Exterior Calculus in Low Dimensions

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Exterior calculus is a very convenient notation whose basic properties embody many of the classical theorems of single and multi-variable calculus: the chain rule, the Leibniz formula, and the symmetry of mixed partial derivatives. When coupled with integration one also obtains the change of variable formula, the Fundamental Theorem of Calculus, and its higher dimensional generalization, Stokes' Theorem. This notation is very useful for doing calculations in complex analysis.

1 Forms in 1-dimension

The types of forms that arise in a given dimension are in 1-1 correspondence with the dimensions of submanifolds that can occur in that dimension: In 1-dimension there are 0-dimensional submanifolds, i.e. collections of points, and 1-dimensional submanifolds, i.e. collections of intervals. Hence in 1-dimension we define 0-forms, which are just functions, and 1-forms. A function $f(x)$ is "integrated" at a point x_0 by evaluation:

$$\int_{\{x_0\}} f \stackrel{d}{=} f(x_0). \tag{1}$$

A 1-form is written symbolically as $\alpha = g(x)dx$; it is integrated over an *oriented* interval $[a, b]$ using the usual definition of the Riemann integral

$$\int\limits_{[a,b]} \alpha \stackrel{d}{=} \int\limits_a^b g(x)dx. \tag{2}$$

Note how the value of the integral depends upon the orientation for the interval: In 1-dimension an interval $[a, b]$ is "positively oriented" if $a \leq b$, and

$$\int\limits_a^b g(x)dx = -\int\limits_b^a g(x)dx. \tag{3}$$

If $\alpha = g(x)dx$ with $g \geq 0$, and $a \leq b$, then

$$\int_{[a,b]} \alpha \geq 0. \tag{4}$$

The idea of orientation is of central importance in exterior calculus.

The pairing between forms and submanifolds is a duality pairing, since these are linear functionals in both the form and the submanifold. From the form side this is of course a well known property of both functional evaluation and Riemann integration:

$$\begin{gathered}(af_1 + bf_2)(x_0) = af_1(x_0) + bf_2(x_2) \\ \int_a^b (cg_1(x) + eg_2(x))dx = c\int_a^b g_1(x)dx + e\int_a^b g_2(x)dx.\end{gathered} \tag{5}$$

To make sense of "linearity" from the submanifold side we need to introduce a linear structure into the space of submanifolds. Here we just use a formal definition: if $\{x_1, \ldots, x_n\}$ a set of points and $\{a_1, \ldots, a_n\}$ a set of numbers then we define the "0-chain" c as the formal sum

$$c = a_1x_1 + \cdots + c_nx_n. \tag{6}$$

The integral of 0-forms is then extended linearly:

$$\int_c f \stackrel{d}{=} \sum_{j=1}^n a_j \int_{\{x_j\}} f = \sum_{j=1}^n a_j f(x_j). \tag{7}$$

Similarly a 1-chain is a formal sum of oriented intervals

$$C = a_1[x_1, y_1] + \cdots + a_n[x_n, y_n], \tag{8}$$

with a single relation

$$-[x, y] = [y, x]. \tag{9}$$

The integral is extended to 1-chains by linearity

$$\int_C \alpha \stackrel{d}{=} \sum_{j=1}^n a_j \int_{[x_j,y_j]} \alpha; \tag{10}$$

where we see that the relation (9) is required for consistency with (3).

The final operation we define is the exterior derivative, which is a map from 0-forms to 1-forms:

$$df(x) = f'(x)dx. \tag{11}$$

For reasons that will become clearer latter we define $d\alpha = 0$ for any 1-form. With this definition, the Fundamental Theorem of Calculus gives the following relationship between a 0-form and its exterior derivative:

$$\int_{[a,b]} df = \int_a^b f'(x)dx = f(b) - f(a). \tag{12}$$

If $[a, b]$ is an oriented interval, then we define its boundary to be the 0-cochain

$$\partial[a, b] = b - a. \tag{13}$$

With this understood we can rewrite the formula in (12) as

$$\int_{[a,b]} df = \int_{\partial[a,b]} f \tag{14}$$

To complete our discussion of the 1-dimensional case, we need to examine how all these concepts behave under differentiable changes of variable. This is where the chain rule/change of variable formula make their appearance. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly monotone, differentiable map. We'll let $x = \varphi(t)$. The pull-back of the 0-form f via this map is defined to be:

$$\varphi^*(f)(t) \stackrel{d}{=} f(\varphi(t)); \tag{15}$$

so 0-forms are pulled back just by substitution of the new variable.

If $\alpha = g(x)dx$ is a 1-form, then its pull-back under $x = \varphi(t)$ is defined to be.

$$\varphi^*(\alpha) \stackrel{d}{=} \varphi^*(g)d\varphi = g(\varphi(t))\varphi'(t)dt. \tag{16}$$

One might say that if $g(x)$ is the representation of α with respect to the coordinate x (i.e. the coefficient of dx), then $g(\varphi(t))\varphi'(t)$ is the representative of $\varphi^*(\alpha)$ with respect to the coordinate t. The pullback operation is clearly linear: for 0-forms

$$\varphi^*(a_1 f_1 + a_2 f_2) = a_1\varphi^*(f_1) + a_2\varphi^*(f_2), \tag{17}$$

and 1-forms

$$\varphi^*(a_1\alpha_1 + a_2\alpha_2) = a_1\varphi^*(\alpha_1) + a_2\varphi^*(\alpha_2), \tag{18}$$

Note that the chain rule implies that

$$f(\varphi(t))\varphi'(t) = \partial_t(f(\varphi(t)), \tag{19}$$

which in our current notation, can be rewritten:

$$\varphi^*(df) = d(\varphi^*(f)), \tag{20}$$

that is: exterior differentiation commutes with the pull-back operation.

To complete our discussion of the 1-dimensional case we need to consider how integration interacts with changes of variable. Elementary calculus tells us that

$$\int_a^b g(\varphi(t))\varphi'(t)dt = \int_{\varphi(a)}^{\varphi(b)} g(x)dx. \tag{21}$$

Exercise: Check that this formula properly handles the issue of orientation.

We can rewrite this in form notation as

$$\int_{[a,b]} \varphi^*(\alpha) = \int_{\varphi_*([a,b])} \alpha, \tag{22}$$

where we use the definition of the push-forward, φ_*:

$$\varphi_*([a,b]) \stackrel{d}{=} [\varphi(a), \varphi(b)], \tag{23}$$

as an oriented interval. If C is the 1-chain given in (8), then the pushforward operation can be extended linearly to give:

$$\varphi_*(C) = \sum_{j=1}^{n} a_j[\varphi(x_j), \varphi(y_j)]. \tag{24}$$

The change of variables extends by linearity as well:

$$\int_C \varphi^*(\alpha) = \int_{\varphi_*(C)} \alpha, \tag{25}$$

It is easy to see that we can define the pullback operation for maps that are merely differentiable and not 1-1. The formulas given above continue to make sense

$$\varphi^*(f)(t) = f(\varphi(t)) \text{ and } \varphi^*(gdx)(t) = g(\varphi(t))\varphi'(t)dt. \tag{26}$$

The fundamental theorem of calculus shows that the change of variables formula continues to be correct, though one should think over a bit how it relates to the formula in (25).

Exercise: Suppose that φ, ψ are differentiable maps of $\mathbb{R}$ to itself, and α is a 1-form. Show that

$$(\varphi \circ \psi)^*(\alpha) = \psi^*(\varphi^*(\alpha)). \tag{27}$$

2 Forms in 2-dimensions

We now generalize to 2-dimensions. In this case we have submanifolds of 0, 1 and 2 dimensions, so we will need to define 0-forms, 1-forms and 2-forms. Let (x, y) denote coordinates on $\mathbb{R}^2$, then 0-forms are simply functions $f(x, y)$; 1-forms are expressions of the form

$$\alpha = g_1(x, y)dx + g_2(x, y)dy; \tag{28}$$

and 2-forms are expressions of the form

$$\omega = h(x, y)dx \wedge dy. \tag{29}$$

We need to explain the meaning of the wedge-product, $\wedge$. It is defined to be a skew symmetric product, that is

$$dx \wedge dy = -dy \wedge dx. \tag{30}$$

This requires that $dx \wedge dx = 0$ and $dy \wedge dy = 0$.

As before we define the exterior derivative as a map from j forms to $j + 1$ forms by

$$\begin{aligned} df(x, y) &= \partial_x f(x, y)dx + \partial_y f(x, y)dy \\ d(g_1(x, y)dx + g_2(x, y)dy) &= \partial_x g_1(x, y)dx \wedge dx + \partial_y g_1(x, y)dy \wedge dx \\ &\quad + \partial_x g_2(x, y)dx \wedge dy + \partial_y g_2(x, y)dy \wedge dy \\ &= (\partial_x g_2(x, y) - \partial_y g_1(x, y))dx \wedge dy \\ d(h(x, y)dx \wedge dy) &= 0. \end{aligned} \tag{31}$$

The wedge product can then be extended to define a map from a pair of 1-forms to a 2-form:

$$\begin{aligned} &(g_1(x, y)dx + g_2(x, y)dy) \wedge (h_1(x, y)dx + h_2(x, y)dy) \\ &\qquad = g_1h_1dx \wedge dx + g_1h_2dx \wedge dy + g_2h_1dy \wedge dx + g_2h_2dy \wedge dy \\ &\qquad = (g_1h_2 - g_2h_1)dx \wedge dy. \end{aligned} \tag{32}$$

To get to the second line we use the relations satisfied by $\wedge$. This extended product is also skew symmetric $\alpha_1 \wedge \alpha_2 = -\alpha_2 \wedge \alpha_1$.

Note the following important fact: if f is a 0-form, then the skew symmetry of the wedge produce implies that

$$d^2 f = d(\partial_x f dx + \partial_y f dy) = (\partial_x \partial_y f - \partial_y \partial_x f)dx \wedge dy = 0, \tag{33}$$

because $\partial_x \partial_y f = \partial_y \partial_x f$, i.e. mixed partial derivatives commute.
Exercise: Show that if f, g are a 0-forms, then

$$d(fg) = fdg + gdf. \tag{34}$$

Show that if f is a 0-form and α is a 1-form, then

$$d(f\alpha) = df \wedge \alpha + fd\alpha. \tag{35}$$

The motivation for the definition of the wedge product comes out of the change of variables formula in calculus. The idea is that $dx \wedge dy$ is the *oriented area element*. If we change variables with $x = \psi(s,t),\ y = \phi(s,t)$ then

$$\begin{aligned} dx \wedge dy &= d\psi(s,t) \wedge d\phi(s,t) \\ &= (\partial_s \psi ds + \partial_t \psi dt) \wedge (\partial_s \phi ds + \partial_t \phi dt) \\ &= (\partial_s \psi \partial_t \phi - \partial_t \psi \partial_s \phi) ds \wedge dt. \end{aligned} \tag{36}$$

Recall that $|\partial_s \psi \partial_t \phi - \partial_t \psi \partial_s \phi|$ is the coefficient of dsdt in the usual change of variables formula for 2-dimensional integrals. Leaving out the absolute values allows us to keep track of the orientation.

Let us consider how forms in 2-dimensions behave under the pull-back operation. Let $\varphi = (\varphi_1, \varphi_2)$ be a differentiable map from $\mathbb{R}^2$ to itself. If f is a 0-form, then

$$\varphi^*(f) \stackrel{d}{=} f \circ \varphi; \tag{37}$$

if $\alpha = g_1 dx + g_2 dy$ is a 1-form, then

$$\begin{aligned} \varphi^*(\alpha) &\stackrel{d}{=} g_1 \circ \varphi d\varphi_1 + g_2 \circ \varphi d\varphi_2 \\ &= g_1 \circ \varphi(s,t)(\partial_s \varphi_1 ds + \partial_t \varphi_1 dt) + g_2 \circ \varphi(s,t)(\partial_s \varphi_2 ds + \partial_t \varphi_2 dt); \end{aligned} \tag{38}$$

finally if $\omega = hdx \wedge dy$ is a 2-form then

$$\begin{aligned} \varphi^*(\omega) &\stackrel{d}{=} h \circ \varphi d\varphi_1 \wedge d\varphi_2 \\ &= h \circ \varphi(s,t)(\partial_s \varphi_1 \partial_t \varphi_2 - \partial_t \varphi_1 \partial_s \varphi_2) ds \wedge dt. \end{aligned} \tag{39}$$

Exercise: Suppose that α_1 and α_2 are 1-forms and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable map. Show that

$$\varphi^*(\alpha_1 \wedge \alpha_2) = \varphi^*(\alpha_1) \wedge \varphi^*(\alpha_2). \tag{40}$$

Let φ be a differentiable map. It is once again a consequence of the chain rule that for a 0-form, f, 1-form, α, and a 2-form, ω,

$$\begin{aligned} \varphi^*(df) &= d(\varphi^* f) \\ \varphi^*(d\alpha) &= d(\varphi^* \alpha) \\ \varphi^*(d\omega) &= d(\varphi^* \omega) \end{aligned} \tag{41}$$

Exercise:Verify the formulæ in (41).

Following our development in 1-dimension, we now need to describe how to integrate forms of varying degree over submanifolds (and chains) of the corresponding dimensions. 0-chains are just finite collections of points

$$c = a_1(x_1, y_1) + \cdots + a_n(x_n, y_n). \tag{42}$$

A 0-form $f(x, y)$ is "integrated" over this 0-chain by setting

$$\int_c f = \sum_{j=1}^{n} a_j f(x_j, y_j). \tag{43}$$

To begin with, say that an oriented differentiable arc is defined by a differentiable map

$$\gamma : [a, b] \longrightarrow \mathbb{R}^2 \text{ where } a < b. \tag{44}$$

Such a map is defined by its coordinate functions $(\gamma_1(t), \gamma_2(t))$, which are assumed to be differentiable. The parametrization of the arc defines its orientation. The arc with opposite orientation is defined to

$$\gamma^-(t) \stackrel{d}{=} \gamma((1-t)((b-a)+a). \tag{45}$$

If $\varphi : [a, b] \to [c, d]$ is monotone increasing, the $\gamma \circ \varphi = \varphi^*(\gamma)$ is another representation of the same oriented differentiable arc; if φ is monotone decreasing, then $\gamma \circ \varphi$ is another representation of γ^-.

The integral of the 1-form $\alpha = g_1 dx + g_2 dy$ over the oriented arc γ is defined to be

$$\int_\gamma \alpha \stackrel{d}{=} \int_a^b [g_1 \circ \gamma(t)\gamma_1'(t) + g_2 \circ \gamma(t)\gamma_2'(t)]dt. \tag{46}$$

Note that this can also be thought of as the integral of the pulled-back 1-form $\gamma^*(\alpha)$ over the oriented interval $[a, b]$. Thinking of the integral of α over γ in this way, it is easy to see that it is parametrization independent. If $\varphi : [c, d] \to [a, b]$, then

$$(\gamma \circ \varphi)^*(\alpha) = \varphi^*(\gamma^*(\alpha)). \tag{47}$$

Using our 1-dimensional result it then follows that

$$\int_{\gamma\circ\varphi} \alpha = \int_c^d \varphi^*(\gamma^*(\alpha)) = \int_{\varphi(c)}^{\varphi(d)} \gamma^*(\alpha) = \pm \int_\gamma \alpha, \tag{48}$$

where we use the $+$ sign if φ is increasing and the $-$ sign if it is decreasing.

A 1-chain is just a formal sum of oriented differentiable arcs with numerical coefficients with the sole relation

$$-\gamma = \gamma^{-}. \tag{49}$$

If $C = a_1\gamma_1 + \cdots a_n\gamma_n$, then

$$\int_C \alpha = \sum_{j=1}^{n} a_j \int_{\gamma_j} \alpha. \tag{50}$$

The relation is explained by the fact that

$$\int_{\gamma^-} \alpha = -\int_{\gamma} \alpha \tag{51}$$

The 2-dimensional submanifolds of $\mathbb{R}^2$ are subsets of $\mathbb{R}^2$, which for the purposes of this discussion, we take to be bounded sets with a well defined Riemannian area. Such an open set U can be written as a union of countably many positively oriented rectangles

$$U = \bigcup_{i=1}^{\infty} R_i, \tag{52}$$

where $R_i = [a_i, b_i] \times [c_i, d_i]$, so that $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ if $i \neq j$. The rectangles are oriented by the assumption that $a_i \leq b_i$ and $c_i \leq d_i$. Of course the area of R_i is $|R_i| = (b_i - a_i)(d_i - c_i)$ and

$$|U| = \sum_{i=1}^{\infty} |R_i|. \tag{53}$$

We need to define the integral of a 2-form $\omega = h(x, y)dx \wedge dy$, over an oriented rectangle $R = [a, b] \times [c, d]$:

$$\int_R \omega \stackrel{d}{=} \int_c^d \int_a^b h(x, y)dxdy, \tag{54}$$

which we think of as an iterated Riemann integral. A rectangle with the opposite orientation is given by $R^- = [b, a] \times [c, d]$, and it is immediate from the 1-dimensional theory that

$$\int_{R^-} \omega = -\int_R \omega. \tag{55}$$

The integral of ω over the oriented set U defined in (52) is defined by linearity:

$$\int_U \omega = \sum_{i=1}^{\infty} \int_{R_i} \omega. \tag{56}$$

Exercise: Show that if D is bounded Riemann integrable domain with the positive orientation, then the area is given by

$$|D| = \int_D dx \wedge dy. \tag{57}$$

The linearization of 1-1 differentiable map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$D\varphi = \begin{pmatrix} \partial_s \varphi_1 & \partial_t \varphi_1 \\ \partial_s \varphi_2 & \partial_t \varphi_2 \end{pmatrix}. \tag{58}$$

If we let $(x, y) = \varphi(s, t)$, then the usual change of variables formula for integrals states that

$$dxdy = |\det(D\varphi)| dsdt; \tag{59}$$

as noted above

$$\det(D\varphi) ds \wedge dt = d\varphi_1 \wedge d\varphi_2. \tag{60}$$

A map is called orientation preserving if $\det(D\varphi) \geq 0$ and orientation reversing if $\det(D\varphi) \leq 0$.

Suppose that φ is an orientation preserving, 1-1, differentiable map and R is an oriented rectangle. From advanced calculus we know that $\varphi(R)$ is a bounded set with a decomposition of the form given in (52). Let $\omega = h(x, y)dx \wedge dy$ be a 2-form. The usual change of variables formula for integrals in the plane implies that

$$\int_R \varphi^*(\omega) = \int_{\varphi(R)} h(x, y) dxdy \tag{61}$$

If φ is orientation reversing then

$$\int_R \varphi^*(\omega) = -\int_{\varphi(R)} h(x, y) dxdy. \tag{62}$$

A formula of this general character holds without the assumption that φ is 1-1 and orientation preserving, but we then need to make a greater effort to define $\varphi_*(R)$. We will not do this for the moment.

3 Exact forms and Stokes' theorem

We now consider special properties of integrals of 1- and 2- forms that are "exact." A 1-form, α, is exact if there is a function f so that

$$\alpha = df; \tag{63}$$

a 2-form, ω, is exact if there is a 1-form β so that

$$\omega = d\beta. \tag{64}$$

Let $\gamma : [a, b] \to \mathbb{R}^2$ be an oriented arc, then the fundamental theorem of calculus easily shows that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)). \tag{65}$$

If, as in 1-dimension, we define the boundary of the oriented arc γ to be the 0-chain

$$\partial\gamma = \gamma(b) - \gamma(a), \tag{66}$$

then we can rewrite this formula as

$$\int_\gamma df = \int_{\partial\gamma} f. \tag{67}$$

The boundary of a rectangle is a 1-chain: if $R = [a, b] \times [c, d]$, then

$$\partial R = [a, b] \times \{c\} + \{b\} \times [c, d] - [a, b] \times \{d\} - \{a\} \times [c, d]. \tag{68}$$

Let $\beta = g_1(x, y)dx + g_2(x, y)dy$, if we use the fundamental theorem of calculus 4 times, then we can easily derive the relation

$$\begin{aligned} \int_R d\beta &= \int_a^b \int_c^d (\partial_x g_2(x, y) - \partial_y g_1(x, y))dxdy \\ &= \int_a^b g_1(x, c)dx + \int_c^d g_2(b, y)dy - \int_a^b g_1(x, d)dx - \int_c^d g_2(a, y)dy. \end{aligned} \tag{69}$$

Comparing this with the definition of ∂R, we see that this can be rewritten as

$$\int_R d\beta = \int_{\partial R} \beta. \tag{70}$$

This formula is a simple case of Stokes' Theorem.

Let $D \subset \mathbb{R}^2$; the boundary of D is defined as the set of points

$$\partial D = \{(x, y) : B_r(x, y) \cap D \neq \emptyset \text{ and } B_r(x, y) \cap D^c \neq \emptyset \text{ for all } r > 0\}. \quad (71)$$

Stokes Theorem can be generalized to domains in the plane with piecewise differentiable boundaries. This means that the ∂D is the union of a finite number simple closed curves that are each made of a finite number of differentiable arcs. The most important issue is how to orient the set ∂D, as the boundary of D.

An oriented arc γ is given by specifying a parametrization: $\gamma : [0, 1] \to \mathbb{R}^2$. If $\gamma'(t) \neq 0$ for any t, then it is clear that the tangent vector $\boldsymbol{t}(t) = \gamma'(t)$ points in the positive direction, that is the direction we move as the parameter increases. An orientation for a differentiable arc can therefore be specified by a choice of non-vanishing, continuous tangent vector $\boldsymbol{\tau}(t)$. A parametrization then gives the specified orientation if $\gamma'(t)$ points in the same direction as $\boldsymbol{\tau}(t)$ at every point. We see from the example of a rectangle that if one is on the boundary facing in the "positive direction," then the domain should lie to your left.

A general definition is given by considering the outward pointing normal vector. At each point of the $p \in \partial D$ which is differentiable, there is a well defined outward pointing normal vector $\boldsymbol{n}(p)$. The positive direction, $\boldsymbol{t}(p)$ tangent to the boundary at p is obtained by rotating $\boldsymbol{n}(p)$ counterclockwise through 90°. This specifies the orientation for each of the differentiable arcs that comprise ∂D. Henceforth ∂D refers to the set defined in (71) with this "induced" orientation. With this understood, then Stokes theorem states that

$$\int_D d\beta = \int_{\partial D} \beta. \quad (72)$$

This theorem is one of the foundations of modern mathematics.

Exercise: Show that if D is a positively oriented bounded region with a piecewise differentiable boundary, then

$$|D| = \int_{\partial D} x dy. \quad (73)$$

A very important problem in the theory of exterior forms is to decide when a k-form α is exact, that is when does there exist a $(k-1)$-form β so that $\alpha = d\beta$. In 2-dimensions this question is most interesting for $k = 1$ and 2. If α is a 1-form, then it follows from (33) that in order for $\alpha = df$, it is necessary that $d\alpha = 0$. A form of any degree that satisfies this condition is called a *closed form.* A 0-form is closed if and only if it is constant and a 2-form is always closed.

Whether or not a closed 1-form is exact depends to some extent on the domain in which we are working. For example the 1-form

$$\alpha = \frac{xdy - ydx}{x^2 + y^2}, \tag{74}$$

which is defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ is closed. If follows from elementary geometry that $\alpha = d\theta$, where $\theta = \tan^{-1}(y/x)$. Of course θ is not a single valued function in all of $\mathbb{R}^2 \setminus \{(0,0)\}$, but is in $\mathbb{R}^2 \setminus (-\infty, 0]$.

Let's prove a simple result to see what is going on here:

Lemma 1. *Suppose that D is a connected open subset of $\mathbb{R}^2$, and that α is a closed 1-form in D. Let $\gamma : [0,1] \times [0,1] \to D$ be a $\mathscr{C}^1$-map such that*

$$\gamma(0,t) = p \text{ and } \gamma(1,t) = q. \tag{75}$$

If we set $\gamma_t(s) = \gamma(s,t)$, then

$$\int_{\gamma_t} \alpha \tag{76}$$

does not depend on t.

Before we prove this result, which is usually called the *Monodromy Theorem*, we observe that $\{\gamma_t\}$ can be thought of as a differentiable family of oriented arcs, lying in D, which join the point p to the point q. The lemma says that, at least within such a family the value of the integral $\int_p^q \alpha$ only depends on the endpoints.

Proof. The proof of the lemma is a simple application of the definitions. Let $R = [0,1] \times [0,1]$ be the parameter domain for the family $\gamma(s,t)$. If we trace back through the definitions we easily deduce that

$$\int_{\gamma_0} \alpha - \int_{\gamma_1} \alpha = \int_{\partial R} \gamma^*(\alpha). \tag{77}$$

By Stokes theorem (for a rectangle) we know that

$$\begin{aligned} \int_{\partial R} \gamma^*(\alpha) &= \int_R d\gamma^*(\alpha) \\ &= \int_R \gamma^*(d\alpha) \\ &= 0. \end{aligned} \tag{78}$$

To go from the first to second line we use (41). The last line follows as $d\alpha = 0$. □

Clearly if a 1-form α in a domain D is exact, $\alpha = df$, then the integral over an oriented, differentiable arc γ depends only on the endpoints p, q

$$\int_\gamma \alpha = \int_\gamma df = f(q) - f(p). \tag{79}$$

Hence the question of when a closed 1-form is exact in a domain D is therefore a question about the topology of the domain D. For example if D is simply connected, so that every path joining a pair of points can be deformed to any other such path, then the lemma shows that any closed 1-form is exact. On the other hand the domain $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not simply connected, and the closed 1-form $d\theta$ is not exact.
Exercise: Let D be a simply connected domain in $\mathbb{R}^2$, and α a closed 1-form. If $\gamma : [0, 1] \to D$ is a closed, oriented differentiable arc, then prove that

$$\int_\gamma \alpha = 0. \tag{80}$$

The remaining material is for your entertainment and is not required for Math 608 this semester. A more detailed discussion can be found in *Calculus on Manifolds* by Michael Spivak.

4 Forms in 3-dimensions

In 3-dimensions we have 0-, 1-, 2-, and 3-forms. 0-forms are just scalar valued functions; 1-forms are of the form $a_1dx_2 + a_2dx_2 + a_3dx_3$; 2-forms are of the form $a_1dx_2 \wedge dx_3 + a_2dx_3 \wedge dx_1 + a_3dx_1 \wedge dx_2$; and 3-forms are expressions of the form $adx_1 \wedge dx_2 \wedge dx_3$. We recall the definition of d on forms defined on $\mathbb{R}^3$:

$$\begin{aligned}
&\text{0-forms } \Lambda^0\mathbb{R}^3: \quad da = \partial_{x_1}adx_1 + \partial_{x_2}adx_2 + \partial_{x_3}adx_3 \\
&\text{1-forms } \Lambda^1\mathbb{R}^3: \quad d(a_1dx_1 + a_2dx_2 + a_3dx_3) = (\partial_{x_1}a_2 - \partial_{x_1}a_1)dx_1 \wedge dx_2 \\
&\quad + (\partial_{x_3}a_1 - \partial_{x_1}a_3)dx_3 \wedge dx_1 + (\partial_{x_2}a_3 - \partial_{x_3}a_2)dx_2 \wedge dx_3 \\
&\text{2-forms } \Lambda^2\mathbb{R}^3: \quad d(a_1dx_1 \wedge dx_2 + a_2dx_3 \wedge dx_1 + a_3dx_2 \wedge dx_3) = \\
&\quad (\partial_{x_1}a_1 + \partial_{x_2}a_2 + \partial_{x_3}a_3)dx_1 \wedge dx_2 \wedge dx_3 \\
&\text{3-forms } \Lambda^3\mathbb{R}^3: \quad d(adx_1 \wedge dx_2 \wedge dx_3) = 0.
\end{aligned} \tag{81}$$

The sequence (106) becomes:

$$\mathscr{C}^\infty(U) \xrightarrow{d} \mathscr{C}^\infty(U; \Lambda^1\mathbb{R}^3) \xrightarrow{d} \mathscr{C}^\infty(U; \Lambda^2\mathbb{R}^3) \xrightarrow{d} \mathscr{C}^\infty(U; \Lambda^3\mathbb{R}^3). \tag{82}$$

All of the classical differential relations for vector differentials ($\nabla \times \nabla = 0$ and $\nabla \cdot \nabla \times = 0$) are simply $d^2 = 0$.

4.1 Exterior forms on a manifold

Generally we can define the exterior p-forms on an n-dimensional manifold, M, as sections of the vector bundle $\Lambda^p T^*M$. The exterior derivative is a canonical map

$$d : \mathscr{C}^\infty(M; \Lambda^p T^*M) \to \mathscr{C}^\infty(M; \Lambda^{p+1} T^*M). \tag{83}$$

If $(x_1, \ldots, x_n)$ are local coordinates, then a p-form can be expressed as

$$\boldsymbol{\alpha} = \sum_{I \in \mathscr{I}_p} a_I(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \quad a_I \in \mathscr{C}^\infty. \tag{84}$$

Here $\mathscr{I}_p$ is the set of increasing p-multi-indices $1 \leq i_1 < \cdots < i_p \leq n$. In these local coordinates, the exterior derivative of a function $f(x)$ is defined to be

$$df = \sum_{j=1}^{n} \partial_{x_j} f(x) dx_j, \tag{85}$$

and of a p-form

$$d\boldsymbol{\alpha} = \sum_{I \in \mathscr{L}_p} da_I(x) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \tag{86}$$

The remarkable fact is that this is invariantly defined; though it is really nothing more than the chain rule. The fact that mixed partial derivatives commute easily applies to show that $d^2 = 0$. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are forms, then we have the Leibniz Formula:

$$d(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (d\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + (-1)^{\deg \boldsymbol{\alpha}} \boldsymbol{\alpha} \wedge d\boldsymbol{\beta}. \tag{87}$$

If $\boldsymbol{\alpha}$ is a k-form defined in an open subset, U of M then we can integrate it over any smooth, oriented compact submanifold $\Sigma^k \subset\subset U$, of dimension k, with or without boundary. We denote this pairing by

$$\langle \boldsymbol{\alpha}, [\Sigma] \rangle = \int_\Sigma \boldsymbol{\alpha}. \tag{88}$$

If $\boldsymbol{\alpha}$ is an exact form, that is, $\boldsymbol{\alpha} = d\boldsymbol{\beta}$, and Σ is a smooth, oriented submanifold with or without boundary, then Stokes' theorem states that

$$\int_\Sigma d\boldsymbol{\beta} = \int_{\partial\Sigma} \boldsymbol{\beta}. \tag{89}$$

The boundary must be given the induced orientation. Note, in particular, that if $\partial\Sigma = \emptyset$, then the integral of $d\boldsymbol{\beta}$ over Σ vanishes.

There is a second natural operation on exterior forms that satisfies a Leibniz formula. If $\boldsymbol{v}$ is a vector field, then the *interior product* of $\boldsymbol{v}$ with a k-form, ω, is a $(k-1)$-form, $i_{\boldsymbol{v}}\omega$, defined by:

$$i_{\boldsymbol{v}}\omega(\boldsymbol{v}_2, \ldots, \boldsymbol{v}_k) \stackrel{d}{=} \omega(\boldsymbol{v}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k). \tag{90}$$

If ω and η are exterior forms, then

$$i_{\boldsymbol{v}}[\omega \wedge \eta] = [i_{\boldsymbol{v}}\omega] \wedge \eta + (-1)^{\deg \omega}\omega \wedge [i_{\boldsymbol{v}}\eta]. \tag{91}$$

Using forms simplifies calculations considerably because forms can be automatically integrated over submanifolds of the "correct" dimension, keep track of orientation, and all the differential relationships follow from the fact that $d^2 = 0$. Moreover, Stokes' theorem subsumes all the classical integration by parts formulæ is one simple package.

4.2 Hodge star-operator

To write the Maxwell equations we need one further operation, called the *Hodge star-operator*. This operation can be defined on an oriented Riemannian manifold. Suppose that $\omega_1, \ldots, \omega_n$ is an local orthonormal basis of one forms, and

$$dV = \omega_1 \wedge \cdots \wedge \omega_n, \tag{92}$$

defines the orientation. If $1 \leq i_1 < \cdots < i_p \leq n$, and $j_1 < \cdots < j_{n-p}$ are complementary indices, then we define

$$\star[\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}] = (-1)^{\epsilon}\omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-p}}, \tag{93}$$

with $\epsilon = 0$ or 1, chosen so that

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \star[\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}] = dV. \tag{94}$$

For example, the Hodge operator is defined on the standard orthonormal basis of exterior forms for $\mathbb{R}^3$ by setting:

$$\begin{gathered} \star 1 = dx_1 \wedge dx_2 \wedge dx_3 \\ \star dx_1 = dx_2 \wedge dx_3 \quad \star dx_2 = dx_3 \wedge dx_1 \quad \star dx_3 = dx_1 \wedge dx_2 \\ \star dx_1 \wedge dx_2 = dx_3 \quad \star dx_3 \wedge dx_1 = dx_2 \quad \star dx_2 \wedge dx_3 = dx_1 \\ \star dx_1 \wedge dx_2 \wedge dx_3 = 1 \end{gathered} \tag{95}$$

From these formulæ it is clear that, in 3-dimensions, $\star^2 = I$. Notice that applying the $\star$-operator exchanges the two correspondences between vector fields and forms in (108).

The star-operator is simply related to the metric: If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are real p-forms, then

$$[\boldsymbol{\alpha} \wedge \star\boldsymbol{\beta}]_x = (\boldsymbol{\alpha}, \boldsymbol{\beta})_x dV_x, \tag{96}$$

where $(\cdot, \cdot)$ is the (real) inner product defined by the metric on p-forms. Generally,

$$\star^2 = (-1)^{p(n-p)}, \tag{97}$$

on p-forms defined on an n-dimensional manifold. Using this observation and (96) we easily show that $\star$ is a pointwise isometry:

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_x = (\star\boldsymbol{\alpha}, \star\boldsymbol{\beta})_x \tag{98}$$

A fundamental role of the Hodge $\star$-operator is to define a Hilbert space inner product on forms. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are (possibly complex) forms of the same degree defined in U, then $\boldsymbol{\alpha} \wedge \star\overline{\boldsymbol{\beta}}$ is a n-form, which can therefore be integrated:

$$\begin{aligned} \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle &= \int_U \boldsymbol{\alpha} \wedge \star\overline{\boldsymbol{\beta}}. \\ &= \int_U (\boldsymbol{\alpha}, \boldsymbol{\beta})_x dV(x). \end{aligned} \tag{99}$$

We assume that $(\cdot, \cdot)$ is extended to define an Hermitian inner product on complex valued forms. The extended metric continues to satisfy (98).

4.3 Adjoints and Integration-by-parts

On an n-dimensional manifold the expression for the formal adjoint, with respect to the pairing in (99), of the d-operator, acting on a p-form $\boldsymbol{\beta}$, is:

$$d^*\boldsymbol{\beta} = \begin{cases} -\star d \star \boldsymbol{\beta} & \text{if } n \text{ is even} \\ (-1)^p \star d \star \boldsymbol{\beta} & \text{if } n \text{ is odd.} \end{cases} \tag{100}$$

Let G be a bounded domain with a smooth boundary. Let r be a function that is negative in G and vanishes on bG. Suppose moreover that $(dr, dr)_x \equiv 1$ for $x \in bG$, and $\boldsymbol{n}$ is the outward pointing unit normal along bG. The basic integration by parts formulæ for d and d^* can be expressed in terms of this inner product by

$$\begin{aligned} \langle d\boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_G &= \int_{bG} (dr \wedge \boldsymbol{\alpha}, \boldsymbol{\beta})_x dS(x) + \langle \boldsymbol{\alpha}, d^*\boldsymbol{\beta} \rangle_G \\ \langle d^*\boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_G &= -\int_{bG} (i_{\boldsymbol{n}}\boldsymbol{\alpha}, \boldsymbol{\beta})_x dS(x) + \langle \boldsymbol{\alpha}, d\boldsymbol{\beta} \rangle_G. \end{aligned} \tag{101}$$

Here we use dS to denote surface measure on bG. It is important to recall that, with respect to the pointwise inner product,

$$(dr \wedge \boldsymbol{\alpha}, \boldsymbol{\beta})_x = (\boldsymbol{\alpha}, i_{\boldsymbol{n}}\boldsymbol{\beta})_x. \tag{102}$$

The (positive) Laplace operator, acting on any form degree is given by formula

$$dd^* + d^*d. \tag{103}$$

In $\boldsymbol{R}^3$ this would give $-(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2)$. To avoid confusion with standard usage in E&M, we use Δ to denote the negative operator $-(dd^* + d^*d)$.

5 Maxwell's Equations in Differential Form Notation

In the traditional approach to electricity and magnetism Maxwell's equations are expressed in terms of relationships between four vector fields $\boldsymbol{E}$, $\boldsymbol{D}$ and $\boldsymbol{B}$, $\boldsymbol{H}$ defined on $\mathbb{R}^3 \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial \boldsymbol{D}}{\partial t} = c\nabla \times \boldsymbol{H} - 4\pi\, \boldsymbol{J} \quad \frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \\ \nabla \cdot \boldsymbol{D} = 4\pi\rho \quad \nabla \cdot \boldsymbol{B} = 0; \end{aligned} \tag{104}$$

c is the speed of light. Here $\boldsymbol{J}$ is the current density and ρ is the charge density, they satisfy the conservation of charge:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{J}. \tag{105}$$

The differential symmetries of this system of equations are rooted in the exactness of the sequence:

$$\mathscr{C}^\infty(U) \xrightarrow{\nabla} \mathscr{C}^\infty(U; T\mathbb{R}^3) \xrightarrow{\nabla\times} \mathscr{C}^\infty(U; T\mathbb{R}^3) \xrightarrow{\nabla\cdot} \mathscr{C}^\infty(U), \tag{106}$$

here $U \subset \mathbb{R}^3$ is an open set, and $\mathscr{C}^\infty(U; T\mathbb{R}^3)$ are the smooth vector fields defined in U :

$$\mathscr{C}^\infty(U; T\mathbb{R}^3) = \{a_1(x)\partial_{x_1} + a_2(x)\partial_{x_2} + a_3(x)\partial_{x_3} : \ a_1, a_2, a_3 \in \mathscr{C}^\infty(U)\}. \tag{107}$$

The exactness of the sequence is equivalent to the classical identities $\nabla \times \nabla = 0$, and $\nabla \cdot \nabla\times = 0$.

While this representation is traditional, physically and geometrically it makes more sense to regard Maxwell's equations as a relationship amongst differential, or exterior forms. In this paper we usually work with the fields $\boldsymbol{E}$ and $\boldsymbol{H}$. It turns

out to be convenient to use a 1-form to represent $\boldsymbol{E}$ and a 2-form to represent $\boldsymbol{H}$. We use the correspondences

$$\begin{aligned} \boldsymbol{H} &= h_1\partial_{x_1} + h_2\partial_{x_2} + h_3\partial_{x_3} \leftrightarrow h_1 dx_2 \wedge dx_3 + h_2 dx_3 \wedge dx_1 + h_3 dx_1 \wedge dx_2 = \boldsymbol{\eta} \\ \boldsymbol{E} &= e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} \leftrightarrow e_1 dx_1 + e_2 dx_2 + e_3 dx_3 = \boldsymbol{\xi} \end{aligned} \tag{108}$$

It is natural to think of the electric field as a 1-form, for the electric potential difference is then obtained by integrating this 1-form:

$$\phi_P - \phi_Q = \int_\gamma \boldsymbol{\xi}, \tag{109}$$

with γ a path from P to Q. Similarly, it is reasonable to think of the magnetic field as a 2-form, for the flux of $\boldsymbol{H}$ through a surface Σ is then obtained by integrating:

$$\text{Flux of } \boldsymbol{H} \text{ through } \Sigma = \int_\Sigma \boldsymbol{\eta}. \tag{110}$$

While these are the most basic measurements associated to electric and magnetic fields, there are times when it is natural to integrate the $\boldsymbol{E}$-field over a surface, or the $\boldsymbol{H}$-field over a curve. This is done, in the form language, by using the Hodge-star and interior product operations, $\star$, i_v, introduced below. A detailed exposition of this approach to Maxwell's equations can be found in [**?**].

5.1 Maxwell's Equations in terms of exterior forms

With these preliminaries we can state the correspondences between the differential operators, ∇, $\nabla\times$, and $\nabla\cdot$ and the corresponding objects acting on forms. For a scalar function ϕ, $\nabla\phi$ corresponds to $d\phi$. An elementary calculation shows that if $\boldsymbol{E} \leftrightarrow \boldsymbol{\xi}$, a 1-form, then $\nabla \times \boldsymbol{E} \leftrightarrow d\boldsymbol{\xi}$, and $\nabla \cdot \boldsymbol{E} \leftrightarrow d^*\boldsymbol{\xi}$. Moreover with $\boldsymbol{H} \leftrightarrow \boldsymbol{\eta}$, a 2-form, we have $\nabla \times \boldsymbol{H} \leftrightarrow d^*\boldsymbol{\eta}$, and $\nabla \cdot \boldsymbol{H} \leftrightarrow d\boldsymbol{\eta}$. The operator d^* also acts on 3-forms.

If we let $\boldsymbol{E} \leftrightarrow \boldsymbol{\xi}$, $\boldsymbol{H} \leftrightarrow \boldsymbol{\eta}$, and $\boldsymbol{J} \leftrightarrow \boldsymbol{j}$ (a 1-form) as in (108), then Maxwell's equations in a vacuum become:

$$\begin{aligned} \frac{\partial \boldsymbol{\xi}}{\partial t} &= c d^*\boldsymbol{\eta} - 4\pi\, \boldsymbol{j} \quad \frac{\partial \boldsymbol{\eta}}{\partial t} = -c d\boldsymbol{\xi} \\ d^*\boldsymbol{\xi} &= 4\pi\rho \quad d\boldsymbol{\eta} = 0 \quad \frac{\partial \rho}{\partial t} = -d^*\boldsymbol{j}. \end{aligned} \tag{111}$$

If $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are time harmonic with time dependence $e^{-it\omega}$, then in the absence of sources, we easily derive the Helmholtz equations:

$$c^2\Delta\boldsymbol{\xi} + \omega^2\boldsymbol{\xi} = 0 \quad c^2\Delta\boldsymbol{\eta} + \omega^2\boldsymbol{\eta} = 0. \tag{112}$$

We let $D \subset \mathbb{R}^3$ denote a bounded set with smooth boundary and let

$$\Omega = \mathbb{R}^3 \setminus \overline{D} \quad \Gamma = bD. \tag{113}$$

In this paper D is usually taken to be a perfect conductor, lying in a bounded domain with smooth boundary, and Ω a dielectric. We assume that ϵ is the electrical permittivity, μ is the magnetic permeability and σ the electrical conductivity of Ω. As above, we identify the $\boldsymbol{E}$-field with a 1-form, $\boldsymbol{\Xi}$:

$$e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} = \boldsymbol{E} \leftrightarrow \boldsymbol{\Xi} = e_1dx_1 + e_2dx_2 + e_3dx_3, \tag{114}$$

and $\boldsymbol{H}$ with a 2-form, $\boldsymbol{N}$:

$$h_1\partial_{x_1} + h_2\partial_{x_2} + h_3\partial x_3 = \boldsymbol{H} \leftrightarrow \boldsymbol{N} = h_1dx_2 \wedge dx_3 + h_2dx_3 \wedge dx_1 + h_3dx_1 \wedge dx_2. \tag{115}$$

In terms of exterior forms, we set

$$\boldsymbol{\Xi}(x,t) = \left[\frac{\omega}{\omega\epsilon + i\sigma}\right]^{\frac{1}{2}} \boldsymbol{\xi}(x)e^{-i\omega t} \quad \boldsymbol{N}(x,t) = \mu^{-\frac{1}{2}}\boldsymbol{\eta}(x)e^{-i\omega t}; \tag{116}$$

the time harmonic Maxwell equations become:

$$\begin{aligned} d\boldsymbol{\xi} &= ik\boldsymbol{\eta} \quad d^*\boldsymbol{\xi} = 0 \\ d^*\boldsymbol{\eta} &= -ik\boldsymbol{\xi} \quad d\boldsymbol{\eta} = 0. \end{aligned} \tag{117}$$

Here k is the square root of $\mu(\epsilon\omega^2 + i\sigma\omega)$, with non-negative imaginary part. In most of the paper we use the somewhat less symmetric, though more standard normalization, (**??**).

With this choice of correspondence between the vector and form representations, we can write the Maxwell equations in a very succinct and symmetric form:

$$(d + d^*)(\boldsymbol{\xi} + \boldsymbol{\eta}) = ik\Lambda(\boldsymbol{\xi} + \boldsymbol{\eta}); \tag{118}$$

here Λ is the operation defined on forms by

$$\Lambda(\boldsymbol{\alpha}) = (-1)^{\deg\boldsymbol{\alpha}}\boldsymbol{\alpha}. \tag{119}$$

Simple calculations shows that

$$(d + d^*)\Lambda = -\Lambda(d + d^*) \text{ and } \Lambda^2 = I, \tag{120}$$

implying that

$$(d + d^* - ik\Lambda)^2 = -(\Delta + k^2). \tag{121}$$

Thus, acting on forms, the operator $\Delta + k^2$ has a local square root. Or, put differently, $d + d^* - ik\Lambda$ is an operator of Dirac-type, see [**?**]. Indeed, we could write the vacuum Maxwell equations in the form

$$[c(d + d^*) + \Lambda\partial_t](\boldsymbol{\xi} + \boldsymbol{\eta}) = 4\pi c\rho, \tag{122}$$

noting that

$$[c(d + d^*) + \Lambda\partial_t]^2 = \partial_t^2 - c^2\Delta. \tag{123}$$