Reading: There are many excellent references for this material; I especially like Real Analysis by Elias Stein and Rami Shakarchi. Standard problems: The solutions to the following problems do not need to be handed in.

1. Let $X$ be a set and $\mathcal{M}$ a collection of subsets of $X$. Show that if $X \in \mathcal{M}$ and $\mathcal{M}$ is closed under complements and countable disjoint unions, then $\mathcal{M}$ is a $\sigma$-algebra.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $E, F \in \mathcal{M}$, then

$$
\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F). \quad (1)
$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Show that $f$ is Borel measurable.

4. Prove that the Borel-Cantelli Lemma holds in any measure space $(X, \mathcal{M}, d\mu)$: If $\{E_n\}$ is a collection of measurable sets with

$$
\sum_{n=1}^{\infty} \mu(E_n) < \infty, \quad (2)
$$

then the set consisting of points belonging to infinitely many of these subsets has measure zero.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Show that a $\sigma$ algebra is either finite or uncountable.

2. Last semester we defined the Lebesgue outer measure $m_*$ on all subsets of $\mathbb{R}^d$. A set $E$ is defined to be Lebesgue measurable if for any $\epsilon > 0$ there is an open set $E \subset U$, such that $m_*(U \setminus E) < \epsilon$. A set $E$ is Caratheodory measurable provided for any subset $A$ of $\mathbb{R}^d$ we have

$$
m_*(A) = m_*(E \cap A) + m_*(E^c \cap A). \quad (3)
$$

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Prove that a set is Lebesgue measurable if and only if it is Carathéodory measurable. Hint: If \( E \) is Lebesgue measurable and \( A \) is any subset, choose a \( G_\delta \)-set \( G \) so that \( G \supset A \) and \( m(G) = m_*(A) \). Conversely, if \( E \) is Carathéodory measurable and \( m_*(E) < \infty \), then show that we can choose a \( G_\delta \)-set \( G \) so that \( G \supset E \) and \( m_*(G) = m_*(E) \).

3. Let \( \mathcal{C}^0([a, b]) \) be the space of continuous functions on the closed, bounded interval \([a, b]\), and let \( d\mu \) be a finite Borel measure on \([a, b]\). We define a linear functional by setting

\[
\ell(f) = \int_{[a, b]} f(x) d\mu(x). \tag{4}
\]

Show that this linear functional is positive in the sense that \( \ell(f) \geq 0 \) if \( f \geq 0 \), and therefore there is a constant \( C \) so that, for all \( f \in \mathcal{C}^0([a, b]) \) we have the estimate \( |\ell(f)| \leq C \|f\|_{L^\infty} \).

Conversely prove that for any linear functional \( \ell \) on \( \mathcal{C}^0([a, b]) \), which is positive in this sense, there is a unique finite Borel measure \( d\mu \) so that this formula holds. Hint: Let \( x \in [a, b] \). Define

\[
F(x) = \lim_{\epsilon \to 0^+} \ell(f_\epsilon), \tag{5}
\]

where the function

\[
f_\epsilon(y) = \begin{cases} 
1 & \text{for } y \in [a, x] \\
\frac{x+\epsilon-y}{\epsilon} & \text{for } y \in [x, x+\epsilon] \\
0 & \text{for } y \in [x+\epsilon, b].
\end{cases} \tag{6}
\]

Show that \( F \) is increasing, right continuous, and \( \ell(f) = \int_{[a,b]} f(x)dF(x) \).

4. Let \( F \) be a monotone increasing, right-continuous function on \( \mathbb{R} \) and \( \mu_F \) the measure it defines. We let

\[
F(x^-) = \lim_{y \to x^-} F(y). \tag{7}
\]

Recall that, by definition, \( \mu_F([a, b]) = F(b) - F(a) \); prove the following formulæ:

(a) \( \mu_F([a]) = F(a) - F(a^-) \).
(b) \( \mu_F([a, b)) = F(b^-) - F(a^-) \).
(c) \( \mu_F([a, b]) = F(b) - F(a^-) \).
5. Let \( g \) be a function in \( \mathcal{C}^1([a, b]) \), and \( F \) a monotone increasing, right continuous function on \( \mathbb{R} \). Let \( a < a_1 < \cdots < a_N = b \), with \( \{a_1, \ldots, a_{N-1}\} \) points of continuity of \( F \). This is a partition, \( P \), of \([a, b]\) with \( |P| = \max\{a_{j+1} - a_j, a_1 - a\} \). Prove that as \( |P| \to 0^+ \), we have

\[
\sum_{j=1}^{N-1} F(a_j) \frac{g(a_{j+1}) - g(a_j)}{a_{j+1} - a_j} (a_{j+1} - a_j) \to \int_a^b g'(x) F(x) \, dx. \tag{8}
\]

6. Suppose that \( \{f_n\} \subset L^2(X; d\mu) \) with \( \|f_n\|_{L^2} \leq 1 \). Prove that \( f_n(x)/n \to 0 \) for a.e. \( x \).

7. Let \((X, \mathcal{M}, d\mu)\) be a measure space and let \( w \) be a non-negative measurable function on \( X \). Define a new measure by setting

\[
\nu(E) = \int_E w(x) \, d\mu(X) \quad \text{for} \quad E \in \mathcal{M}. \tag{9}
\]

If \( f \) is a non-negative measurable function on \( X \), then show that

\[
\int_X f(x) \, dv(x) = \int_X f(x) w(x) \, d\mu(x). \tag{10}
\]

8. Let \( \mu \) be a Borel measure on \([0, 1] \), for \( 0 < x \leq 1 \), set \( \mu((0, x]) = F(x) \), and let \( F(0) = 0 \). Suppose that \( E \) is a Borel set contained in \([0, F(1)] \). Show that

\[
m(E) = \mu(F^{-1}(E)), \tag{11}
\]

where \( m \) is Lebesgue measure on \( \mathbb{R} \). Hint: Do this first assuming that \( F \) is continuous.

9. Let \( f \) be a non-negative integrable function on the measure space \((X, \mathcal{M}, d\mu)\). For \( \epsilon > 0 \) show that there is a set \( E \in \mathcal{M} \) such that \( \mu(E) < \infty \), and

\[
\int_E f(x) \, d\mu(x) > \int_X f(x) \, d\mu(x) - \epsilon. \tag{12}
\]

Even if \((X, \mathcal{M}, d\mu)\) is not \( \sigma \)-finite, show that if \( f \in L^1(X; d\mu) \), then the supp \( f \) is a \( \sigma \)-finite set, that is

\[
\text{supp } f = \bigcup_{j=1}^{\infty} E_j \quad \text{where} \quad \{E_j\} \subset \mathcal{M} \quad \text{and} \quad \mu(E_j) < \infty. \tag{13}
\]
10. A real valued function $\varphi$ defined on $\mathbb{R}$ is convex if

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y)$$

for every $\lambda \in [0, 1]$. \hfill (14)

Let $(X, \mathcal{M}, d\mu)$ be a measure space with $\mu(X) = 1$. Let $f$ be an integrable function. Show that

$$\varphi \left( \int_X f(x)d\mu(x) \right) \leq \int_X \varphi \circ f(x)d\mu(x).$$

(15)

Show that the function

$$m(p) = \left[ \int_X |f(x)|^p d\mu(x) \right]^{\frac{1}{p}}$$

is monotonically increasing. Suppose $m(p) < \infty$ for $1 \leq p$; what is $\lim_{p \to \infty} m(p)$?