**Reading:** There are many excellent references for this material; several I especially like are *Complex Analysis* by Elias Stein and Rami Shakarchi, *Complex Analysis* by Lars V. Ahlfors, and *Conformal Mapping* by Zeev Nehari.

**Homework assignment:** The solutions to the following problems should be carefully written up and handed in.

1. Suppose that $D$ is a bounded connected region in the plane, with a $C^1$ boundary, and $u, v$ are twice continuously differentiable functions in $D$, whose first derivatives have continuous extensions to $\overline{D}$. We let $n$ denote the outer unit normal vector along $bD$, and $t = in$, the oriented unit tangent vector. The normal and tangential derivatives of $u$ along the boundary are defined by:

$$\frac{\partial u}{\partial n} = \langle \nabla u, n \rangle \quad \frac{\partial u}{\partial s} = \langle \nabla u, t \rangle. \tag{1}$$

(a) Show that *Stokes’ Theorem* (for 1-forms) implies that

$$\int_D [u_x v_x + u_y v_y] dx dy + \int_D u \Delta v dx dy = \int_{bD} u \frac{\partial v}{\partial n} ds. \tag{2}$$

Here $ds$ denotes arclength measure along $bD$.

(b) Use this formula to deduce that if $u$ is also harmonic in $D$, then

$$\int_{bD} \frac{\partial u}{\partial n} ds = 0. \tag{3}$$

(c) Show that equation (2) implies that

$$\int_D [u \Delta v - v \Delta u] dx dy = \int_{bD} [u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}] ds. \tag{4}$$
2. Suppose that \( h \) is a continuous function on \( \mathbb{R} \) with support in the finite interval \([-1, 1]\), and define
\[
g(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{h(x)dx}{x - z}\]

(a) Prove that \( g \) is an analytic function in \( \mathbb{C} \setminus [-1, 1] \), which tends to zero as \( |z| \) tends to infinity.

(b) Let \( D_1^+(0) (D_1^-(0)) \) denote the upper (resp. lower) half of the unit disk. If \( h \) is the restriction of an analytic function defined in \( D_1(0) \), then show that \( g \) has analytic continuations, across the interval \((-1, 1)\), from \( D_1^+(0) \) to \( D_1(0) \) and from \( D_1^-(0) \) to \( D_1(0) \). Hint: Deform the contour of integration \([-1, 1]\) to a curve lying below (resp. above) \((-1, 1)\).

(c) Does the continuation of \( g \) from \( D_1^+(0) \) to \( D_1^-(0) \), defined in (b), ever agree with \( g \mid_{D_1^-(0)} \), as defined by (5)?

3. In class we proved that for any \( \varphi \in \mathcal{C}_c^1(\mathbb{R}^2) \) the equation
\[
\bar{\partial} u = \varphi
\]
has a solution given by
\[
u_0(z, \bar{z}) = \frac{1}{\pi} \iint \varphi(w, \bar{w})dw \, d\bar{w}.\]

(a) \( u_0 \) is one solution; what are all the other solutions?

(b) Show that this solution satisfies \( \lim_{z \to \infty} u_0(z, \bar{z}) = 0 \), and it is uniquely determined by this condition.

(c) What conditions must \( \varphi \) satisfy for there to exist a solution with compact support? Hint: The solution \( u_0 \) is analytic outside the support of \( \varphi \). Find a representation, for \( z \) with large modulus, that reflects this fact. Note that \( \varphi \) must satisfy infinitely many conditions.

4. Using (4) show that if \( u \) is a \( C^2 \)-function with compact support then
\[
\frac{1}{4\pi} \int_{\mathbb{C}} \log(x^2 + y^2) \Delta u(x, y) \, dx \, dy = u(0).\]
If $u$ is a compactly supported, $C^2$-function, then show that the function

$$U(x, y) = \frac{1}{4\pi} \int_C \log(x'^2 + y'^2)u(x - x', y - y')dx'dy'$$

is a twice differentiable function satisfying

$$\Delta U = u. \quad (10)$$

Hint: Be careful because $\log(x^2 + y^2)$ is singular at $(0, 0)$.

Under what condition is

$$\lim_{(x,y)\to\infty} U(x, y) = 0? \quad (11)$$

5. Suppose that $u$ is a harmonic function defined in a simply connected domain, $D$ with $C^1$-boundary, and let $v$ denote a harmonic conjugate to $u$. Suppose that the first derivatives of $u$ and $v$ extend continuously to the $bD$. For a differentiable function $f$ defined along $bD$ let $\frac{\partial f}{\partial s}$ denote the derivative of $f$ with respect to arclength along $bD$. If $t$ is the unit tangent vector to $bD$, oriented in the positive direction, then the tangential derivative is:

$$\frac{\partial (v |_{bD})}{\partial s} = \langle \nabla v, t \rangle |_{bD}. \quad (12)$$

(a) Show that along $bD$ we have the relation:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}. \quad (13)$$

(b) Let $g(s)$ be a continuous function defined along $bD_1(0)$ with $s$ the arclength parameter and

$$\int_{bD_1(0)} g(s)ds = 0. \quad (14)$$

Explain how to use (13) to prove that there is a harmonic function $u$ defined in $D_1(0)$ such that

$$\frac{\partial u}{\partial n}(s) = g(s). \quad (15)$$

Hint: The function $G(s) = \int_{s_0}^{s} g(\sigma)d\sigma$ is a continuous function on $bD_1$. 

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6. Let $g(x, y) = \frac{1}{4\pi} \log(x^2 + y^2)$. Let $\gamma : [0, L] \to \mathbb{C}$ give an arclength parametrization of, $\Gamma$, a simple closed $C^1$-curve. Let $D$ be the domain bounded by $\Gamma$. For $\sigma$, a continuous function defined on $\Gamma$, and $ds$ the arclength measure along $\Gamma$ we define the function

$$u(x, y) = \int_0^L g(x - \gamma_1(s), y - \gamma_2(s))\sigma(s)ds, \quad (x, y) \in \mathbb{C}. \quad (16)$$

(a) Show that $u$ is a continuous function in $\mathbb{C}$, which is harmonic in $\mathbb{C} \setminus \Gamma$.

(b) Let $n$ denote the outward unit normal vector field along $\Gamma$. The normal lines to $\Gamma$ foliate a neighborhood of $\Gamma$, see Figure 1. We can therefore extend $n$ to a neighborhood of $\Gamma$ as the unit tangent directions to these lines. For $p \in \Gamma$, let $\partial_n^+ u(p)$ denote the limit of $\partial_n u(q)$ from the inside of $D$, as $q \to p$, and $\partial_n^- u(p)$ the analogous limit from $D^c$. What is

$$\partial_n^+ u(p) - \partial_n^- u(p)? \quad (17)$$

Figure 1. Figure showing contour $\Gamma$ with several normal lines, and outward normal vectors.