Reading: Read Chapters 1, 2, and 3 in Lax, Functional Analysis.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Suppose that $K, L \subset X$, a real vector space, are convex sets. Prove that $K + L$ is also convex.

2. Let $X, Y$ be real vector spaces and $M : X \to Y$ a linear map. Prove that if $K \subset X$ is convex, then $M(K)$ is convex, and if $L \subset Y$ is convex, then $M^{-1}(L)$ is convex.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. A linear function from a real vector space $X$ to $\mathbb{R}$ is just a linear map $\ell : X \to \mathbb{R}$. Show that a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ is continuous with respect to the topology defined by any norm on $\mathbb{R}^n$.

2. Let $X$ be a finite dimensional vector space, and $Y \subset X$ a proper subspace. Let $\{y_1, \ldots, y_k\}$ be a basis for $Y$. If $\dim X = n$, then show that there are vectors $\{x_1, \ldots, x_{n-k}\}$ so that $\{y_1, \ldots, y_k, x_1, \ldots, x_{n-k}\}$ is a basis for $X$. Conclude that

$$\dim X = \dim Y + \dim(X/Y).$$

(1)

3. Suppose that $X$ is a finite dimensional real vector space.

   (a) Show that the set, $X'$, of linear functions on $X$, with its natural vector space structure, has the same dimension as $X$. If $Y \subset X$ is a subspace, then the $\dim(X/Y)$ is called the codimension of $Y$, and

$$Y^\perp = \{\ell \in X' : \ell(y) = 0 \text{ for all } y \in Y\}.$$  

(2)

(b) Show that $Y^\perp$ is a subspace of $X'$ and $\dim(X/Y) = \dim Y^\perp$. 

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(c) Let \( d \in \mathbb{N} \), and \( \mathcal{P}_d \) denote polynomials with real coefficients of order at most \( d \). Show that the functionals

\[ \ell_j(p) = \partial^j_x p(0) \text{ for } j = 0, \ldots, d \]

are a basis for \( \mathcal{P}_d' \). For \( 0 \leq d' < d \) use this basis to describe \( \mathcal{P}_{d'} \).

4. Let \( X \) be a finite dimensional vector space over \( \mathbb{C} \), and let \( X_{\mathbb{R}} \) denote the vector space \( X \), but with the scalar multiplication restricted to the real numbers. Prove that \( \dim_{\mathbb{R}} X_{\mathbb{R}} = 2 \dim_{\mathbb{C}} X \). Show that \( z \mapsto \bar{z} \) is a linear map from \( \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}} \), but not from \( \mathbb{C} \to \mathbb{C} \).

5. Suppose that \( K \subset \mathbb{R}^2 \) is a convex set. A point \( x \) lies on the boundary of \( K \), \( bK \), if, for any \( \epsilon > 0 \), \( B_{\epsilon}(x) \cap K \neq \emptyset \), and \( B_{\epsilon}(x) \cap K^c \neq \emptyset \). Show that if \( x \in bK \), then there is a linear function \( \ell_x : \mathbb{R}^2 \to \mathbb{R} \) so that

\[ \ell_x(x) \geq \ell_x(y) \text{ for all } y \in K \setminus \{x\}. \]  

When does the strict inequality hold for all \( y \in K \setminus \{x\} \)? The set \( \{y : \ell_x(y) = \ell_x(x)\} \) is called a supporting line. Is the supporting line always unique?

6. Let \( \ell : \mathbb{R}^2 \to \mathbb{R} \) be a linear function. A set of the form

\[ H_{\ell,c} = \{ x \in \mathbb{R}^2 : \ell(x) > c \} \]  

is called an open half space. If \( K \subset \mathbb{R}^2 \) is a closed convex set, then show that

\[ K = \bigcap_{H_{\ell,c} \supseteq K} H_{\ell,c}. \]  

That is, \( K \) is the intersection of all the open half spaces that contain it. Prove that a closed unbounded, proper convex subset of \( \mathbb{R}^2 \) satisfies exactly one of the following criteria:

(a) \( K \) is a closed half space.

(b) \( K \) is the region between two parallel lines.

(c) \( K \) lies in a proper cone (the intersection of two half-spaces with non-parallel boundaries).
7. Let \( X = \mathbb{R}^2 \) and \( Y = \{ (x, 0) : x \in \mathbb{R} \} \), be a subspace. Suppose that we define a linear function \( \ell \) on \( Y \) by setting \( \ell((1, 0)) = 1 \). For \( 1 \leq p < \infty \), define the norms

\[
\|(x, y)\|_p = (x^p + y^p)^{\frac{1}{p}},
\]

and

\[
\|(x, y)\|_\infty = \max\{|x|, |y|\}.
\]

This linear function on \( Y \) satisfies

\[
|\ell((x, 0))| \leq \|(x, 0)\|_p,
\]

for all \( 1 \leq p \leq \infty \). We can linearly extend \( \ell \) to all of \( \mathbb{R}^2 \) by setting

\[
\ell((0, 1)) = \beta.
\]

Denote this extension by \( \ell_\beta \). For each \( 1 \leq p \leq \infty \), find the values of \( \beta \) so that

\[
|\ell_\beta((x, y))| \leq \|(x, y)\|_p, \text{ for all } (x, y) \in \mathbb{R}^2.
\]

We can define another family of norms, for \( 0 < a < \infty \), by setting

\[
N_a(x, y) = \sqrt{x^2 + a^2 y^2}.
\]

For each \( 0 < a < \infty \), find the values of \( \beta \) so that

\[
|\ell_\beta((x, y))| \leq N_a(x, y), \text{ for all } (x, y) \in \mathbb{R}^2.
\]

8. Show for \( 0 < q < 1 \), the function \( d_q : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty) \) defined by

\[
d_q(x, y) = \sum_{j=1}^{n} |x_j - y_j|^q
\]

defines a metric on \( \mathbb{R}^n \). How about \( d_q(x, y)^{\frac{1}{q}} \)? What is

\[
\lim_{q \to 0^+} d_q(x, y)^{\frac{1}{q}}
\]
9. Let $V$ be a vector space, possibly infinite dimensional.

(a) Show that if $\mathcal{B} = \{x_\alpha : \alpha \in \mathcal{A}\} \subset V$ is a set of linearly independent vectors, then there is a basis for $V$ of the form $\{x_\alpha : \alpha \in \mathcal{A}\} \cup \{y_\beta : \beta \in \mathcal{B}\}$. Hint: Let $\mathcal{W}$ consists of sets of linearly independent vectors in $V$, with the partial ordered defined by inclusion, then apply Zorn’s lemma to prove this assertion.

(b) Use this result to show that if $U \subset V$ is a subspace of $V$, then there exists another subspace $W$ of $V$ so that $V = U \oplus W$, and an isomorphism

$$\varphi : W \longrightarrow V/U.$$ (16)