Reading: Please be sure to get Trefethen and Bau: *Numerical Linear Algebra*, which is published by SIAM. Page numbers below refer to this book.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Show that an $m \times n$ matrix is rank 1 if and only if $A = uv^*$, where $u$ is an $m$-vector and $v$ is an $n$-vector.

2. Show that a matrix that is both orthogonal and upper triangular is diagonal. What are all such matrices?


5. Page 24, problem 3.2. When is this inequality strict?

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $A = (a_{ij})$ be an $n \times n$ matrix. The trace of $A$ is defined to be

\[
\text{tr} \ A = \sum_{i=1}^{n} a_{ii}.
\] (1)

If $A$ and $B$ are $n \times n$ matrices then show that $\text{tr} \ AB = \text{tr} \ BA$. Suppose that $A(t)$ satisfies the differential equation

\[
\frac{dA}{dt} = [X(t), \ A(t)],
\] (2)

for a matrix valued function $X(t)$. Recall the $[X, \ A] = XA - AX$, is the commutator. What can you say about $\text{tr} \ A(t)$?
2. Recall that if \( \{x_1, \ldots, x_n\} \) are complex numbers, then the Vandermonde matrix is given by
\[
V(x_1, \ldots, x_n) = \begin{pmatrix}
1 & \cdots & 1 \\
x_1 & \cdots & x_n \\
\vdots & \ddots & \vdots \\
x_1^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}.
\tag{3}
\]
Compute the \( \det V(x_1, \ldots, x_n) \). From this formula it is clear that \( V(x_1, \ldots, x_n) \) is invertible whenever the points are distinct. Give an independent proof of this fact using the connection between the Vandermonde matrix and polynomial interpolation.

3. Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be norms on \( \mathbb{C}^m \). Prove that there is a constant \( C \) so that
\[
\frac{1}{C} \| x \|_1 \leq \| x \|_2 \leq C \| x \|_1 \quad \text{for all} \quad x \in \mathbb{C}^m.
\tag{4}
\]

4. Let \( \mathcal{P}_d = \{ \sum_{j=0}^d a_j x^j : a_j \in \mathbb{R} \} \), be the vector space of real polynomials of degree at most \( d \). Show that
\[
\| p \|_1 = \max \{|p(x)| : x \in [10, 100]\}
\tag{5}
\]
defines a norm on \( \mathcal{P}_d \) and
\[
\langle p, q \rangle_2 = \int_0^1 p(x)q(x)dx
\tag{6}
\]
defines an inner product on \( \mathcal{P}_d \). Prove that there is a constant \( C_d \) so that
\[
\| p \|_1^2 \leq C_d \langle p, p \rangle.
\tag{7}
\]
How does this constant behave as \( d \to \infty \).

5. Let \( V \) be a real vector space and \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \), inner products on \( V \). Show that there are unique matrices \( A \) and \( B \) so that
\[
\langle x, y \rangle_2 = \langle Ax, y \rangle_1 \quad \text{and} \quad \langle x, y \rangle_1 = \langle Bx, y \rangle_2.
\tag{8}
\]
Show that these matrices are symmetric. How are \( A \) and \( B \) related to one another?
If
\[
\left\langle \sum_{j=0}^{d} a_j x^j, \sum_{j=0}^{d} b_j x^j \right\rangle_1 = \sum_{j=0}^{d} a_j b_j
\] (9)
and
\[
\langle p, q \rangle_2 = \int_{0}^{1} p(x) q(x) dx,
\] (10)
then what is \( A \)? What does the first part of this problem tell us about \( A \)?

6. Define the Hermitian inner product on \( \mathbb{C}^m \):
\[
\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j,
\] (11)
and let \( \|z\| = \sqrt{\langle z, z \rangle} \) be the norm it defines. Show that
\[
\langle z, w \rangle = \frac{1}{4} \left[ \|z + w\|^2 - \|z - w\|^2 + i \|z + iw\|^2 - i \|z - iw\|^2 \right].
\] (12)

7. Page 16, problems 2.5, and 2.6.

8. Page 24, problems 3.3, and 3.6