AMCS/MATH 609
Problem set 3 due February 17, 2015
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi. **Standard problems:** The solutions to the following problems do not need to be handed in.

1. Let X be a set and $\mathcal{M}$ a collection of subsets of X. Show that if $X \in \mathcal{M}$ and $\mathcal{M}$ is closed under complements and countable *disjoint* unions, then $\mathcal{M}$ is a σ-algebra.

2. Let $(X\mathcal{M}, \mu)$ be a measure space, and $E, F \in \mathcal{M}$, then
$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F). \tag{1}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Show that f is Borel measurable.

4. Prove that the Borel-Cantelli Lemma holds in any measure space $(X, \mathcal{M}, d\mu)$: If $\{E_n\}$ is a collection of measurable sets with
$$\sum_{n=1}^{\infty} \mu(E_n) < \infty, \tag{2}$$
then the set consisting of points belonging to infinitely many of these subsets has measure zero.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Show that a σ algebra is either finite or uncountable.

2. Last semester we defined the Lebesgue outer measure m_* on all subsets of $\mathbb{R}^d$. A set E is defined to be Lebesgue measurable if for any $\epsilon > 0$ there is an open set $E \subset U$, such that $m_*(U \setminus E) < \epsilon$. A set E is Caratheodory measurable provided for *any* subset A of $\mathbb{R}^d$ we have
$$m_*(A) = m_*(E \cap A) + m_*(E^c \cap A). \tag{3}$$

Prove that a set is Lebesgue measurable if and only if it is Caratheodory measurable. Hint: If E is Lebesgue measurable and A is any subset, choose a G_δ-set G so that $G \supset A$ and $m(G) = m_*(A)$. Conversely, if E is Caratheodory measurable and $m_*(E) < \infty$, then show that we can choose a G_δ-set G so that $G \supset E$ and $m_*(G) = m_*(E)$.

3. Let $\mathscr{C}^0([a,b])$ be the space of continuous functions on the closed, bounded interval $[a,b]$, and let $d\mu$ be a finite Borel measure on $[a,b]$. We define a linear functional by setting

$$\ell(f) = \int_{(a,b]} f(x) d\mu(x). \tag{4}$$

Show that this linear functional is positive in the sense that $\ell(f) \geq 0$ if $f \geq 0$, and therefore there is a constant C, so that, for all $f \in \mathscr{C}^0([a,b])$ we have the estimate $|\ell(f)| \leq C\|f\|_{L^\infty}$.

Conversely prove that for any linear functional ℓ on $\mathscr{C}^0([a,b])$, which is positive in this sense, there is a unique finite Borel measure $d\mu$ so that this formula holds. Hint: Let $x \in [a,b]$. Define

$$F(x) = \lim_{\epsilon \to 0^+} \ell(f_\epsilon), \tag{5}$$

where the function

$$f_\epsilon(y) = \begin{cases} 1 \text{ for } y \in [a,x] \\ \frac{x+\epsilon-y}{\epsilon} \text{ for } y \in [x, x+\epsilon] \\ 0 \text{ for } y \in [x+\epsilon, b]. \end{cases} \tag{6}$$

Show that F is increasing, right continuous, and $\ell(f) = \int_{(a,b]} f(x) dF(x)$.

4. Let F be a monotone increasing, right-continuous function on $\mathbb{R}$ and μ_F the measure it defines. We let

$$F(x^-) = \lim_{y \to x^-} F(y). \tag{7}$$

Recall that, by definition, $\mu_F((a,b]) = F(b) - F(a)$; prove the following formulæ:

(a) $\mu_F(\{a\}) = F(a) - F(a^-)$.

(b) $\mu_F([a,b)) = F(b^-) - F(a^-)$.

(c) $\mu_F([a,b]) = F(b) - F(a^-)$.

(d) $\mu_F((a,b)) = F(b^-) - F(a)$.

5. Let g be a function in $\mathscr{C}^1([a,b])$, and F a monotone increasing, right continuous function on $\mathbb{R}$. Let $a < a_1 < \cdots < a_N = b$, with $\{a_1, \ldots, a_{N-1}\}$ points of continuity of F. This is a partition, P, of $[a,b]$ with $|P| = \max\{a_{j+1} - a_j, a_1 - a\}$. Prove that as $|P| \to 0^+$, we have

$$\sum_{j=1}^{N-1} F(a_j) \frac{g(a_{j+1}) - g(a_j)}{a_{j+1} - a_j}(a_{j+1} - a_j) \longrightarrow \int_a^b g'(x)F(x)dx. \tag{8}$$

6. Suppose that $\{f_n\} \subset L^2(X; d\mu)$ with $\|f_n\|_{L^2} \leq 1$. Prove that $f_n(x)/n \to 0$ for a.e. x.

7. Let $(X, \mathscr{M}, d\mu)$ be a measure space and let w be a non-negative measurable function on X. Define a new measure by setting

$$\nu(E) = \int_E w(x) d\mu(X) \text{ for } E \in \mathscr{M}. \tag{9}$$

If f is a non-negative measurable function on X, then show that

$$\int_X f(x) d\nu(x) = \int_X f(x) w(x) d\mu(x) \tag{10}$$

8. Let μ be a Borel measure on $[0,1]$, for $0 < x \leq 1$, set $\mu((0,x]) = F(x)$, and let $F(0) = 0$. Suppose that E is a Borel set contained in $[0, F(1)]$. Show that

$$m(E) = \mu(F^{-1}(E)), \tag{11}$$

where m is Lebesgue measure on $\mathbb{R}$. Hint: Do this first assuming that F is continuous.

9. Let f be a non-negative integrable function on the measure space $(X, \mathscr{M}, d\mu)$. For $\epsilon > 0$ show that there is a set $E \in \mathscr{M}$ such that $\mu(E) < \infty$, and

$$\int_E f(x) d\mu(x) > \int_X f(x) d\mu(x) - \epsilon. \tag{12}$$

Even if $(X, \mathscr{M}, d\mu)$ is not σ-finite, show that if $f \in L^1(X; d\mu)$, then the supp f is a σ-finite set, that is

$$\operatorname{supp} f = \bigcup_{j=1}^{\infty} E_j \text{ where } \{E_j\} \subset \mathscr{M} \text{ and } \mu(E_j) < \infty. \tag{13}$$

10. A real valued function φ defined on $\mathbb{R}$ is convex if

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) \text{ for every } \lambda \in [0,1]. \tag{14}$$

Let $(X, \mathcal{M}, d\mu)$ be a measure space with $\mu(X) = 1$. Let f be an integrable function. Show that

$$\varphi\left(\int_X f(x)d\mu(x)\right) \leq \int_X \varphi \circ f(x)d\mu(x). \tag{15}$$

Show that the function

$$m(p) = \left[\int_X |f(x)|^p d\mu(x)\right]^{\frac{1}{p}} \tag{16}$$

is monotonically increasing. Suppose $m(p) < \infty$ for $1 \leq p$; what is $\lim_{p\to\infty} m(p)$?