Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi.

**Standard problems:** The solutions to the following problems do not need to be handed in.

1. If $f$ is a function then we define
   \[ f^+(x) = \max\{0, f(x)\} \quad \text{and} \quad f^-(x) = \min\{0, f(x)\}. \]  

   Show that if $f$ is measurable, then so are $f^+$ and $f^-$, and therefore so is $|f|$.

2. Show that $f$ is measurable if and only if the sets \( \{ x : f(x) \geq a \} \) are measurable for every $a \in \mathbb{R}$.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Prove that a $\sigma$-algebra is either finite or uncountable. Give an example of a finite $\sigma$-algebra.

2. Prove that every measurable function is the limit a.e. of a sequence of continuous functions.

3. Let $D \subseteq \mathbb{R}$ be a dense subset. Let $f$ be an extended real-valued function defined on $\mathbb{R}$. Show that if the sets \( \{ x : f(x) > a \} \) are measurable for all $a \in D$, then $f$ is measurable.

4. Let $E \subseteq \mathbb{R}^d$ be a measurable set and $f$ a function defined on $E$. We define the function
   \[ g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases} \]

   Show that $f$ is measurable if and only if $g$ is measurable.
5. Let \( \{ f_n \} \) be a sequence of measurable functions on \([0, 1]\) with \( |f_n(x)| < \infty \) for a.e. \( x \). Show that there is a sequence of \( \{ c_n \} \) of positive real numbers such that

\[
\frac{f_n(x)}{c_n} \longrightarrow 0 \text{ for a.e. } x.
\]  

(3)

Hint: Pick \( c_n \) such that \( m(\{ x : \frac{|f_n(x)|}{c_n} > 1/n \}) < 2^{-n} \) and apply the Borel-Cantelli lemma.

6. Let \( \mathcal{C} \) be the middle thirds Cantor set. Show that \( x \in \mathcal{C} \) if and only if it has a ternary expansion of the form

\[
x = \sum_{j=1}^{\infty} \frac{t_j}{3^j} \text{ where } t_j \in \{0, 2\}.
\]  

(4)

Note that the ternary expansion is not unique. Define \( F : \mathcal{C} \rightarrow [0, 1] \) by letting

\[
F(x) = \sum_{j=1}^{\infty} \frac{t_j/2}{2^j}.
\]  

(5)

Show that \( F \) is well defined and continuous on \( \mathcal{C} \), and that \( F(0) = 0 \), and \( F(1) = 1 \), then show that \( F \) is surjective. Finally show that if \( (a, b) \) is a maximal open subset in \( \mathcal{C}^c \), then \( F(a) = F(b) \), and thereby extend \( F : [0, 1] \rightarrow [0, 1] \), as a continuous map.

Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set. Hint: If \( \mathcal{N} \subset [0, 1] \) is the non-measurable subset constructed in class, then consider \( F^{-1}(\mathcal{N}) \cap \mathcal{C} \).

7. Let \( \mathcal{N} \) be the non-measurable subset constructed in class. Show that any measurable set \( E \subset \mathcal{N} \) has measure zero. Show that if \( G \) is a set with \( m_*(G) > 0 \), then \( G \) has a non-measurable subset.

8. In this problem we prove the following theorem: A bounded function \( f \) defined on an interval \( J = [a, b] \) is Riemann integrable if and only if its set of discontinuities has measure zero.

To prove this we use the following concept: For a bounded function \( f \) defined on a compact interval \( J \) and \( 0 < r \) let

\[
osc(f, c, r) = \sup\{|f(x) - f(y)| : x, y \in J \cap (c - r, c + r)\}.
\]  

(6)
This is a non-decreasing function of $r$ and therefore \( \text{osc}(f, c) = \lim_{r \to 0^+} \text{osc}(f, c, r) \) is well defined; $f$ is continuous at $c$ if and only if $\text{osc}(f, c) = 0$. To prove the statement above, prove the following assertions:

(a) For every $\epsilon > 0$ the set of points $A_\epsilon = \{x \in J : \text{osc}(f, x) \geq \epsilon \}$ is compact.

(b) If the set of discontinuities of $f$ has measure 0, then $f$ is Riemann integrable.
   Hint: Cover $A_\epsilon$ by a finite collection of open intervals of length less than $\epsilon$, then construct appropriate partitions of $J$ on which to estimate the difference between the upper and lower Riemann sums.

(c) Conversely, if $f$ is Riemann integrable on $J$, then its set of discontinuities has measure zero. Hint: The set of discontinuities of $f$ is contained in $\bigcup_n A_{\frac{\epsilon}{n}}$. Construct a partition $P$ so that

\[
U(f, P) - L(f, P) \leq \frac{\epsilon}{n}. \tag{7}
\]

Show that the total length of the intervals in $P$ that intersect $A_{\frac{\epsilon}{n}}$ is at most $\epsilon$.  