AMCS 610
Problem set 2 due February 11, 2014
Dr. Epstein

Reading: Read Chapters 3.2-3.3 (especially the proof of Theorem 8), 4.2, 5.1-5.2, 6.1-6.3 in Lax, *Functional Analysis*.

Standard problem: The following problems should be done, but do not have to be handed in.

1. Prove Theorem 4 in §3.2 of Lax.

2. Suppose that \((X, d)\) is a metric space. Show that if \(\lim_{n \to \infty} x_n = x^*\), then, for any \(x \in X\), we also have that \(\lim_{n \to \infty} d(x, x_n) = d(x, x^*)\).

3. For \(1 \leq p \leq \infty\), prove that the normed vector space \(\ell_p\) is a Banach space.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. Prove that in any real normed linear space \((X, \| \cdot \|)\), the open and closed unit balls

\[
\mathcal{B}_1 = \{x \in X : \|x\| < 1\}, \quad \overline{\mathcal{B}}_1 = \{x \in X : \|x\| \leq 1\}
\]

are convex and have non-empty interior. The unit ball is *strictly convex*, if, whenever \(\|x\| = \|y\| = 1\) and \(x \neq y\), then

\[
\left\| \frac{x + y}{2} \right\| < 1. \tag{2}
\]

Show that the unit ball in \(\ell_2\) is strictly convex, but the unit ball in \(\ell_1\) is not.

2. A bounded sequence \(<c_j>\) is Cesaro summable if

\[
\lim_{n \to \infty} \frac{c_1 + \cdots + c_n}{n} \text{ exists.} \tag{3}
\]

Show that a Banach limit \(\text{LIM}\) can be defined on \(\ell_\infty\) so that if \(<c_j>\) is Cesaro summable then

\[
\text{LIM}c_j = \lim_{j \to \infty} \frac{c_1 + \cdots + c_n}{n}. \tag{4}
\]
3. Suppose that $X$ is a Banach space and $Y \subset X$ is a closed subspace. Show that the quotient space $X/Y$, with the quotient norm

$$\| [x] \|_{X/Y} = \inf_{x \in [x]} \| x \|_X,$$

is complete.

4. Prove that every finite dimensional subspace of a normed vector space is closed. Hint: Use the fact that all norms on a finite dimensional vector space are equivalent to show that every finite dimensional subspace is complete.

5. Let $\mathcal{P}$ denote the subspace of $C^0([0, 1])$ defined by polynomials restricted to $[0, 1]$. Suppose that $\ell : \mathcal{P} \to \mathbb{R}$ is a linear function with the property that

$$p(x) \geq 0 \text{ for } x \in [0, 1] \Rightarrow \ell(p) \geq 0. \quad (6)$$

Show that $\ell$ extends to define a linear functional $\tilde{\ell}$, on all of $C^0([0, 1])$, satisfying an estimate of the form

$$|\tilde{\ell}(f)| \leq C\|f\|_{\infty}. \quad (7)$$

Can you find a closed form expression for $C$?

6. Let $Y \subset \ell_\infty$ be the subspace of sequences that are eventually zero (only finitely many terms non-zero). Find the closure of $Y$ with respect to the $\ell_\infty$-norm.

7. Prove that $\ell_1$ has a countable dense subset, but $\ell_\infty$ does not.

8. [This problem assumes an elementary knowledge of holomorphic functions of one complex variable.] Let $H^2(D_1)$ denote the closure of bounded holomorphic functions on the unit disk with respect to the $L^2$-norm

$$\| f \|_2^2 = \int_{D_1} |f(x, y)|^2 dx dy = \lim_{r \to 1^+} \iint_{D_r} |f(x, y)|^2 dx dy < \infty. \quad (8)$$

$L^2(D_1)$ is defined as the closure of $C^0(\overline{D}_1)$ with respect to the $L^2$-norm.

(a) Show that $f \in H^2(D_1)$ is holomorphic in int $D_1$. That is, every element of $H^2(D_1)$ has a representative that is holomorphic in int $D_1$.

(b) Show that if $f$ is a square integrable function in $D_1$, which is holomorphic in the interior of $D_1$, then $f \in H^2(D_1)$. (You need to show that $f$ is an $L^2$-limit of functions in $C^0(\overline{D}_1)$.)
(c) Prove that for any $k \in \mathbb{N}$, there is a bounded linear functional $\ell_k$ defined on $L^2(D_1)$, so that if $f \in \mathcal{H}^2(D_1)$, then

$$\ell_k(f) = \partial_z^k f(0).$$

(9)