Math 508 Lecture 10/26/17

Continuous functions:

3 definitions:

1. $\varepsilon - \delta$
2. limit definition
3. Inverse image of open sets

Algebraic properties of cts lims

Thm: Let $f + g$ be cts in a set D then

1. $f+g$ is cts in D
2. $f \cdot g$ is cts in D
3. If $g(x) \neq 0$ $\forall x \in D$ then $\frac{f}{g}$ is cts in D

If $\lim_{x \to x_0} f(x) = f(x_0)$, $\lim_{x \to x_0} g(x) = g(x_0)$

then $\lim_{x \to x_0} f(x) \cdot g(x) = f(x_0) \cdot g(x_0)$

Simple cts fcns can be used to make complicated ones

Ex. $f(x) = c$ $\forall x \in \mathbb{R}$

$f(x) = x$ is cts

Given $\varepsilon > 0$, take $\delta = \varepsilon$, so $|x-y| < \delta$

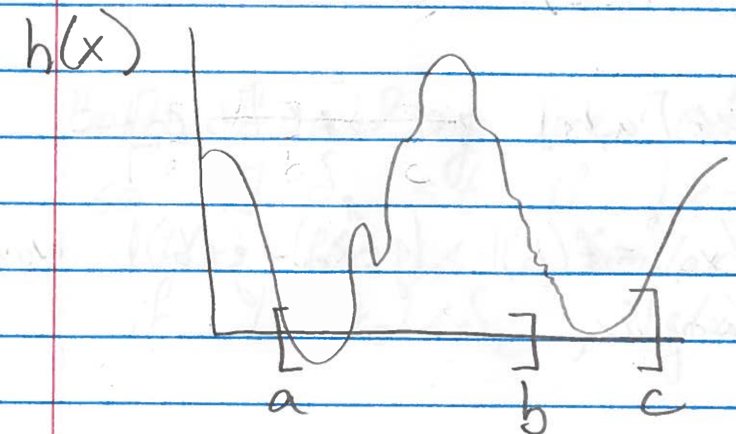
$\Rightarrow |x-y| < \varepsilon$

From our thm, by induction, then

$p(x) = \sum_{i=0}^{n} a_j x^j$ is cts $\forall (a_j, \dots, a_n) \in \mathbb{R}^{n-1}$

$q(x) = \sum_{j=0}^{m} b_j x^j$, let $D = \{x : q(x) = 0\}^c$

Then fcn $\frac{p(x)}{q(x)}$ is cts on D.



(Look on nxt page)

f is cts on $[a,b]$
g is cts on $[b,c]$

and $f(b)=g(b)$
then

$$h(x) = \begin{cases} f(x) & x\in[a,b] \\ g(x) & x\in[b,c] \end{cases}$$

then h is cts on $[a,c]$

$$\lim_{x\to b} h(x) = h(b)$$

Pf idea take $\langle x_n \rangle$ st $\lim_{n\to\infty} x_n = b$

NTS: Given $\varepsilon>0$, $\exists N$ st
$|h(x_n)-h(b)| < \varepsilon$ if $n \geq N$

either $x_n \in [a,b]$, or $x_n \in [b,c]$

we know $|f(x_n)-f(b)|$, $|g(x_n)-g(b)|$ would be less than ε

Thm: Let f, g be cts on D

then $h_+(x) = \max\{f(x), g(x)\}$

$h_-(x) = \min\{f(x), g(x)\}$

are cts too.

Pf: Give proof for $h_+(x)$.

Let $x_0 \in D$ and we'll suppose
$h_+(x_0) = f(x_0)$, WLOG

NTS: given $\varepsilon>0$, $\exists\delta$ st

$0<|x-x_0|<\delta$, then $|\max\{f(x),g(x)\} - f(x_0)| < \varepsilon$

Both f and g are cts @ x_0
so $\exists \delta_1$ st if $|x-x_0|<\delta_1$ then
$|f(x)-f(x_0)|<\varepsilon$. Similarly, $\exists\delta_2$ st
if $|x-x_0|<\delta_2$, $|g(x)-g(x_0)|<\varepsilon$

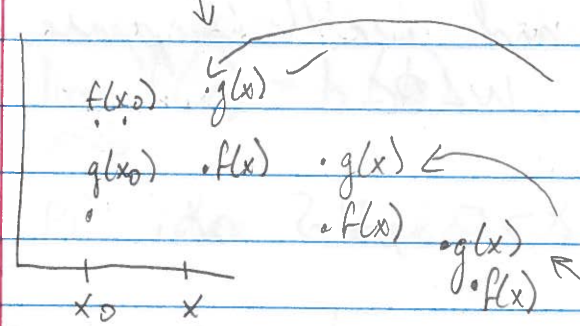
Let $\delta = \min\{\delta_1, \delta_2\}$

If $|x - x_0| < \delta$ and $\max\{f(x), g(x)\} = f(x)$

then $|f(x) - f(x_0)| < \varepsilon$ ✓

Now, if $\max\{f(x), g(x)\} = g(x)$

Say $g(x_0) \le g(x)$



$|f(x_0) - g(x)| \le |g(x_0) - g(x)| < \varepsilon$

$|f(x_0) - g(x)| \le |f(x_0) - f(x)| < \varepsilon$

$|f(x_0) - g(x)| \le |f(x_0) - f(x)| < \varepsilon$

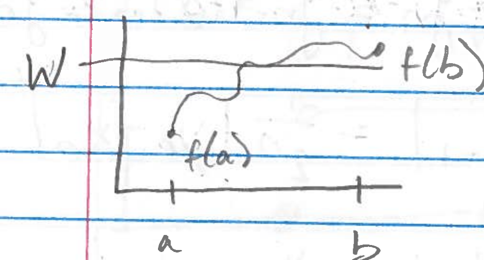
Above, 3 cases where is $g(x)$ relative to $f(x_0)$, which can be estimated by things we do know

Ex. $|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$

$|x| = \max\{x, -x\}$, which is cts, by what we've just proved.

$\boxed{\text{Intermediate Value Theorem}}$

Let f be a cts function on $[a,b]$. Suppose $f(a) < f(b)$. If $w \in (f(a), f(b))$ then $\exists\, x \in (a,b)$ st $f(x) = w$.



Pf: $f\left(\frac{a+b}{2}\right) = w$ or ①

$f\left(\frac{a+b}{2}\right) > w$ or ②

$f\left(\frac{a+b}{2}\right) < w$ ③

If (2), let $x_0 = a$, $y_0 = \frac{a+b}{2}$

If (3), let $x_0 = \frac{a+b}{2}$, $y_0 = b$

In both cases, $f(x_0) < w < f(y_0)$

Now, $f\left(\frac{x_0+y_0}{2}\right) < w \quad x_1 = \frac{x_0+y_0}{2}, \; y_1 = y_0$

$f\left(\frac{x_0+y_0}{2}\right) > w \quad x_1 = x_0, \; y_1 = \frac{x_0+y_0}{2}$

$x_0, x_1, x_2, \ldots$

$y_0, y_1, \ldots$

$$x_0 \le x_1 \le x_2 \le \cdots \le y_n \le y_{n-1} \le \cdots \le y_0$$

$$y_n - x_n = \frac{1}{2}|y_{n-1} - x_{n-1}| = \frac{1}{2^n}|y_0 - x_0|$$

$$f(x_n) < w < f(y_n)$$

$\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$, b/c bounded monotone sequences

$$\lim_{n\to\infty} |x_n - y_n| \le \frac{1}{2^{n+1}}|b-a|$$

$$||$$

$$|x^* - y^*| \le \lim_{n\to\infty} \frac{1}{2^{n+1}}|b-a| = 0$$

$$x^* = y^*$$

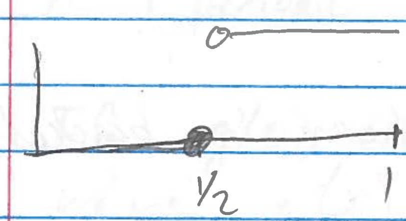
$$w \ge \lim_{n\to\infty} f(x_n) = f(x^*)$$

$$w \le \lim_{n\to\infty} f(y_n) = f(x^*)$$

$$\Rightarrow \boxed{w = f(x^*)}$$

Ex:

$$f(x) = \begin{cases} 0 & x \in [0, \frac{1}{2}] \\ 1 & x \in (\frac{1}{2}, 1] \end{cases}$$



Continuity is necessary for IVT

$$\sum_{j=0}^{2n+1} a_j x^j \qquad a_{2n+1} \neq 0$$

$$a_{2n+1} x^{2n+1}\left[1 + \sum_{j=0}^{2n} \frac{a_j x^{j-(2n+1)}}{a_{2n+1}}\right]$$

$$\left|\sum_{j=0}^{2n} \frac{a_j x^{j-(2n+1)}}{a_{2n+1}}\right| \qquad |x| \geq 1$$

$$= \frac{1}{|x|} \sum_{j=0}^{2n} \frac{|a_j||x|^{j-2n}}{|a_{2n+1}|}$$

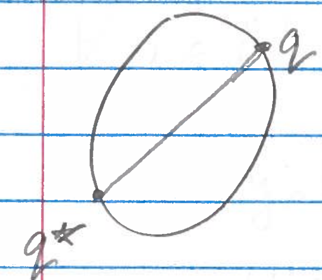
$$\leq \frac{1}{|x|} \underbrace{\sum_{j=0}^{2n} \left|\frac{a_j}{a_{2n+1}}\right|}_{A}$$

If $|x| \geq 2A$ then term in brackets is $\geq \frac{1}{2}$

sign $f(x)$ for $x \geq 2A$ is the opposite of sign $f(x)$ for $x < -2A$

IVT $\Rightarrow$ $\exists\, x \in [-2A, 2A]$ where $f(x) = 0$

circle



$f(x + nP) = f(x)$

periodic function on circle

If f is a periodic function on the circle, then there is a pair of antipodal points q, q^* such that $f(q) = f(q^*)$.

$$\begin{array}{cc} x & x + \frac{P}{2} \\ \wr & \wr \\ q & q^* \end{array}$$

$$g(x) = f(x) - f\left(x + \frac{P}{2}\right)$$

$$g(0) = f(0) - f\left(\frac{P}{2}\right)$$

$$g(P/2) = f(P/2) - f(P)$$

b/c P periodic $\qquad = f(P/2) - f(0) = -g(0)$

If $g(0) \neq 0$, then $\exists\, x \in (0, P/2)$ where $g(x) = 0$, $f(x) - f(x + P/2) = 0$

f is cts at x_0 if
given $\varepsilon>0$ $\exists\ \delta>0$ st
if $x \in D$ and

$|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\varepsilon$

A function is uniformly continuous on D
if $(\forall)$ given $\varepsilon>0$, $\exists\ \delta>0$ st $\forall x,y \in D$
with $|x-y|<\delta$, we have

$|f(x)-f(y)|<\varepsilon$.

Thm: Let f be a continuous function
on a compact set. Then f is
uniformly cts on D.

Pf: (by contradiction)

Not uniformly cts means $\exists\ \varepsilon>0$ st
$\forall \delta>0$, $\exists\ x,y \in D$ with $|x-y|<\delta$ but

$|f(x)-f(y)|\geq\varepsilon$

$\forall n \in \mathbb{N}$, $\exists\ x_n, y_n \in D$ st

$|x_n-y_n|<\frac{1}{n}$ but $|f(x_n)-f(y_n)|\geq\varepsilon$

$\langle x_n\rangle$, $\langle y_n\rangle$ sequences

By compactness, $\exists\ \langle x_{n_i}\rangle$ st

$\lim_{i\to\infty} x_{n_i} = x^* \in D$

$|y_{n_i}-x_{n_i}|<\frac{1}{n_i}$ $n_i \geq i$

$\Rightarrow \lim_{i\to\infty} y_{n_i} = x^*$ b/c equivalent Cauchy seq

$\lim_{i\to\infty} |f(x_{n_i})-f(y_{n_i})| = |f(x^*)-f(x^*)| = 0$

$f(x)=x^2$, $x \in \mathbb{R}$

$|x^2-y^2| = |(x+y)||(x-y)|$

$|(x+y)||(x-y)|<\varepsilon \Rightarrow |x-y|<\frac{\varepsilon}{|x+y|}$

Not uniformly cts, since same δ won't work $\forall x,y$

Another example: $\frac{1}{x}$ on $(0, 1]$

As we approach 0, we have to take smaller and smaller δ.

Thm: Let f be a cts function on a compact set D, then $\exists\, x^+, x^- \in D$ st

$$f(x^+) = \sup\{f(x), x \in D\}$$

$$f(x^-) = \inf\{f(x), x \in D\}$$

In particular ① a cts function on a compact set is bounded above and below. ② cts function on compact set assumes its max & min

Let $\langle x_n \rangle \subseteq D$ be a sequence such that

$$\lim_{n\to\infty} f(x_n) = \sup\{f(x) ; x \in D\}$$

Let $\langle x_{n_i} \rangle$ be a convergent subsequence

$$\lim_{i\to\infty} x_{n_i} = x^* \in D \text{ by compactness}$$

$$\lim_{i\to\infty} f(x_{n_i}) = f(x^*) = \sup\{f(x) : x \in D\}$$
by continuity

Thm: f is a cts function on a compact set D. Then $f(D)$ is also compact.

$$f(D) \subseteq [f(x^-), f(x^+)]$$

Let $\langle y_n \rangle \subseteq f(D)$ be a convergent sequence.

$$\lim_{n\to\infty} y_n = y^* \quad \text{NTS } y^* \in f(D)$$

$\exists \langle x_n \rangle \subset D$ st

$$f(x_n) = y_n$$

$\exists \langle x_{n_i} \rangle$ st $\lim_{i\to\infty} x_{n_i} = x^* \in D$

$$f(x^*) = \lim_{i\to\infty} f(x_{n_i}) = \lim_{i\to\infty} y_{n_i} = y^*$$

Ex: $D = \mathbb{R}$

$f(x) = \frac{1}{x^2+1}$ $\quad f(D) = (0, 1]$

$f(x) = \frac{x}{\sqrt{1+x^2}}$ $\quad f(\mathbb{R}) = (-1, 1)$

$f(x) = \frac{x}{\sqrt{1-x^2}}$ on $(-1, 1)$, $f((-1,1)) = \mathbb{R}$

All hypotheses of compactness are necessary.