AMCS 610
Problem set 9 due April 14, 2015
Dr. Epstein

Reading: Read Chapters 10 and 11 in Lax, Functional Analysis. You might also want to look at the sections in Royden, Real Analysis, or Rudin, Real and Complex Analysis on the Baire Category Theorem and the Uniform Boundedness Principle.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Let $(X, d)$ be a metric space and $U, V$ open dense subsets of $X$. Show that $U \cap V$ is also dense.

2. A subset $C$ of a Banach space is called weakly sequentially compact if any sequence of points in $C$ has a subsequence that converges weakly to a point in $C$. Show that that a weakly sequentially compact subset is bounded.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Let $X$ be a complete, countable metric space. Show that $X$ has a discrete subset $Y$ so that $\overline{Y} = X$. A subset $Y$ is discrete if for each $y \in Y$ the set $\{y\}$ is open as a subset of $X$.

2. Let $\{q_n\}$ be an enumeration of the $\mathbb{Q} \cap [0, 1]$. For each $m$ we let
   \[ U_m = \bigcup_{n=1}^{\infty} (q_n - \frac{1}{m2^n}, q_n + \frac{1}{m2^n}) \cap [0, 1]. \]  
   Prove that
   \[ \bigcap_{m=1}^{\infty} U_m \neq \mathbb{Q} \cap [0, 1]. \]  

3. Prove: If $< x_n >$ is a sequence in $\ell_1$ that converges weakly to 0, then
   \[ \lim_{n \to \infty} \|x_n\|_1 = 0, \]  
   that is: $< x_n >$ also converges strongly to zero. Hints: Argue by contradiction, choose an appropriate subsequence, and use the fact that $\ell'_1 = \ell_\infty$ is a very big vector space.
4. Suppose that \(< b_j >\) is a sequence of real numbers so that, for every real sequence \(< a_j >\), converging to zero, the limit
\[
\ell(a) = \lim_{N \to \infty} \sum_{j=1}^{N} a_j b_j
\]
exists. Prove that
\[
\sum_{j=1}^{\infty} |b_j| < \infty. \tag{5}
\]

5. Let \((a_{ij})\) be an infinite matrix with complex entries, \(1 \leq i, j < \infty\). Suppose that for every convergent sequence \(< s_j >\), and \(1 \leq i\), we define
\[
\sigma_i = \lim_{N \to \infty} \sum_{j=1}^{N} a_{ij} s_j, \tag{6}
\]
if the limit exists.
Show that these limits exist, for all convergent sequences \(< s_j >\), and define a sequence \(< \sigma_i >\), with the same limit, if and only if the following conditions hold:

(a) \[
\lim_{i \to \infty} a_{ij} = 0 \text{ for each } j.
\]
(b) \[
\sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty.
\]
(c) \[
\lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij} = 1.
\]
Give an example of such a matrix for which there exists a non-convergent sequence, \(< s_j >\), so that \(\sigma_j\) exists for every \(j \in \mathbb{N}\), and the sequence \(< \sigma_j >\) is convergent.

6. Let \(\{f_n\}\) be a sequence of continuous, real valued functions defined on \([0, 1]\), such that \(f(x) = \lim_{n \to \infty} f_n(x)\) exists for every \(x \in [0, 1]\).
(a) Prove that there is a non-empty open set $V \subset [0, 1]$, and a number $M$ such that
\[ |f_n(x)| < M \text{ for all } x \in V. \] (7)

(b) If $\epsilon > 0$, show that there is a nonempty open set, $V$ and an integer $N$ so that if $n \geq N$, then
\[ |f(x) - f_n(x)| < \epsilon \text{ for all } x \in V. \] (8)

Hint: For each $N$ define $A_N = \{x : |f_n(x) - f_m(x)| \leq \epsilon \text{ if } N \leq n, m\}$, and consider $\cup_N A_N$.

7. If $1 \leq p < q < \infty$, then $\ell_p \subset \ell_q$. For fixed $p < q$, and $n \in \mathbb{N}$, show that the set
\[ B_n = \{(x_j) \in \ell_q : \sum_{j=1}^{\infty} |x_j|^p \leq n\} \] (9)

is closed and nowhere dense, as a subset of $\ell_q$. Hence, as a subset of $\ell_q$, $\ell_p$ is a set of first category.