AMCS/MATH 608
Problem set 0, September, 2014
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Reading: References for this material are *Principles of Mathematical Analysis*, by Walter Rudin, *The Way of Analysis* by Robert Strichartz, *Elementary Classical Analysis*, by Jerrold Marsden, and *Calculus on Manifolds*, by Michael Spivak.

Recall that an $\mathbb{R}^m$-valued function, f, defined in an open set $U \subset \mathbb{R}^n$ is differentiable at $x_0 \in U$, if there is a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$f(x) = f(x_0) + A(x - x_0) + o(\|x - x_0\|). \tag{1}$$

We use $Df(x_0)$ to denote A. The function is differentiable in U if it is differentiable at every $x \in U$, and continuously differentiable if the map $x \mapsto Df(x)$ is a continuous map from U to $m \times n$ matrices.

The function f is twice differentiable if the map $x \mapsto Df(x)$ is differentiable, and continuously differentiable if $x \mapsto D(Df)(x)$ is continuous as a map from U to $\mathbb{R}^{mn^2}$. You should think about this so you understand how to interpret $D(Df)(x)$ as a family of quadratic forms with values in $\mathbb{R}^m$.

These problems, which do not have to be handed in, cover material in advanced calculus and exterior calculus. You are expected to know this material and are strongly encouraged to do these problems carefully.

Standard Problem

1. Suppose that $f(x, y)$, defined in $[0, 1] \times [0, 1]$, has partial derivatives in $(0, 1) \times (0, 1)$. If $\partial_y f(x, y) = 0$ throughout $(0, 1) \times (0, 1)$, show that there is a differentiable function $g(x)$ so that $f(x, y) = g(x)$. How is $\partial_x f(x, y)$ related to $g'(x)$?

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 \quad \text{if } (x, y) = (0, 0). \end{cases} \tag{2}$$

   (a) Show that f is differentiable in a neighborhood of $(0, 0)$ with $\partial_x f(0, y) = -y$ and $\partial_y f(x, 0) = x$.

   (b) Show that $\partial_x \partial_y f(0, 0)$ and $\partial_y \partial_x f(0, 0)$ exist but are not equal.

(c) Why does this example not contradict the theorem that "mixed partials commute?"

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f is not one-to-one. Hint: If, for example, $\partial_x f(x, y)$ is not zero, then show that the map $F : (x, y) \to (f(x, y), y)$ is locally one-to-one and onto.

More challenging problems:

1. Let $\omega = f(x, y)dx + g(x, y)dy$ be a 1-form defined in $U \subset \mathbb{R}^2$ and $H(s, t) : V \to U$ a C^2-map. Show that
$$H^*(d\omega) = d(H^*\omega). \tag{3}$$
Is it necessary for H to be 1-1?

2. Let $\gamma : [0, 1] = \mathbb{R}^2$ be an embedded C^1-curve in the plane, and let $\boldsymbol{p} = \{p_0, \ldots, p_N\}$, where $p_i = \gamma(i/N)$. Finally, let $\gamma_{\boldsymbol{p}}$ be the polygonal curve obtained by joining successive points, p_i, p_{i+1}, by the segments of the straight lines they define. We orient $\gamma_{\boldsymbol{p}}$ so that the vertices come in the given order. We let
$$|\boldsymbol{p}| = \max\{|p_{i+1} - p_i| : i = 0, 1, \ldots, N-1\},$$
and ω be a C^0 1-form defined in a neighborhood of γ. Prove that, as $|\boldsymbol{p}| \to 0$,
$$\int_{\boldsymbol{p}} \omega \quad \text{converges to} \int_{\gamma} \omega. \tag{4}$$

3. Suppose that S_1 is the unit circle in the plane, centered at $(0, 0)$, which we approximate by a polygonal "stair case"-curve, γ_N obtained as follows: let $p_j = e^{\frac{2\pi i j}{N}} = (x_j, y_j)$, $j = 0, \ldots, N$. We join p_j to p_{j+1} by the arc:
$$c(t) = \begin{cases} (1-2t)(x_j, y_j) + 2t(x_j, y_{j+1}) \text{ for } t \in [0, \frac{1}{2}] \\ (2-2t)(x_j, y_{j+1}) + (2t-1)(x_{j+1}, y_{j+1}) \text{ for } t \in [\frac{1}{2}, 1]. \end{cases} \tag{5}$$
Draw this curve for a few values of N. Show that
$$\lim_{N\to\infty} \int_{\gamma_N} ds_N = 8 \text{ and } \lim_{N\to\infty} \int_{\gamma_N} x dy = \pi, \tag{6}$$
where ds_N is arclength along the curve γ_N, and xdy is the 1-form.

4. Define a 1-form in $\mathbb{R}^2 \setminus \{0\}$ by setting

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{7}$$

(a) Prove that $d\omega = 0$, but show that there is no function f defined in $\mathbb{R}^2 \setminus \{0\}$ so that $df = \omega$. Hint: what is

$$\oint_{x^2+y^2=1} \omega? \tag{8}$$

(b) How about in the upper half plane $\{(x, y) : \ y > 0\}$?

(c) Show how to use the 1-form ω in Stokes theorem to compute

$$\int_0^1 \frac{dt}{1 - 2t + 2t^2}. \tag{9}$$

(d) Now let

$$\eta = \frac{(x^2 - y^2)dx + 2xydy}{(x^2 + y^2)^2}. \tag{10}$$

Show that $d\eta = 0$, and that there is a function g defined in $\mathbb{R}^2 \setminus \{0\}$ so that $dg = \eta$.

5. Suppose that $U \subset \mathbb{R}^n$ is an open convex set, and $f : U \to \mathbb{R}^m$ is a continuously differentiable function. Show that for $x_1, x_2 \in U$,

$$f(x_1) - f(x_2) = \int_0^1 Df(tx_1 + (1 - t)x_2)(x_1 - x_2)dt. \tag{11}$$

Use this to show that in any compact subset $K \subset U$, and for any $\epsilon > 0$, there is a $\delta > 0$ so that if $x_1, x_2 \in K$ and $\|x_1 - x_2\| < \delta$, then

$$\|f(x_1) - f(x_2) - Df(x_1)(x_1 - x_2)\| \leq \epsilon \|x_1 - x_2\|. \tag{12}$$

Show that if the operator norm, $\|Df(x)\| < M$ for all $x \in U$, then

$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\|. \tag{13}$$

6. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ a continuously differentiable map, such that, at any every point $x \in U$, $\det Df(x) \neq 0$. Show that $f(U)$ is an open set.

7. Define a map from $\mathbb{R}^2$ to itself by setting

$$F(x, y) = (\sin x \cos y + \sin y \cos x, \cos x \cos y - \sin x \sin y). \tag{14}$$

   Does there exist a point (x_0, y_0) such that F is locally invertible in a neighborhood of $F(x_0, y_0)$. You must prove your answer.

8. Let $U \subset \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a twice continuously differentiable function. Suppose that at $x_0 \in U$ we know that $\nabla f(x_0) = 0$ and the Hessian $\frac{\partial^2 f}{\partial_{x_j} \partial_{x_k}}(x_0)$ is an invertible matrix. Show that for a $\delta > 0$ there is a map $G : B_\delta(0) \to U$ so that $G(0) = x_0$ and $\nabla f(G(y)) = y$. Show that there is a function $g(y)$ defined in a neighborhood of 0 so that $G(y) = \nabla g(y)$.

   Finally suppose that $U = \mathbb{R}$ and there is a constant $c > 0$ so that $f''(x) > c$ for all $x \in \mathbb{R}$. Show that G and g are defined on all of $\mathbb{R}$ as well.