MATH 508

Lecture 12

RECAP

If $\lim_{n\to\infty} \sum_{j=1}^{n} a_j$ exists, then the series converges

If $\lim_{n\to\infty} \sum_{j=1}^{n} |a_j|$ exists, then the series converges absolutely

Convergent but not abs. convergent $\longleftrightarrow$ conditionally convergent

Unconditionally Conv. means for any rearrangement $\langle n_j \rangle$ $\lim_{n\to\infty} \sum_{j=1}^{n} a_{n_j}$ exists & equals to $\sum_{j=1}^{\infty} a_j$.

$\updownarrow$

Abs. convergent

Example : $$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

but $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$ does converge

$$= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \dots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right) - \dots$$

$$= 1 - \underbrace{\frac{1}{2\cdot 3}}_{\leq \frac{1}{4}} - \underbrace{\frac{1}{4\cdot 5}}_{\leq \frac{1}{16}} - \dots - \sum \underbrace{\frac{1}{2n(2n+1)}}_{\leq \frac{1}{(2n)^2}} < 1$$

Reordering we can get :

$$1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{4} + \frac{1}{9}\right) + \cdots$$

$$= 1 + \underbrace{\frac{1}{2\cdot 3\cdot 5}}_{\leq \frac{1}{8}} + \underbrace{\frac{1}{4\cdot 7\cdot 9}}_{\leq \frac{1}{64}} + \cdots$$

$$> 1$$

Idea : Reordering the above sum can allow us to get any number.

Alternating Series Test

Let $\langle a_n \rangle$ be a non-negative decreasing sequence of #s with $\lim_{n\to\infty} a_n = 0$, then

$$\sum_{n=1}^{\infty} a_n (-1)^{n-1} \text{ converges.}$$

Pf :

$$S_n = \sum_{j=1}^{n} a_j (-1)^{j-1}$$

$$S_{2n} = \sum_{j=1}^{2n} a_j (-1)^{j-1}$$

$$S_{2n} - S_{2(n-1)} = a_{2n}(-1)^{2n-1} + a_{2n-1}(-1)^{2n-2}$$

$$= -a_{2n} + a_{2n-1} \geq 0$$

$$S_{2n+1} - S_{2n-1} = a_{2n+1}(-1)^{2n} + a_{2n}(-1)^{2n-1}$$

$$= -a_{2n} + a_{2n+1} \leq 0$$

even partial sums are increasing
odd —"— decreasing

$$S_{2n+1} = S_{2n} + a_{2n+1} \geq S_{2n}$$

$$S_{2n+1} - S_{2n} = a_{2n+1}$$

and $a_{2n+1} \to 0$ as $n \to \infty$

We see that:

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1}$$

and $S_{2n} \leq S \leq S_{2n+1}$ ■

Riemann Rearrangement Th.

Let $\sum_{j=1}^{\infty} a_j$ be a conditionally convergent series that is not abs. convergent then given any number $c \in \mathbb{R}$, there is a rearrangement of this series $\langle n_j \rangle$ s.t.

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} a_{n_j} = c$$

Pf: Let $\langle P_j \rangle$ be indices of positive terms in decreasing (*) order and $\langle q_j \rangle$ be indicies of negative terms in increasing order.

Every ~~and~~ $n \in \mathbb{N}$ belongs to either $\langle P_j \rangle$ or $\langle q_j \rangle$

(*) is possible because $\lim_{j\to\infty} a_j = 0$

$$\sum_{j=1}^{\infty} a_{p_j} = +\infty$$

$$\sum_{j=1}^{\infty} a_{q_j} = -\infty$$

Let $c > 0$

$a_{p_1}, a_{p_2}, \ldots a_{p_j}, \ldots$

$a_{q_1}, a_{q_2}, \ldots a_{q_j}, \ldots$

First take n_1 terms of the positive list s.t :

$$\sum_{j=1}^{n_1} a_{p_j} > c \quad \text{and} \quad \sum_{j=1}^{n_1-1} a_{p_j} \leq c$$

Choose m_1 s.t.

$$\sum_{j=1}^{n_1} a_{p_j} + \sum_{j=1}^{m_1} a_{q_j} < c \quad \text{and}$$

$$\sum_{j=1}^{n_1} a_{p_j} + \sum_{j=1}^{m_1-1} a_{q_j} \geq c$$

Continuing this way, choose

$n_1 < n_2 < \ldots < n_j$

$m_1 < \ldots \ldots < m_j$

③

$$\sum_{k=1}^{n_j} a_{p_k} + \sum_{k=1}^{m_j-1} a_{q_k} \geq C \quad \text{and} \quad \sum_{k=1}^{n_j} a_{p_k} + \sum_{k=1}^{m_j} a_{q_k} < C$$

$$\sum_{k=1}^{n_j-1} a_{p_k} + \sum_{k=1}^{m_j-1} a_{q_k} \leq C \quad \text{and} \quad \sum_{k=1}^{n_j} a_{p_k} + \sum_{k=1}^{m_j-1} a_{q_k} > C$$

$$\sum_{k=1}^{n_j} a_{p_k} + \sum_{k=1}^{m_j-1} a_{q_k} - C \leq a_{p_{n_j}}$$

$$C - \left(\sum_{k=1}^{n_j} a_{p_k} + \sum_{k=1}^{m_j} a_{q_k} \right) \leq |a_{q_{m_j}}|$$

$\Rightarrow$ this rearrangement converges to C. ∎

$$\int_a^b f(x)g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

$$\sum_{n=1}^{N} a_n b_n$$

$$B_n := \sum_{k=1}^{n} b_k \qquad \sum_{n=1}^{N} a_n (B_n - B_{n-1})$$

$$B_0 := 0 \qquad = a_1 B_1 - \cancel{a_1 B_0}^{\,0} + a_2 B_2 - a_2 B_1 + \dots$$

$$= (a_1 - a_2) B_1 + (a_2 - a_3) B_2 + \dots$$

Summation by Parts

$$\Rightarrow \quad \sum_{j=m+1}^{n} a_j b_j = a_n B_n - a_{m+1} B_m + \sum_{j=m+1}^{n-1} (a_j - a_{j+1}) B_j$$

$$\Delta a_j = a_{j+1} - a_j$$

$$\sum_{k=m+1}^{n} a_k b_k = a_n B_n - a_{m+1} B_m - \sum_{j=m+1}^{n-1} \Delta a_j B_j$$

Thm: Suppose $\langle a_j \rangle$ is a decreasing sequence with $\lim_{j\to\infty} a_j = 0$ and $\langle b_j \rangle$ is a sequence with bounded partial sums $\langle B_j \rangle$, then,

$$\sum_1 a_j b_j \text{ converges.}$$

Pf:

$$S_m = \sum_{j=1}^{m} a_j b_j$$

$n > m$

$$S_n - S_m = \sum_{j=m+1}^{n} a_j b_j = a_n B_n - a_{m+1} B_n + \sum_{j=m+1}^{n-1} (-\Delta a_j) B_j$$

need to show $\langle S_n \rangle$ is a Cauchy sequence

Given $\varepsilon > 0$ find N s.t.

$$|S_n - S_m| < \varepsilon \quad \text{if} \quad n, m \geq N$$

We know $\exists$ M s.t. $|B_n| \leq M$.

$$|S_n - S_m| \leq (a_n + a_{m+1}) M + M \Big(\underbrace{\sum_{j=m+1}^{n-1} -\Delta a_j}_{\substack{\text{telescoping sum} \\ = a_{m+1} - a_n}} \Big)$$

$$\leq \underbrace{2 a_{m+1}}_{\substack{\to 0 \\ \text{as } m \to \infty}} M$$

$\Rightarrow \langle S_n \rangle$ is a Cauchy sequence. ■

If $\sum_{j=1}^{\infty} a_j$ converges then $\sum_{j=1}^{\infty} \frac{a_j}{J}$ converges, $\sum_{j=1}^{\infty} \frac{a_j}{\log(j+1)}$ converges,

If $a_n \downarrow 0$ then $\sum_{n=1}^{\infty} a_n e^{inx}$ converges if $x \neq 2\pi n$ $\forall n \in \mathbb{Z}$.