Reading: There are many excellent references for this material; several I especially like are Complex Analysis by Elias Stein and Rami Shakarchi, Complex Analysis by Lars V. Ahlfors, and Conformal Mapping by Zeev Nehari.

Standard problems: The following problems should be done, but do not have to be handed in.

1. Show that there does not exist an analytic function in \( D_1(0) \), which extends continuously to \{ \( z : |z| = 1 \) \}, so that \( f(z) = 1/z \) on the unit circle.

2. Suppose that \( f : D_1(0) \to \mathbb{C} \) is an analytic map and \( f'(0) \neq 0 \). In class we showed that there is an \( r > 0 \) and an analytic map \( g : D_r(f(0)) \to D_1(0) \), satisfying \( g(f(0)) = 0 \), \( f(g(w)) = w \), for \( w \in D_r(f(0)) \), and \( g(f(z)) = z \), for \( z \) closed enough to 0.

   (a) Show that for any \( n \in \mathbb{N} \) and \( w_0 \in \mathbb{C} \setminus \{0\} \) there is an analytic \( n \)th root function \( g_n(w) \) defined in a neighborhood \( U \) of \( w_0 \), so that, for \( w \in U \), we have \( (g_n(w))^n = w \). Explain why, if \( n > 1 \), no such function can be analytic in a neighborhood of 0.

   (b) Show that there is a neighborhood \( U \) of any point \( w_0 \in \mathbb{C} \setminus \{0\} \) in which an analytic function \( l(w) \) is defined that satisfies

   \[ e^{l(w)} = w, \]

   for \( w \in U \). This is a branch of the log. What is the real part of \( l \)? For this problem we can take \( e^{z} \) to be the entire function defined by the power series:

   \[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \]

   You need to prove that \( e^z \) satisfies the hypotheses in part (a).
Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let \{w_1, \ldots, w_m\} be points in the unit circle. Show that there exists a point, \(z\) on the unit circle where
\[
\prod_{j=1}^{m} |z - w_j| > 1.
\] (3)

Conclude that there are also points on the unit circle where
\[
\prod_{j=1}^{m} |z - w_j| = 1.
\] (4)

2. Show that
\[
4\partial_z \partial_{\bar{z}} = 4\partial_{\bar{z}} \partial_z = \Delta,
\] where \(\Delta = \partial_x^2 + \partial_y^2\) is the Laplace operator.

(a) Show that if \(f = u + iv\) is an analytic function in an open subset of \(\mathbb{C}\), then \(u\) and \(v\) are harmonic, that is
\[
\Delta u = \Delta v = 0.
\] (6)

(b) Suppose that \(U\) is a harmonic function in \(B_1(0)\). Show that there is a function \(V\) that satisfies the system of equations:
\[
V_x = -U_y \text{ and } V_y = U_x.
\] (7)

Conclude that \(U + iV\) is analytic in \(B_1(0)\). If \(U = x^5 - 10x^3y^2 + 5xy^4\) what is \(V\)?

3. Suppose that \(f\) is analytic in \(D_{R_0}(0)\). Show that whenever \(0 < R < R_0\) and \(|z| < R\), then
\[
f(z) = \frac{1}{2\pi} \int_{0}^{2\pi} f(Re^{i\theta}) \text{Re} \left( \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) d\theta.
\] (8)

Show that
\[
\text{Re} \left( \frac{Re^{i\theta} + r}{Re^{i\theta} - r} \right) = \frac{R^2 - r^2}{R^2 - 2rR \cos \theta + r^2}.
\] (9)
Finally, if \( f = u + iv \), and \( f(0) \) is real, then show that

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\theta}) \left( \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \right) d\theta.
\]  

(10)

4. Suppose that \( f \) is analytic in the set \( D_1(0)^c = \{ z : |z| > 1 \} \), continuous up to \( |z| = 1 \), and bounded as \( z \to \infty \). Find an analogue of the Cauchy integral formula expressing \( f(z) \) for \( z \in D_1(0)^c \) as a complex weighted average of the values of \( f \) on the unit circle.

Using this formula find a simple expression for \( \lim_{z \to \infty} f(z) \). How do we know that this limit exists?

Find a similar formula for \( f(z) \), \( z \in D_1(0)^c \), if \( f \) satisfies an estimate of the form \( |f(z)| \leq C|z|^n \) as \( z \to \infty \).

You must prove that your formulæ are correct.

5. Let \( f \) be a non-constant analytic function defined in a neighborhood of the closed unit disk. Suppose that \( |f(z)| = 1 \) where \( |z| = 1 \). Show that \( \theta \to f(e^{i\theta}) \) goes counterclockwise around the unit disk, and makes at least one full rotation. Hint: In a neighborhood of any point on the unit disk \( \log f(z) = \log |f(z)| + i \arg f(z) \) is an analytic function. Use the maximum principle.

6. Suppose that \( f \) is an analytic function defined in \( D_1(0) \). We say that \( e^{i\theta} \) is a regular boundary point for \( f \) if there is \( \rho > 0 \), so that \( f \) has an analytic extension to the set \( D_1(0) \cup D_\rho(e^{i\theta}) \). The function

\[
f(z) = \sum_{j=0}^{\infty} z^{2^j},
\]

(11)

is clearly analytic in \( D_1(0) \). Prove that there are no regular boundary points. Hint: consider points of the form \( re^{i\theta} \) where \( \theta = \frac{2\pi p}{2^k} \), with \( p \) and \( k \) positive integers.