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MATH 508 – Lecture 10

Cauchy Sequences

If $\langle x_n \rangle$ is a seq. of #s s.t. given $N \in \mathbb{N}$, $\exists\, M \in \mathbb{N}$ s.t. $\forall\, n, m \geq M$

$$|x_n - x_m| \leq \frac{1}{N}$$

then we say that $\langle x_n \rangle$ is a Cauchy seq.

Thm: If $\langle x_n \rangle$ is a Cauchy seq. then $\lim_{n \to \infty} x_n$ exists.

This property is called completeness.

- The set of Cauchy sequences of rational numbers on which we define an equivalence relation

$$\langle x_n \rangle \sim \langle x_n' \rangle \text{ if } \lim_{n \to \infty} (x_n - x_n') = 0$$

- Real #s, as a set, is defined to be the eq. classes of Cauchy seqs. of rational #s.

Arithmetic on $\mathbb{R}$.

$$x \leftrightarrow \langle x_n \rangle$$
$$y \leftrightarrow \langle y_n \rangle$$
$$-x \leftrightarrow \langle -x_n \rangle$$
$$x+y \leftrightarrow \langle x_n + y_n \rangle$$
$$x \cdot y \leftrightarrow \langle x_n \cdot y_n \rangle$$

$$\langle x_n \rangle \sim \langle x_n' \rangle$$
$$\langle y_n \rangle \sim \langle y_n' \rangle$$
$$\langle -x_n \rangle \sim \langle -x_n' \rangle$$
$$\langle x_n + y_n \rangle \sim \langle x_n' + y_n' \rangle$$
$$\langle x_n \cdot y_n \rangle \sim \langle x_n' \cdot y_n' \rangle$$

Arithmetic is on equivalence classes, not on representations.

$Y \neq 0 \iff \exists$ a $N \in \mathbb{N}$ such that $\forall$ $\langle y_n \rangle$ that ~~corresponds~~ represents y $\exists$ $M \in \mathbb{N}$ s.t. $|y_n| \geq \frac{1}{N}$ for all $n \geq M$.

Thus if $y \neq 0$, then we can choose a representation $\langle y_n \rangle$ s.t. $|y_n| \geq \frac{1}{N}$ $\forall n$. Then we define

$$\frac{1}{y} \leftrightarrow \left\langle \frac{1}{y_n} \right\rangle$$

Then: $$y \cdot \frac{1}{y} \leftrightarrow \left\langle y_n \cdot \frac{1}{y_n} \right\rangle = \langle 1 \rangle$$

ORDER

We say $x \leq y$ if for any representatives $\langle x_n \rangle$, $\langle y_n \rangle$ $\exists$ M such that for all $n \geq M$

$x_n \leq y_n$.

DISTANCE

$$d(x,y) = |x-y| = \langle |x_n - y_n| \rangle$$

- We need to show that the real numbers are a complete ordered field. If $\langle x_n \rangle$ is a cauchy of real #s, then $\exists$ a real x^* s.t. $|x_n - x^*| \longrightarrow 0$ as $n \to \infty$

- $\langle x_k^{(n)} \rangle$ is a cauchy seq. representative of x_n

We need to construct a c.s. $\langle y_n \rangle$ of rational #s that represents $\lim\limits_{n\to\infty} x_n$

Given $N \in \mathbb{N}$, $\exists$ M s.t. $|x_n - x_m| \leq \frac{1}{N}$

if $n, m \geq M$

$\Rightarrow \langle |x_k^{(n)} - x_k^{(m)}| \rangle$, Given $l > 0$, $\exists$ K_l

s.t.

$$|x_k^{(n)} - x_k^{(m)}| \leq \frac{1}{N} + \frac{1}{l} \quad \text{if } k \geq K_l.$$

For each $l \in \mathbb{N}$ we know that there is an M_l such that $|x_n - x_m| \leq \frac{1}{2^l}$ $n, m \geq M_l$

$$|x_k^{(n)} - x_k^{(m)}| \leq \frac{2}{2^l} \quad \text{for } k \geq K$$

$$|x_k^{(n)} - x_k^{(M_l)}| \leq \frac{2}{2^l} \quad , \begin{array}{l} n \geq M_l \\ k \geq K \end{array}$$

~~Throw away $x_1^{(M_l)}, \ldots x_{K-1}^{(M_l)}$~~

$l = 1, 2, \ldots$

$n, m \geq M_l$

let $l = 1$;

$$|x_k^{(n)} - x_k^{(m)}| \leq \frac{2}{2^1} \quad , \quad K \geq K_{n,m}$$

$m = M_l$

$$|x_k^{(n)} - x_k^{(M_l)}| \leq \frac{2}{2^l} \quad \text{for} \quad k \geq K_{n,m}$$

$\langle x_k^{(M_l)} \rangle$ is itself a Cauchy sequence

$\exists K_l$ s.t. $|x_m^{(M_l)} - x_n^{(M_l)}| < \frac{1}{2^l}$ if $m, n \geq K_l$

$$|x_n^{(M_\ell)} - x_m^{(M_\ell)}| \leq \frac{2}{2\ell} \quad \text{if } n, m \geq K_\ell'$$

Choose $y_\ell = x_k^{(M_\ell)}$ for $k \geq K_{n,m}$
and $k \geq K_\ell'$

Idea : Construct a sequence of rational numbers such that the seq. is a Cauchy sequence which represents the limit of the Cauchy sequence of the reals ~~$x_1, x_2, \ldots$~~ $\langle x_n \rangle$

Complete proof as exercise.

- $\langle x_n \rangle$ is a Cauchy seq. Suppose that the sub-seq. $\langle x_{n_j} \rangle$ converges i.e. $\lim_{j \to \infty} x_{n_j} = x^*$.

Given N, $\exists M$ s.t. $|x_{n_j} - x^*| \leq \frac{1}{N}$ $\forall j \geq M$

$\exists M'$ s.t. if $n, m \geq M'$ ~~$x_j - x$~~

then $|x_n - x_m| \leq \frac{1}{N}$ if $n, m \geq M'$

$M'' = \max\{M, n_M\}$

If $n \geq M''$ then

$$|x_n - x^*| \leq |x_n - x_{n_M}| + |x_{n_M} - x^*|$$

$$\leq \frac{2}{N}$$