AMCS 602

Problem set 10 due December 6, 2016

Dr. Epstein

Reading: Page numbers below refer to *Numerical Linear Algebra* by Trefethen and Bau.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Page 218, problem 28.1.
2. Page 218, problem 28.2.
3. Page 218, problem 28.3.
4. Page 218, problem 28.4.
5. Suppose that P is an $m \times m$ matrix with positive entries, and λ is its maximal eigenvalue. Show that if we start with any vector $v^{(0)}$ with non-negative entries such that $\sum_{j=1}^{m} v_j^{(0)} = 1$, and we define

$$\begin{aligned} &\text{For } k = 1, 2, 3, \ldots \\ &w = Pv^{(k-1)} \\ &v^{(k)} = \frac{w}{\sum_{j=1}^{m} w_j}, \end{aligned} \tag{1}$$

then $< v^{(k)} >$ converges to the eigenvector associated to $\lambda(P)$. What is the rate of convergence?

6. Use a MATLAB ODE solver to write a program to solve the Toda system:

$$\begin{aligned} a_k' &= 2(b_k^2 - b_{k-1}^2) \text{ for } k = 1, \ldots, m \text{ with } b_0 = 0; \\ b_k' &= b_k(a_{k+1} - a_k) \text{ for } k = 1, \ldots, m-1, \end{aligned} \tag{2}$$

for a variety of initial conditions with $m = 3, 5, 10$. The initial vector $(a_1, \ldots, a_m)$ can be arbitrary, whereas $(b_1, \ldots, b_{m-1})$ should not have any zero entries. You should plot the trajectories, with the a-curves on one plot and the b-curves on another.

Now use the same initial data you used to solve the Toda system to construct symmetric Jacobi matrices, and apply the QR-iteration. Compare the solution of the Toda system at integer times, with these iterates.

7. The following MATLAB code computes a sequence of examples where we see a transition from a very unstable spectrum, with a lot pseudospectrum far from the spectrum to a very stable spectrum. Can you explain this?

```
%Matrix size
p=20;
%Size of random perturbation
dy=.0005
%
%Perturbation of a tridiagonal matrix with 1 below the
%diagonal and -od above. We can see the transition from
%non-normal toward normal behavior.
%
eps=.025;
figure
for l=1:40
    od=eps*(l-1);
    B=dy*rand(p)+gallery('tridiag',p,1,0,-od);
    db=eig(B);
    scatter(real(db),imag(db))
    dd=eig(zeros(p)+gallery('tridiag',p,1,0,-od));
    hold on
%Plot the spectrum of the unperturbed matrix
    scatter(real(dd),imag(dd),'x','r')
    axis equal
    title(['\epsilon = ',num2str(od),' dy = ',num2str(dy)])
    legend('Perturbed spectrum','Unperturbed spectrum')
    hold off
    pause
end
```