Reading: There are many excellent references for this material; I especially like Real Analysis by Elias Stein and Rami Shakarchi. Standard problems: The solutions to the following problems do not need to be handed in.

1. Suppose that \( L : \mathbb{R}^d \to \mathbb{R}^d \) is a linear transformation. Show that if \( E \) is a Lebesgue measurable set then so is \( L(E) \). Hint: Show that \( L \) maps sets of measure zero to sets of measure zero. Prove that
\[
m(L(E)) = |\det(L)| m(E). \tag{1}
\]

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Suppose that \( \nu, \nu_1, \nu_2 \) are signed measures on \((X, \mathcal{M})\) and \( \mu \) is a positive measure. Prove the following assertions:
   
   (a) If \( \nu_1 \perp \mu \) and \( \nu_2 \perp \mu \), then \( (\nu_1 + \nu_2) \perp \mu \).
   
   (b) If \( \nu_1 \ll \mu \) and \( \nu_2 \ll \mu \), then \( (\nu_1 + \nu_2) \ll \mu \).
   
   (c) If \( \nu_1 \perp \nu_2 \) implies that \( |\nu_1| \perp |\nu_2| \).
   
   (d) \( \nu \ll |\nu| \).
   
   (e) If \( \nu \perp \mu \) and \( \nu \ll \mu \), then \( \nu = 0 \).

2. If \( \nu \ll \mu \), with \( \mu \) a positive, \( \sigma \)-finite measure, then we let \( \frac{d\nu}{d\mu} \) denote the Radon-Nikodym derivative, so that
\[
\int_E d\nu = \int_E \left[ \frac{d\nu}{d\mu} \right] d\mu. \tag{2}
\]

   (a) If \( \nu \ll \mu \) and \( f \) is a non-negative measurable function, then
\[
\int_X f(x) d\nu(x) = \int_X f(x) \left[ \frac{d\nu}{d\mu} \right] (x) d\mu(x). \tag{3}
\]
(b) If \( \nu_1 \ll \mu \) and \( \nu_2 \ll \mu \), then
\[
\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}
\]  
(4)

(c) If \( \lambda \ll \nu \ll \mu \), with \( \nu \) and \( \mu \) positive measures, then
\[
\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}.
\]  
(5)

(d) If \( \nu \ll \mu \), and \( \mu \ll \nu \), with both measures positive, then
\[
\frac{d\nu}{d\mu} = \left[ \frac{d\mu}{d\nu} \right]^{-1}.
\]  
(6)

3. In this problem we give an example to show that the \( \sigma \)-finiteness of \( \mu \) cannot be omitted from the hypotheses of the Radon-Nikodym theorem. Let \( X = [0, 1] \) and \( \mathcal{M} \) be the class of Lebesgue measurable subsets of \( [0, 1] \). Let \( \nu \) be Lebesgue measure restricted to \( X \) and \( \mu \) be the counting measure on subsets of \( X \). Clearly \( \nu \ll \mu \), but show that there is no measurable function \( f \) such that
\[
\nu(E) = \int_E f(x) d\mu(x).
\]  
(7)

4. Suppose that \( \mu_1, \nu_1 \) are \( \sigma \)-finite measures on \( (X_1, \mathcal{M}_1) \) and \( \mu_2, \nu_2 \) are \( \sigma \)-finite measures on \( (X_2, \mathcal{M}_2) \), with \( \mu_1 \) and \( \mu_2 \) positive measures. Show that if \( \nu_1 \ll \mu_1 \) and \( \nu_2 \ll \mu_2 \), then \( \nu_1 \times \nu_2 \ll \mu_1 \times \mu_2 \) and
\[
\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{d\nu_2}{d\mu_2}(x_2).
\]  
(8)

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be a monotone increasing, continuously differentiable function. Show that \( f \) maps Borel measureable sets to Borel measurable sets. Define a Borel measure by setting
\[
\mu(E) = m(f(E)),
\]  
(9)

where \( m \) is Lebesgue measure. Show that \( \mu \ll m \), and compute the Radon-Nikodym derivative \( \frac{d\mu}{dm} \).

6. Let \( (X, \mathcal{M}, \mu) \) be a \( \sigma \)-finite measure space, and \( \mathcal{N} \) a sub-\( \sigma \)-algebra of \( \mathcal{M} \), also \( \sigma \)-finite. We let \( \nu = \mu \restriction \mathcal{N} \).
(a) Show that for any \( f \in L^1(X; d\mu) \) there is a function \( g \in L^1(X; d\nu) \) (which is therefore \( \mathcal{N} \)-measurable) so that for any set \( E \in \mathcal{N} \), we have

\[
\int_E f(x) d\mu(x) = \int_E g(x) d\nu(x). \tag{10}
\]

The point here is that \( g \) is measurable with respect to \( \mathcal{N} \), while in general \( f \) is not. Show that \( g \) is unique modulo sets of \( \nu \) measure zero.

(b) Suppose that \( \mathcal{M} \) is the Lebesgue measurable subsets of \( \mathbb{R} \) and \( \mathcal{N} \) is the \( \sigma \)-algebra generated by the sets \( \{(n, n+1) : n \in \mathbb{Z}\} \). Give a formula for \( g \) in terms of \( f \).