MATH 508 - Lecture 11

Let $(x_1, x_2 \ldots)$ be a sequence of real numbers. want to give a meaning to

$$\sum_{j=1}^{\infty} x_j$$

Define the <u>partial sum</u> S_n as

$$S_n := \sum_{j=1}^{n} x_j$$

Consider the sequence of partial sums $S_1, S_2, \ldots$

$\sum_{j=1}^{\infty} x_j$ converges if $\lim_{n\to\infty} S_n$ exists

and this is the defn of the sum of the sequence

— OR —

$$\sum_{j=1}^{\infty} x_j := \lim_{n\to\infty} \sum_{j=1}^{n} x_j$$

<u>Obs</u> :

- $S_n - S_{n-1} = x_n$

  If $\lim_{n\to\infty} S_n$ exists then $\lim_{n\to\infty} x_n = 0$

  The converse is false. (think about $x_n = 1/n$)

eg: (geometric series) $$\sum_{j=0}^{n} r^j = \begin{cases} \dfrac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \\ n+1 & \text{if } r = 1 \end{cases}$$

This formula shows that

$$\lim_{n\to\infty} \sum_{j=1}^{n} r^j = \begin{cases} \dfrac{1}{1-r} & , |r| < 1 \\ \text{diverges} & , \text{ if } |r| \geq 1 \end{cases}$$

Algebraic Properties

- If $\sum_{j=1}^{\infty} x_j = s$ and $\sum_{j=1}^{\infty} y_j = t$,

  then $\sum_{j=1}^{\infty} (x_j + y_j) = s + t$

- If $\sum_{j=1}^{\infty} x_j = s$, then

  $$\sum_{j=1}^{\infty} c x_j = cs$$

If $\langle x_j \rangle$ is a sequence such that

$\sum_{j=1}^{\infty} |x_j|$ converges, then we say that

$\sum_{j=1}^{\infty} x_j$ <u>converges absolutely</u>.

<u>Prop</u>: Absolute convergence $\Rightarrow$ convergence

<u>Pf</u>:

Cauchy criterion: $\sum_{j=1}^{\infty} |x_j|$ converges iff

$\forall \varepsilon > 0, \ \exists N$ s.t. $\sum_{j=n}^{m} |x_j| < \varepsilon \ \forall n, m \geq N$

$$\left| \sum_{j=n}^{m} x_j \right| \leq \sum_{j=n}^{m} |x_j| \qquad \{ \text{triangle inequality} \}$$

If $m, n \geq N$, $\sum_{j=n}^{m} |x_j| \leq \varepsilon$

$$\Rightarrow \left| \sum_{j=n}^{m} x_j \right| = | S_m - S_{n-1} | \leq \varepsilon$$

i.e. the sequence of partial sums is a

Cauchy sequence, i.e. it converges

<u>Defn</u>: A series that is convergent but <u>not</u> absolutely convergent is called conditionally convergent.

<u>Fact</u>: If $0 \leq x_j \;\; \forall j \geq 1$, then

$\sum_{j=1}^{\infty} x_j$ is convergent iff the partial sums are bounded.

<u>Note</u>: S_n is a monotone increasing seq. in this case.

<u>Pf</u>: Left as exercise.

<u>Defn</u>: A series ~~is~~ unconditionally convergent if it converges no matter how the terms are ~~de~~ ordered

<u>Defn</u>: A rearrangement is a one-to-one, onto map from $\mathbb{N} \to \mathbb{N}$

Equivalently,

A sum is ~~und~~ unconditionally convergent if $\lim_{m \to \infty} \sum_{j=1}^{m} x_{n_j}$ exists for any rearrangement.

Thm : A sum is unconditionally convergent iff it is absolutely convergent.

Pf :

($\Leftarrow$) abs. convergence $\Rightarrow$ unconditional convergence

$$S = \sum_{j=1}^{\infty} x_j$$

Let $\langle n_j \rangle$ be a rearrangement.

Given $\varepsilon > 0$, $\exists$ a N s.t. if $n \geq N$, then

$$\left| \sum_{j=1}^{n} x_j - S \right| < \varepsilon$$

And then $\exists$ an N' s.t.

$$\sum_{j=m}^{\infty} |x_j| \leq \varepsilon \quad \text{if} \quad m \geq N'$$

Fix an $n > \max\{N, N'\}$

$\exists$ J such that

$$\{n_j : j = 1, \dots J\} \supseteq \{1, \dots, n\}$$

$$\left| \sum_{j=1}^{J} x_{n_j} - S \right| = \left| \sum_{j=1}^{n} x_j - S + \sum_{\{n_j \notin [n],\, j \leq J\}} x_{n_j} \right|$$

$$\leq \left| \sum_{j=1}^{n} x_j - S \right| + \underbrace{\sum_{\substack{n_j \notin [n]; \\ j \leq J}} |x_{n_j}|}_{\leq \sum_{j=n+1}^{\infty} |x_{n_j}| \leq \varepsilon}$$

$$\leq 2\varepsilon$$

$\Rightarrow$ This is also true for sufficiently large J

$\Rightarrow$ this series is unconditionally convergent.

($\Rightarrow$) unconditional convergence $\Rightarrow$ absolute convergence

—OR— equivalently

not absolutely convergent $\Rightarrow$ not unconditionally convergent

$$\sum_{j=1}^{\infty} |x_j| = +\infty$$

In order for such a sum to be finite, $\sum_j x_j < \infty$

$$x^+ = \max\{0, x\}$$
$$x^- = \max\{0, x\}$$

Then

$$\sum_{j=1}^{\infty} x_j^+ = +\infty \quad , \quad \sum_{j=1}^{\infty} x_j^- = -\infty$$

in order for $\sum_{j=1}^{\infty} x_j$ to converge.

(on pset 5)

$\exists$ a rearrangement $\langle n_j \rangle$ s.t.

$$\lim_{m\to\infty} \sum_{j=1}^{m} x_{n_j} = +\infty$$

[implies not unconditionally convergent]

■

Tests of convergence

1. M test

If $\langle x_j \rangle$ is a seq. and there is another seq. $\langle y_j \rangle$ s.t. $|x_j| \leq M y_j$ and if $\sum_{j=1}^{\infty} y_j$ converges, then $\sum_{j=1}^{\infty} x_j$ ~~also~~ is absolutely convergent.

2. Ratio test

If $a_n \neq 0 \;\forall n$ and $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = r < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

- If $\frac{|a_{n+1}|}{|a_n|} \geq 1$ for $n \geq N$

  then $\lim_{n\to\infty} a_n \neq 0$

  and the series diverges.

3. Root test

If $\limsup_{n\to\infty} |a_n|^{1/n} = r < 1$

then the series is absolutely convergent.

- $$\limsup_{n\to\infty} |a_n|^{1/n} = \begin{cases} 1 & \text{(no info)} \\ >1 & \text{(diverges)} \end{cases}$$

Note: These tests tell us nothing about

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \quad \forall k$$

Thm: Let $a_n \geq a_{n+1} \geq \dots \geq 0$

$\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{m=1}^{\infty} 2^m a_{2^m}$ converges.

Pf : ~~Consider~~

($\Leftarrow$)

Consider

$$a_1 + \underbrace{a_2 + a_3}_{2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{4} + \overbrace{a_8 + \cdots + a_{15}}^{8} + \cdots + \underbrace{a_{2^{N-1}} + \cdots + a_{2^N - 1}}_{2^{N-1}}$$

$$\leq a_1 + 2a_2 + 2^2 a_4 + \cdots + 2^{N-1} a_{2^{N-1}}$$

$$\leq M \qquad \forall N$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

($\Rightarrow$) Similar argument as above.

- $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges iff $k > 1$.