AMCS 602
Problem set 1, due September 6, 2016
Dr. Epstein

Reading: Please be sure to get Trefethen and Bau: *Numerical Linear Algebra*, which is published by SIAM. Page numbers below refer to this book.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Show that an $m \times n$ matrix is rank 1 if and only if $A = uv^*$, where u is an m-vector and v is an n-vector.
2. Show that a matrix that is both orthogonal and upper triangular is diagonal. What are all such matrices?
3. Page 10, Problem 1.4.
4. Page 16, problem 2.6.
5. Page 24, problem 3.2. When is this inequality strict?

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $A = (a_{ij})$ be an $n \times n$ matrix. The trace of A is defined to be

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}. \tag{1}$$

   If A and B are $n \times n$ matrices then show that $\operatorname{tr} AB = \operatorname{tr} BA$. Suppose that $A(t)$ satisfies the differential equation

$$\frac{dA}{dt} = [X(t), A(t)], \tag{2}$$

   for a matrix valued function $X(t)$. Recall the $[X, A] = XA - AX$, is the commutator. What can you say about $\operatorname{tr} A(t)$?

2. Recall that if $\{x_1, \dots, x_n\}$ are complex numbers, then the Vandermonde matrix is given by

$$V(x_1, \dots, x_n) = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}. \tag{3}$$

Compute the det $V(x_1, \dots, x_n)$. From this formula it is clear that $V(x_1, \dots, x_n)$ is invertible whenever the points are distinct. Give an independent proof of this fact using the connection between the Vandermonde matrix and polynomial interpolation.

3. Let $\|\cdot\|_1$, and $\|\cdot\|_2$ be norms on $\mathbb{C}^m$. Prove that there is a constant C so that

$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \text{ for all } x \in \mathbb{C}^m. \tag{4}$$

4. Let $\mathcal{P}_d = \left\{ \sum_{j=0}^{d} a_j x^j : a_j \in \mathbb{R} \right\}$, be the vector space of real polynomials of degree at most d. Show that

$$\|p\|_1 = \max\{|p(x)| : x \in [10, 100]\} \tag{5}$$

defines a norm on $\mathcal{P}_d$ and

$$\langle p, q\rangle_2 = \int_0^1 p(x)q(x)dx \tag{6}$$

defines an inner product on $\mathcal{P}_d$. Prove that there is a constant C_d so that

$$\|p\|_1^2 \leq C_d\langle p, p\rangle. \tag{7}$$

How does this constant behave as $d \to \infty$.

5. Let V be a real vector space and $\langle\cdot,\cdot\rangle_1$, and $\langle\cdot,\cdot\rangle_2$, inner products on V. Show that there are unique matrices A and B so that

$$\langle x, y\rangle_2 = \langle Ax, y\rangle_1 \text{ and } \langle x, y\rangle_1 = \langle Bx, y\rangle_2. \tag{8}$$

Show that these matrices are symmetric. How are A and B related to one another?

If

$$\left\langle \sum_{j=0}^{d} a_j x^j, \sum_{j=0}^{d} b_j x^j \right\rangle_1 = \sum_{j=0}^{d} a_j b_j \tag{9}$$

and

$$\langle p, q \rangle_2 = \int_0^1 p(x)q(x)dx, \tag{10}$$

then what is A? What does the first part of this problem tell us about A?

6. Define the Hermitian inner product on $\mathbb{C}^m$:

$$\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j, \tag{11}$$

and let $\|z\| = \sqrt{\langle z, z \rangle}$ be the norm it defines. Show that

$$\langle z, w \rangle = \frac{1}{4}\left[\|z + w\|^2 - \|z - w\|^2 + i\|z + iw\|^2 - i\|z - iw\|^2\right]. \tag{12}$$

7. Page 16, problems 2.5, and 2.6.

8. Page 24, problems 3.3, and 3.6