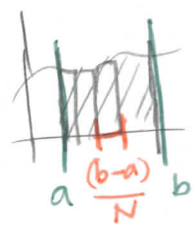
# Integration

want to know about the area under the curve.

we know certainly about the area of a rectangle


Area = l·h

so the idea is to approximate the region under the curve by a collection of rectangles.




If we break up the region into a disjoint union, then the area of the sum is the sum of the area.

$x_0 = a$

$x_1 = a + \frac{b-a}{N}$

$x_2 = a + 2\left(\frac{b-a}{N}\right)$

$x_j = a + j\left(\frac{b-a}{N}\right)$

$x_N = b$.

$a = x_a < x_1 < x_2 < \dots < x_N = b$

this is called a <u>partition</u> of $[a, b]$.

$$P = \{ x_j \mid j = 0, \dots, N \}$$

$$\sum_{j=1}^{N} f(y_j)(x_j - x_{j-1}) = S(f, Y, P)$$

"cauchy sum w.r.t. P"

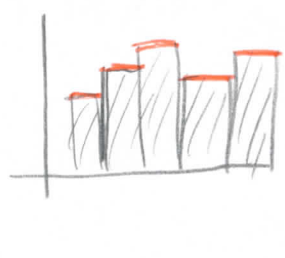
$$Y = \{ y_j \mid x_{j-1} \leq y_j \leq x_j \text{ for } j = 1, \dots, N \}$$

Y is a set of pts "in the partition"

<u>def:</u> <u>Mesh size</u> of P is defined to be

$$|P| = \max \{ |x_j - x_{j-1}| : j = 1, \dots, N \}$$

for a piecewise function that looks like this, it is easy to find the area exactly.



we let $M_j = \max\{f(y) : y \in [x_{j-1}, x_j]\}$

$$m_j = \min\{f(y) : y \in [x_{j-1}, x_j]\}$$

$$S^+(f,P) = \sum_{j=1}^{N} M_j (x_j - x_{j-1})$$ // upper Reimann sum

$$S^-(f,P) = \sum_{j=1}^{N} m_j (x_j - x_{j-1})$$ // lower Reimann sum

Note:

for any choice of $Y$ we have that

$$\underbrace{S^-(f,P)}_{= S(f,Y^-,P)} \leq S(f,Y,P) \leq \underbrace{S^+(f,P)}_{= S(f,Y^+,P)}$$

Theorem: given $\varepsilon > 0$, $\exists\ \delta > 0$ st.

$$S^+(f,P) - S^-(f,P) < \varepsilon$$

if $|P| < \delta$

pf// Recall that a cts. func'n on a compact interval is uniformly cts. That is, given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$ $(x, y \in [a,b])$

Choose the $\delta$ that works for $\frac{\varepsilon}{b-a}$

If $P$ is a partition $\{x_0 < x_1 < \dots < x_N\}$ w/

$|P| < \delta$

then $x_j - x_{j-1} < \delta \quad \forall j$

let $y_j^+ \in [x_{j-1}, x_j]$ be the pt where $f(y_j^+) = M_j$

$y_j^- \in [x_{j-1}, x_j]$ " " where $f(y_j^-) = m_j$

$$S^+(f,P) - S^-(f,P) = \sum_{j=1}^{N} (f(y_j^+) - f(y_j^-))(x_j - x_{j-1})$$

$$|y_j^+ - y_j^-| \leq x_j - x_{j-1} < \delta$$

so, $f(y_j^+) - f(y_j^-) < \frac{\varepsilon}{b-a}$

$$S^+(f,P) - S^-(f,P) \leq \sum_{j=1}^{N} \frac{\varepsilon}{b-a}(x_j - x_{j-1})$$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$ ✓ ∎

def: $P > P'$

the underlying sets satisfy.

$$P \supseteq P'$$

P is called a refinement of $P'$

$$\underbrace{[x_{j-1}, x_j]}_{P'} = \underbrace{[\tilde{x}_{\ell-1}, \tilde{x}_\ell] \cup [\tilde{x}_\ell, \tilde{x}_{\ell+1}] \cup \dots \cup [\tilde{x}_{\ell+k+1}, \tilde{x}_{\ell+k}]}_{P}$$

$$M_j = \max\{f(x) : x \in [x_{j-1}, x_j]\}$$

$$\tilde{M}_i = \max\{f(x) : x \in [\tilde{x}_{\ell+k+i}, \tilde{x}_{\ell+i}]\}$$

$$\tilde{M}_i \leq M_j$$

so we have:

$$S^+(f, P) \leq S^+(f, P')$$

$$S^-(f, P) \geq S^-(f, P')$$

"the smaller we take the width of the partitions the closer to the actual value" integral

$$I = \sup_P S^-(f,P) = \inf_P S^+(f,P)$$

$$S^-(f,P) \le S^+(f,\tilde{P}) \quad \forall P, \tilde{P}$$

we can define a new partition called <u>the common refinement</u> by

$$P'' = P \cup \tilde{P}$$

and so,

$$S^-(f,P) \le S^-(f,P'') \le S^+(f,P'') \le S^+(f,\tilde{P})$$

given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $|P| < \delta$

then $S^+(f,P) - S^-(f,P) < \varepsilon$

this must mean that the $\sup S^- = \inf S^+$

we could define a seq of partitions 11/14/17 pg7

- $P_j \subseteq P_{j+1}$ st.

  $\lim_{j \to \infty} S^-(f, P_j) = I$

and

- $\tilde{P}_j \subseteq \tilde{P}_{j+1}$ st.

  $\lim_{j \to \infty} S^+(f, \tilde{P}_j) = I$

and

- $P_j'' = P_j \cup \tilde{P}_j$ st.

  $\lim_{j \to \infty} S^{\pm}(f, P_j'') = I$

If $\langle P_j \rangle$ is any seq. of partitions st. $|P_j| \to 0$

then $\lim_{j \to \infty} S(f, Y_j, P_j) = I$

we denote it

$$\boxed{I = \int_a^b f(x)\,dx}$$ (summa)

(this is the theory for cts. functions)

Properties :

(i) If $f, g$ are cts funct'ns on $[a,b]$,

then

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

if $\alpha \in \mathbb{R}$

then $$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

the integral is a linear operation b/c it has these 2 properties.

so,

$$S(\alpha f + \beta g, \gamma, P_j)$$

$$= \alpha S(f, \gamma, P_j) + \beta S(g, \gamma, P_j).$$

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(ii)

If $a < b < c$ if $f \in C^0[a, c]$

then $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

If $m \leq f(x) \leq M$

then $$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(iii) Δ- inequality

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

$$|S(f, y, P)| = \left| \sum_{k=1}^{N_j} f(y_k^j)(x_{k-1} - x_k^j) \right|$$

$$\leq \sum_{k=1}^{N_j} |f(y_k^j)| (x_{k-1}^j - x_k^j)$$

$|P_j| \to 0$ $\quad j \to \infty \to \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

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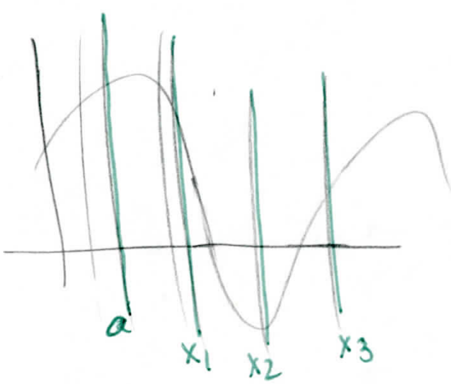
def: $\int_a^b f(x)\,dx$ is called a definite integral




define $\boxed{F(x) = \int_a^x f(y)\,dy}$

where $x$ is any pt in $[a,b]$

→

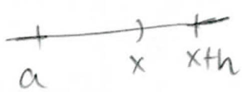
## Fundamental theorem of calculus (I)

If $\quad f' \in C^0[a,b]$

then $\quad F \in C^1[a,b]$ and $F'(x) = f(x)$.

pf// $\quad h > 0$

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left[ \int_a^{x+h} f(y)\,dy - \int_a^{x} f(y)\,dy \right]$$




$$= \frac{1}{h}\int_x^{x+h} f(y)\,dy$$

make sure you use different variables here!

$f$ is cts at $x$, this means that given $\varepsilon > 0$ $\exists\, \delta$ st

$$|f(x) - f(y)| < \varepsilon \quad \text{if} \quad |x-y| < \delta$$

$$f(x) - \varepsilon \le f(y) \le f(x) + \varepsilon \quad \text{if } h < \delta$$

this tells us that;

$$(f(x) - \varepsilon) \le \frac{1}{h}\int_x^{x+h} f(y)\,dy \le (f(x) + \varepsilon) \quad \text{if } h < \delta$$

(continue next)

$$\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = f(x)$$

WTS $$\lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h} = f(x)$$

well, $h > 0$; $\dfrac{F(x-h) - F(x)}{-h}$

$$= \frac{F(x) - F(x-h)}{h} = \frac{1}{h}\int_{x-h}^{x} f(y)\,dy$$

$$\lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h} = f(x) \checkmark$$

Hence, F is differentiable at X and

$$\boxed{F'(x) = f(x)}$$

If you can find an $F(x)$ s.t.

$$F'(x) = f(x)$$

then $\int_a^b f(y)\,dy = F(b) - F(a)$

## Fundamental theorem of calculus (II)

If $f \in C'([a,b])$

then $\int_a^b f'(x)\,dx = f(b) - f(a)$

Pf// Choose a partition $x_0 < x_1 < \dots < x_N$ + $y_j \in [x_{j-1}, x_j]$

$$S(f; Y, P) = \sum_{j=1}^{N} f'(y_j)(x_j - x_{j-1})$$

By the M.V.T.

$$\exists\ y_j \in (x_{j-1}, x_j)$$

$$f'(y_j)(x_j - x_{j-1}) = f(x_j) - f(x_j - 1)$$

with this, choose:

$$S(f, Y, P) = \sum_{j=1}^{N} (f(x_j) - f(x_{j-1}))$$

$$= f(x_N) - f(x_0) = f(b) - f(a) \checkmark$$

$$\boxed{\int_a^b f'(y)\,dy = f(b) - f(a)}$$

$\blacksquare$

**def:** A function $F$ s.t. $F'(x) = f(x)$

is called a primitive of $f$

or indefinite integral of $f$.

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Diff. formulas produce:

- $\frac{d}{dx} f g = f'g + g'f$ ↙ integration by parts:

$$\to \int_a^b \frac{d}{dx}(fg)(y)\,dy = \int_a^b f'(y)\,g(y)\,dy + \int_a^b f(y)\,g'(y)\,dy$$

$$fg(b) - fg(a) - \int_a^b f(y)\,g'(y)\,dy = \int_a^b f'(y)\,g(y)\,dy$$

chain rule:

$$\frac{d}{dx} f \circ g(x) = f' \circ g(x) \cdot g'(x)$$

$$\to f \circ g(b) - f \circ g(a) = \int_a^b \frac{d}{dx}(f \circ g(x))\,dx = \int_a^b f' \circ g(x)\,g'(x)\,dx$$

↙ "change of variable formula"

we can rewrite it:

let $f$ be $C^0(g([a,b]))$, $g \in C^1([a,b])$

$$\int_a^b f \circ g(x)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy.$$

(*)

(continue next)

that formula comes from:

$$F(x) = \int^{x} f(y)\,dy$$

$$\frac{d}{dx} F \circ g(x) = F' \circ g(x) \cdot g'(x) = \overset{**}{f \circ g(x)\, g'(x)}$$

and we get

$$F(g(b)) - F(g(a)) = \int_a^b f \circ g(x)\, g'(x)\, dx$$

this equals (*) + (**)

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Consider integration by parts in the following way:

let $f \in C^n([a,b])$ & let $x_0 \in (a,b)$

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(y)\,dy$$

$\frac{d}{dy}(y-x) = 1$

$$= f(x_0) - (x-y) f'(y) \Big|_{x_0}^{x} + \int_{x_0}^{x} f''(y)(x-y)\,dy$$

$$= f(x_0) - f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(y)(x-y)\,dy$$

⋮

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doing this many times (integrating by parts many times)

we get the formula

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{[j]}(x_0)(x-x_0)^j}{j!} + \underbrace{\int_{x_0}^{x} \frac{(x-y)^{n-1} f^{[n]}(y)\,dy}{(n-1)!}}$$

(this gives us a new formula for the remainder)

$$f(x) - T_{n-1}(f, x_0, x) = \int_{x_0}^{x} \frac{f^{[n]}(y)(x-y)^{n-1}}{(n-1)!}\,dy \quad (*)$$

$$m \leq f^{[n]}(y) \leq M_x$$

$$(*) = d \int_{x_0}^{x} \frac{(x-y)^{n-1}}{(n-1)!}\,dy \quad \text{where } m \leq d \leq M$$

$\exists\ \xi \in [a,b]$ where $f^{[n]}(\xi) = d$

so, $$d \int_{x_0}^{x} \frac{(x-y)^{n-1}}{(n-1)!}\,dy = \frac{(x-x_0)^n}{n!} f^{[n]}(\xi)$$

this is nice b/c its just an integral & not evaluated at a certain pt.