Reading: Read Chapters 8.1-3, 9.1, and 10.1-3 in Lax, *Functional Analysis*.

Standard problems: The following problems should be done, but do not have to be handed in:

1. Let $Y \subset X$, a normed linear space. Show that $Y^\perp$ is a closed subspace of $X'$.

2. Prove that $L^2([0,1])$, the closure of $C^0([0,1])$ with respect to
\[
\|f\|_2^2 = \int_0^1 |f(x)|^2 \, dx,
\]
(1)
is a separable space.

3. Suppose that $(M, \Sigma, d\mu)$ is a measure space, with $\mu(M) = 1$. Show that
\[
\|f\|_p = \left[ \int_M |f(m)|^p d\mu(m) \right]^{\frac{1}{p}}
\]
(2)
is an increasing function of $p$.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. For $f \in \mathcal{C}_0^\infty(\mathbb{R})$, show that $|f|$ has a weak derivative, which can be represented by a function $g(x)$ that satisfies
\[
|g(x)| \leq |\partial_x f(x)|.
\]
(3)
The statement that the weak derivative is *represented by* $g$ means that for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ we have
\[
\int_{\mathbb{R}} |f(x)| \partial_x \varphi(x) dx = - \int_{\mathbb{R}} g(x) \varphi(x) dx.
\]
(4)
Hint: Consider $\sqrt{f^2(x) + \epsilon^2}$. Compute the weak derivative of $|x|$.
2. Suppose that we define a weak solution of the wave equation,
\[ \partial^2_x u(x, t) - \partial^2_t u(x, t) = 0, \]  
(5)
to be a function that is square integrable in \([-R, R] \times [-R, R]\) for any \(R\), and such that
\[ \int_{\mathbb{R}^2} u(x, t) (\partial^2_x \varphi(x, t) - \partial^2_t \varphi(x, t)) dx dt = 0, \]  
(6)
for any function \(\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)\). Show that if \(f \in L^2(\mathbb{R})\), then
\[ u(x, t) = f(x - t) \quad \text{and} \quad v(x, t) = f(x + t) \]  
(7)
are weak solutions of the wave equation. Hint: Approximate!

3. The space \(H_1(D_1)\) is the closure of \(\mathcal{C}_0^\infty(\overline{D_1})\) with respect to the norm
\[ \|u\|_1^2 = \int_{D_1} [|u(x)|^2 + |\nabla u(x)|^2] dx. \]  
(8)
In class we proved that the map \(R : u \mapsto u|_{\partial D_1}\) has a continuous extension as a map \(R : H_1(D_1) \to L^2(\partial D_1)\). Prove that there are functions \(f \in L^2(\mathbb{R})\) for which there does not exist a function \(u \in H_1(D_1)\) for which \(f = R(u)\). Hint: The argument given in class actually showed that \(R(u)\) belongs to a subspace of \(L^2(\partial D_1)\).

4. Let \(Y \subset X\) be a closed subspace of a normed linear space. Show that
\[ Y^\perp = \{ \ell \in X' | \ell(y) = y \text{ for all } y \in Y \}. \]  
(9)
is isometrically isomorphic to \((X/Y)'\). Show that \(Y'\) is isometrically isomorphic to \(X'/Y^\perp\).

5. Suppose that \(\ell\) is a bounded linear functional on a Hilbert space, and \(\{e_j\}\) is a collection of orthonormal vectors. Show that
\[ \lim_{j \to \infty} \ell(e_j) = 0. \]  
(10)