Consider the semi-simple algebra $\mathrm{End}(A_0)^\circ := \mathrm{End}_{\mathbb{F}}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$ with the positive definite involution $*$ induced by the principal polarization λ_0. Then $\mathrm{End}(A_0)^\circ$ decomposes into a product of $\mathrm{End}(A_0)^\circ = E_1 \times \cdots E_m$ of semisimple algebras, such that each E_i is isotypic and $\mathrm{Hom}(E_i, E_j) = (0)$ if $i \neq j$. The positive definite involution $*$ stabilizes each E_i and induces a positive definite involution $*_i$ on E_i. Let C_i be the center of E_i. Write $[E_i : \mathbb{Q}]$ as $[E_i : \mathbb{Q}] = [C_i : \mathbb{Q}] \cdot d_i^2$, and let $g_i = \frac{1}{2}[C_i : \mathbb{Q}] d_i$. One knows that each g_i is an integer, and $g_1 + \cdots + g_m = g$.

It is well-known that for each i, there exists a commutative semisimple subalgebra $S_i \subset E_i$, such that elements of S_i are fixed under $*_i$ and $[S_i : \mathbb{Q}] = g_i$. Since $*_i$ is a positive definite involution on E_i, S_i is a product of totally real number fields. Actually one can choose S_i to be a totally real field; we will not need this though.

Let F be the product of the S_i's, so F is a product of totally real number fields $F_1 \times \cdots \times F_s$ embedded in $\mathrm{End}(A_0)^\circ$ and fixed under the Rosati involution $*$. Then there exist an invertible $\mathcal{O}_F$-module $\Lambda = \Lambda_1 \times \cdots \times \Lambda_s$ with a notion of positivity, a triple $(A_1, \iota_1, \phi_1) \in \mathcal{M}_F^\Lambda(\mathbb{F})$ and an isogeny $\beta : A_1 \to A_0$, such that $\beta \in \mathrm{Hom}(A_1, A_0)^\circ$ is F-linear and the pull-back of the principal polarization λ_0 on A_0 is the image of a totally positive element λ_1 of Λ under ϕ_1. Let d^2 be the degree of the polarization λ_1 on A_1, that is the symmetric homomorphism $\lambda_1 : A_1 \to A_1^t$ has degree d^2. Using the element $\lambda_1 \in \Lambda$ to polarize the universal abelian variety over $\mathcal{M}_F^\Lambda$, we obtain a morphism i from $\mathcal{M}_F^\Lambda$ to the moduli stack $\mathcal{A}_{g,d}$ of g-dimensional abelian varieties with polarization of degree d^2. It is known that i is a finite morphism which sends $(A_1, \iota_1, \phi_1) \in \mathcal{M}_F^\Lambda(\mathbb{F})$ to $(A_1, \lambda_1) \in \mathcal{A}_{g,d}(\mathbb{F})$. For any prime number $\ell \neq p$, i maps the reduced ℓ-power F-Hecke orbit of (A_1, ι_1, ϕ_1) into the ℓ-power Hecke orbit of (A_1, λ_1). Moreover we have an isogeny correspondence κ between $\mathcal{A}_{g,d}$ and $\mathcal{A}_g$ which contains β; κ is finite over the open subset of $\mathcal{A}_{g,d}$ and $\mathcal{A}_g$ where the p-corank is at most one. The ℓ-power Hecke orbit of (A_1, λ_1) and the ℓ-power Hecke orbit of (A_0, λ_0) correspond under κ; especially their Zariski closure have the same dimension.

Let Z_1 be the Zariski closure of the reduced ℓ-power F-Hecke orbit of the point $x_1 = [(A_1, \iota_1, \phi_1)]$ in $\mathcal{M}_F^\Lambda$. By Thm. 3, there exists a unique degree-one place $\wp_1$ of F above p such that x_1 lies in the zero locus $Z_{\wp_1} \subset \mathcal{M}_F^\Lambda$ of the Hasse invariant attached to $\wp_1$, and Z_1 is a union of irreducible components of the divisor $Z_{\wp_1}$ on $\mathcal{M}_F^\Lambda$.

As in the proof of [3, Prop. 6], the main result of [8] implies that Z_1 contains a supersingular point z_1. Let $y = i(z_1) \in \mathcal{A}_{g,d}(\mathbb{F})$. Let Y_1 be the Zariski closure of the ℓ-power Hecke orbit of (A_1, λ_1) in $\mathcal{A}_{g,d}$, so $i(Z_1) \subset Y_1$. Write $y = (A_y, \lambda_y)$, and let $E_y := \mathrm{End}_{\mathbb{F}}(A_y)^\circ$. The Rosati involution for λ_1 gives a positive definite involution $* = *_y$ on $E = E_y$. The action of $\mathcal{O}_F$ on A_1 gives us an embedding $j : F \hookrightarrow E$ of F which is fixed under $*$. Since Y_1 is stable under all ℓ-power Hecke correspondences, the formal completion $Y_1^{/y}$ of Y_1 at y is stable under the natural action of an open subgroup $K_1 \subset \mathrm{U}(E, *)(\mathbb{Q}_p)$. We are now exactly in the situation of §2, and

Corollary 2.1 implies that there exists an element $u \in K_1$, a commutative finite dimensional semisimple algebra F' over $\mathbb{Q}$ which splits into a product $F' = F'_1 \times F'_2$ with $F'_1 \cong \mathbb{Q}$, an embedding $j' : F' \hookrightarrow E$ fixed under $*$, such that $u \cdot j(F \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cdot u^{-1} = j'(F') \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $u \cdot j(F_{\wp_1} \otimes_{\mathbb{Q}} \mathbb{Q}_p) \cdot u^{-1} = j'(F'_1) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Since $Y_1^{/y}$ is stable under the action of K_1 and Y_1 contains $i(Z_1)$, $Y_1^{/y}$ contains the image $u \cdot i(Z_1)^{/y}$ of the formal completion $i(Z_1)^{/y}$ of $i(Z_1)$ at y under the action of $u \in K_1$. Clearly the universal abelian scheme over the coordinate ring of $i(Z_1)^{/y}$ has endomorphisms by $j(\mathcal{O}_F) \subseteq \mathrm{End}_{\mathbb{F}}(A_y)$, and its Barsotti-Tate group has endomorphisms by $j(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \mathrm{End}_{\mathbb{F}}(A_y) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. So the Barsotti-Tate group of the universal abelian scheme over the coordinate ring of $u \cdot i(Z_1)^{/y}$ has endomorphism by

$$u \cdot j(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cdot u^{-1} = j'(\mathcal{O}_{F'}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \subseteq \mathrm{End}_{\mathbb{F}}(A_y) \otimes_{\mathbb{Z}} \mathbb{Z}_p \ .$$

By the Serre-Tate theorem, the universal abelian scheme over the coordinate ring of $u \cdot i(Z_1)^{/y}$ has endomorphisms by $j'(\mathcal{O}_{F'}) \cap \mathrm{End}_{\mathbb{F}}(A_y)$, an order of $\mathcal{O}_{F'}$ which happens to be maximal at all primes above p. It follows that Y_1 contains a $(g-1)$-dimensional subvariety Z'_1, the image of a finite morphism i' from a Hilbert-Blumenthal modular stack $\mathcal{M}_{F'}^{\Lambda'}$ to $\mathcal{A}_g$, such that the the restriction $A_{/Z'_1}$ of the universal abelian variety to Z'_1 have endomorphisms by an order of $\mathcal{O}_{F'}$. In particular Z'_1 contains an element $x'_2 = (A'_2, \lambda'_2) \in \mathcal{A}_{g,d}(\mathbb{F})$ of p-corank one such that $\mathrm{End}_{\mathbb{F}}(A'_2)^{\circ}$ contains a commutative semisimple algebra isomorphic to F', such that the factor F'_1 of F' cuts out a one-dimensional supersingular abelian subvariety in A'_2 up to isogeny. This completes the step 3) of the proof as outlined in §1.

The argument used above actually proves a little more. As the statement may be useful in other situations, a short digression seems warranted. The proof is omitted because it is entirely similar to that in the previous paragraph.

Proposition 4. *Let (A, λ) be a principally polarized abelian variety over $\mathbb{F}$ of dimension g. Let Z be the Zariski closure of the ℓ-power Hecke orbit of $[(A, \lambda)] \in \mathcal{A}_g(\mathbb{F})$ in $\mathcal{A}_g$ over $\mathbb{F}$, where ℓ is a prime number different from p. Then there exists a point $(A', \lambda') \in Z(\mathbb{F})$ such that*

- *A' has the same slopes as A, with the same multiplicities,*
- *A' is isogenous to a product of abelian varieties $B_1 \times \cdots \times B_a$,*
- *each B_i has at most two distinct slopes, $i = 1, \ldots, a$,*

- *B_i and B_j do not have common slopes if $i \neq j$, $1 \leq i, j \leq a$.*

We resume the proof of main result. From the isogeny correspondence κ between $\mathcal{A}_{g,d}$ and $\mathcal{A}_g$, we conclude that the Zariski closure Z of the ℓ-power Hecke orbit of $x_0 = [(A_0, \lambda_0)] \in \mathcal{A}_g(\mathbb{F})$ contains a point $x_2 = [(A_2, \lambda_2)] \in \mathcal{A}_g(\mathbb{F})$ such that A_2 is isogenous to the product of a supersingular elliptic curve and a $g-1$-dimensional ordinary abelian variety of dimension $g-1$. Hence there exists an isogeny correspondence κ' between $\mathcal{A}_g$ and $\mathcal{A}_{g,d'}$ for some positive integer d' such that there exists a point $x_3 = [(E_3 \times X_3, \lambda_3)] \in \mathcal{A}_{g,d'}(\mathbb{F})$ which corresponds with x_2 under κ', where E_3 is a supersingular elliptic curve over $\mathbb{F}$ and X_3 is an ordinary abelian variety of dimension $g-1$ over $\mathbb{F}$. This forces the polarization λ_3 to be the product of a polarization λ_{E_3} on E_3 and a polarization λ_{X_3} on X_3. So $\mathcal{A}_{g,d'}$ contains a "diagonally embedded" copy of $\mathcal{A}_1 \times \mathcal{A}_{g-1,d_1}$ passing through x_3, where d_1 is the degree of the induced polarization on X_3. By [3, Thm. 2], the Zariski closure of the ℓ-power Hecke orbit of $[(X_3, \lambda_{X_3})]$ in $\mathcal{A}_{g-1,d_1}$ contains the irreducible component $\mathcal{A}_{g-1,\delta}$ of $(\mathcal{A}_{g-1,d_1})_{\mathrm{red}}$ which contains $[(X_3, \lambda_{X_3})]$. Here δ denotes $(\delta_1, \ldots, \delta_{g-1})$ for suitable positive integers $\delta_1, \ldots, \delta_{g-1}$ with $\delta_1 | \cdots \delta_{g-1}$. Since the ℓ-power Hecke orbit of x_3 and the ℓ-power Hecke orbit of x_2 correspond under κ', $\dim(Z) = \dim(Z_3)$, where Z_3 is denotes the Zariski closure of x_3 in $\mathcal{A}_{g,d'}$. Now we use another isogeny correspondence. There exist a point $[(X_4, \lambda_{X_4})] \in \mathcal{A}_{g-1,\delta}(\mathbb{F})$, a point $[(X_5, \lambda_{X_5})] \in \mathcal{A}_g(\mathbb{F})$, and an isogeny $\alpha : X_4 \to X_5$ such that $\lambda_{X_4} = \alpha^*(\lambda_{X_5})$. Similarly for (E_3, λ_{E_3}). So we can find

- a point $x_4 = [(A_4, \lambda_4)] \in Z_3(\mathbb{F})$ of p-corank one
- a point
$$x_5 = [(E_{\mathrm{ss}}, \lambda_{\mathrm{ss}}) \oplus (E_{\mathrm{or}}, \lambda_{\mathrm{or}})^{\oplus(g-1)}] \in \mathcal{A}_g(\mathbb{F})$$
with $(E_{\mathrm{ss}}, \lambda_{\mathrm{ss}})$ a principally polarized supersingular elliptic curve over $\mathbb{F}$ and $(E_{\mathrm{or}}, \lambda_{\mathrm{or}})$ an ordinary principally polarized elliptic curve over $\mathbb{F}$
- an isogeny correspondence κ'' between $\mathcal{A}_{g,d'}$ and $\mathcal{A}_g$

such that x_4 and x_5 correspond under κ''. As before the dimension of Z_3 is equal to the dimension of the Zariski closure Z_5 of the ℓ-power Hecke orbit of x_5 in $\mathcal{A}_g$. Moreover Z_5 contains a copy of 'diagonally embedded' $\mathcal{A}_{g-1}$ passing through x_5.

By [4, Thm. 4.4, Prop. 4.7], the formal completion $V^{/x_5}$ of the p-corank-one locus V in $\mathcal{A}_g$ at x_5 has a natural fibration structure; it is a torsor under a formal torus T of dimension $\frac{g(g-1)}{2}$ over a smooth formal group G of dimension $g-1$. The smooth formal group G is canonically isomorphic to $N \otimes_{\mathbb{Z}_p} E_{\mathrm{ss}}$, where N is the p-adic Tate module of the maximal étale quotient of $E_{\mathrm{or}}^{\oplus(g-1)}$. The fact that Z_5 contains a copy of A_{g-1} means that $V^{/x_5}$ contains the fiber T over the base point of G. The stabilizer subgroup at x_5 is quite big even integrally, for it contains the stabilizer subgroup for the point $(E_{\mathrm{or}}, \lambda_{\mathrm{or}})^{\oplus(g-1)}$ in $\mathcal{A}_{g-1}$, which is equal to $\mathrm{GL}(N) \cong \mathrm{GL}_{g-1}(\mathbb{Z}_p)$; see [3, p. 443].

It is easy to see that $\dim(Z_5) > \frac{g(g-1)}{2}$, for instance by considering the action of the stabilizer subgroup at a supersingular point of Z_5. If $g-1=1$, then $\dim(G)=1$ and $\dim(Z_5) = \frac{g(g+1)}{2} - 1$ trivially, so we may assume that $g > 2$. As shown in [8], V is irreducible. So if we can show that the tangent space of $Z_5^{/x_5}$ is equal to the tangent space of $V^{/x_5}$, then Z_5 has dimension $\frac{g(g+1)}{2} - 1$ and therefore equal to V.

Recall that Z_5 is smooth at x_5, as explained on [3, p. 443], so $Z_5^{/x_5}$ is formally smooth. We already know that $Z_5^{/x_5}$ contains T. The tangent space of $Z_5^{/x_5}$ is stable under the action of $\mathrm{GL}(N)$. From the definition of G one sees that the action of $\mathrm{GL}(N)$ on the tangent space of G, identified with the quotient of the tangent space of $V^{/x_5}$ by the tangent space of T, is given by the composition of the canonical map $\mathrm{GL}(N) \to \mathrm{GL}(N \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ and the standard representation of $\mathrm{GL}(N \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ on $N \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Since the tangent space of $Z_5^{/x_5}$ properly contains the tangent space of T, the tangent space of $Z_5^{/x_5}$ must be equal to the tangent space of $V^{/x_5}$. This shows that $\dim(Z) = \dim(Z_5) = \dim(V)$, so $Z_5 = V = Z$. We have proved the first statement of

Theorem 5. *Let k be an algebraically closed field of characteristic $p > 0$, let ℓ be a prime number different from p. let $g \geq 2$ and d be two positive integers.*

1. *Suppose that $x_0 = (A_{x_0}, \lambda_{x_0}) \in \mathcal{A}_g(k)$ is a point of $\mathcal{A}_g$, then the ℓ-power Hecke orbit of x_0 in $\mathcal{A}_g$ is dense in the zero locus of the Hasse invariant on $\mathcal{A}_g \to \operatorname{Spec} k$.*

2. *Suppose that $y_0 = (A_{y_0}, \lambda_{y_0}) \in \mathcal{A}_{g,d}(k)$ is a point in $\mathcal{A}_{g,d}$, then the Zariski closure of the ℓ-power Hecke orbit of y_0 in $\mathcal{A}_{g,d}$ has dimension*

$\frac{g(g+1)}{2}-1$. *It is an irreducible component of the zero locus of the Hasse invariant on* $\mathcal{A}_{g,d} \to \operatorname{Spec} k$ *if* $(\ell, d) = 1$*; otherwise it is a union of irreducible components of the zero locus of the Hasse invariant.*

Proof. Let Z_{y_0} be the Zariski closure of the ℓ-power Hecke orbit of y_0. Using a suitable isogeny correspondence, we deduce that $\dim(Z_{y_0}) = \frac{g(g+1)}{2}-1$ from the principally polarized case. It remains only to prove the last assertion of part 2 of the theorem. By standard argument, we may and do assume that $d = p^m$, a power of p. We know that each irreducible component of the zero locus $\mathcal{V}_{g-1}$ of the Hasse invariant has dimension $\frac{g(g+1)}{2} - 1$ according to [14, Thm. 4.1]. From the description of the irreducible components of $\mathcal{V}_{g-1}$ by discrete invariants given in [5, Prop. 3.3], it follows that Z_{y_0} is equal to the irreducible component of $\mathcal{V}_{g-1}$ determined by the type of the étale part (or equivalently the multiplicative part) of $Ker(\lambda_{y_0})$. More precisely, let $\bar{\delta}_1, \cdots, \bar{\delta}_{g-1}$ be the elementary divisor of the étale part of $Ker(\lambda_{y_0})$, with $\bar{\delta}_1 | \cdots | \bar{\delta}_{g-1}$. Using the notation as in [5], it is clear that Z_{y_0} is contained in the subvariety $\mathcal{Z}(p^m, \bar{\delta})^c$ of $\mathcal{V}_{g-1}$. Since $\mathcal{Z}(p^m, \bar{\delta})^c$ has codimension one in $\mathcal{A}_g$ and is irreducible according to [5, Prop. 3.3], it is equal to Z_{y_0}. □

Remark. Near the end of the proof, we produced a 'very symmetric' point x_5 and explored the action of its stabilizer subgroup. It is possible to use a point which is only isogenous to a product like $E_{\mathrm{ss}} \oplus E_{\mathrm{or}}^{\oplus(g+1)}$, but the proof would be more complicated. It is also possible to use a boundary point in the closure of the Hecke orbit in the minimal compactification of $\mathcal{A}_g$ which lies in the one-dimensional cusp. Then one explores the action of the stabilizer subgroup at this boundary point just as in [3, §2], where the zero-dimensional cusp was used. Again the proof would be more involved.

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