10/12/17

MATH 508

Lecture 13

Topology of $\mathbb{R}$

$D_r(x) = \{y : |x-y| < r\}$ ("Disc" of radius r around x)

Types of Intervals : (a,b) – open

$[a,b]$ – closed

$(a,b]$, $[a,b)$ } half open

Defn : A set $U \subseteq \mathbb{R}$ is <u>open</u> if $\forall x \in U$, $\exists$ $r > 0$ s.t. $Dr(x) \subseteq U$

Thm : An open interval is an open set.

Pf : Left to reader. (use defn)

- Clearly, half-open / closed intervals are <u>not</u> open.

Set operations

- If U_1, U_2 are open sets then $U_1 \cup U_2$ is also open. More generally if U_α, $\alpha \in A$ are open sets then so is $\bigcup_\alpha U_\alpha$.

- A finite intersection of open sets is open (not true over infinite intersections)

- $(a,b)^c = (-\infty, a] \cup [b, +\infty)$

$\Rightarrow (a,b)^c$ is not an open set.

Defn

A set $F \subseteq \mathbb{R}$ is <u>closed</u> if F^c is open.

Proposition: (1) If $F_1, \dots F_n$ are closed sets, then so is $\bigcup_{i=1}^{n} F_i$.

(2) If F_α, $\alpha \in A$ are closed sets, then $\bigcap_{\alpha \in A} F_\alpha$ is closed

Proof:

(1) Consider:

$$\left(\bigcup_{i=1}^{n} F_i \right)^c = \bigcap_{i=1}^{n} F_i^c$$

→ each F_i^c is open.

→ a finite intersection of open sets is open.

→ thus $\bigcap_{i=1}^{n} F_i^c$ is open

→ $\bigcup_{i=1}^{n} F_i$ is closed

(2) Verify!

Let $A \subseteq \mathbb{R}$ be a set.

Defn: x is a limit point of A if $\forall r > 0$, the set $(D_r(x) \cap A)$ has infinitely many points in it.

—OR—

$\forall r > 0, \quad D_r(x) \cap A \setminus \{x\} \neq \phi$

Thm: x is a limit point of A iff there is a sequence $\langle x_j \rangle \subseteq A$ s.t. $x_j \neq x_k$ if $j \neq k$ and $\lim_{j \to \infty} x_j = x$

($\Rightarrow$)

Pf: Let $r_1 = 1$ choose

$x_1 \in D_1(x) \cap A \setminus \{x\}$

$r_2 = \dfrac{|x - x_1|}{2}$

$D_{r_2}(x) \cap A \setminus \{x\} \neq \phi$ {defn of limit point}

Choose $x \in D_{r_2}(x) \cap A \setminus \{x\}$

$r_3 = \dfrac{|x - x_2|}{2}$

Similarly $\exists \; x_3 \in D_{r_3}(x) \cap A \setminus \{x\}$

Inductively, we have

$r_j = \cancel{x_j} \; \dfrac{|x - x_{j-1}|}{2} > 0$

$x_j \in D_{r_j}(x) \cap A \setminus \{x\}, \quad r_{j+1} = \dfrac{|x - x_{j}|}{2} > 0$

The points $\langle x_j \rangle$ are all distinct and

$$|x - x_j| \leq r_j \leq \frac{r_{j-1}}{2} \leq \cdots \leq \frac{1}{2^j}$$

$$|x - x_k| < |x - x_j| \qquad \text{if } j < k$$

(points are getting closer to x)

This shows that

$$\lim_{j \to \infty} x_j = x$$

($\Leftarrow$) The other direction is easier.

Suppose $\exists \langle x_j \rangle \subseteq A \setminus \{x\}$

s.t. $\lim_{j \to \infty} x_j = x$

Since $x_j \neq x_k$ if $j \neq k$

the sets $|\{x_j, x_{j+1}, \ldots\}| = \infty$

Given $r > 0$, $\exists N$ such that

if $j \geq N$ then $|x - x_j| < r$

because $\lim_{j \to \infty} x_j = x$

$D_r(x) \cap A \setminus \{x\} = \{x_N, x_{N+1}, \ldots\}$

which by defn implies that x is a limit point. ∎

<u>Thm</u> A set $F \subseteq \mathbb{R}$ is closed iff F contains all of its limit points.

<u>Pf</u>:

($\Rightarrow$) Suppose F is closed.

F^c is open. Let x^* be any fixed, arbitrary limit point of F.

Either $x^* \in F$ or $x^* \in F^c$.

If $x^* \in F$, we are done.

If $x^* \in F^c$, then $\exists\, r > 0$ s.t.

$D_r(x^*) \subseteq F^c$. Hence

$D_r(x^*) \cap F = \emptyset$, a contradiction.

($\Leftarrow$) Suppose F contains all of its limit points.

If $y \in F^c$, then y cannot be a limit point of F. This means that $\exists\, r > 0$ s.t.

$D_r(y) \cap F$ has at most finitely many points, say $\{x_1, \ldots x_n\}$

Let $\delta = \min\{|y - x_1|, \ldots, |y - x_n|\} > 0$

$D_\delta(y) \cap F = \emptyset$. This shows that $\forall\, y \in F^c$ $\exists\, \delta > 0$ s.t. $D_\delta(y) \subseteq F^c \Rightarrow F^c$ is open, F is closed

■

Let $\mathcal{U} \subseteq \mathbb{R}$ be an open set. For each $x \in \mathcal{U}$

let $A_x = \{ a < x : (a, x] \subseteq \mathcal{U} \}$

$B_x = \{ b > x : [x, b) \subseteq \mathcal{U} \}$

we know that for each $x \in \mathcal{U}$, $A_x \neq \phi$

$B_x \neq \phi$

Let $\alpha = \inf A_x$

$\beta = \sup B_x$

Claim : $(\alpha, \beta) \subseteq \mathcal{U}$

Pf : $\forall\, a \in A_x,\ b \in B_x$ $[a, b) = (a, x] \cup [x, b) \subseteq \mathcal{U}$

So $\bigcup_{\substack{a \in A_x \\ b \in B_x}} [a, b) \subseteq \mathcal{U}$

Choose any $y \in (\alpha, \beta)$ $\exists$ an $a \in A_x$
$b \in B_x$ s.t.

$y \in (a, b)$

If $\varepsilon > 0$, then

$(\alpha - \varepsilon, \beta) \not\subseteq \mathcal{U}$

and $(\alpha, \beta + \varepsilon) \not\subseteq \mathcal{U}$

Let $x \in \mathcal{U}$.
Let $I_x = (\alpha, \beta)$ } ~~from b.~~ as defined earlier

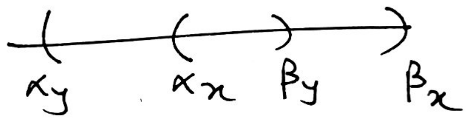
<u>Claim</u>: If $x \neq y$ both belong to $\mathcal{U}$ then either
$I_x = I_y$, or $I_x \cap I_y = \phi$.

<u>Pf</u>: We need to see that if $I_x \cap I_y \neq \phi$
then $I_x = I_y$

$I_x = (\alpha_x, \beta_x)$
$I_y = (\alpha_y, \beta_y)$

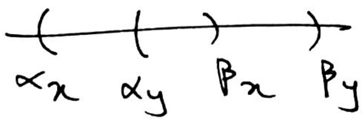
If $I_x \cap I_y \neq \phi$ then:

(1) $\alpha_y < \alpha_x$ but
$\alpha_x < \beta_y < \beta_x$



$\Rightarrow (\alpha_y, \beta_x) \subseteq \mathcal{U}$
but $(\alpha_x - \varepsilon, \beta_x) \not\subseteq \mathcal{U}$, a contradiction.

(2)



(3) / (4) $(\alpha_x, \beta_x) \subseteq (\alpha_y, \beta_y)$
$(\alpha_y, \beta_y) \subseteq (\alpha_x, \beta_x)$

All arrive at a similar contradiction.

$\Rightarrow I_x = I_y$.

∎

$$\mathcal{U} = \bigcup_{x \in \mathcal{U}} I_x$$

Each I_x has positive length which means $I_x \cap \mathbb{Q} \neq \phi$ $\forall x \in \mathcal{U}$

for each equivalence class of intervals (where $I_x \sim I_y$ if $I_x = I_y$)

we choose an $r_j \in I_x \cap \mathbb{Q}$

$I_{r_j} = I_x$ $\forall x$ with $I_x \sim I_{r_j}$

<u>Thm</u> : If $\mathcal{U} \subseteq \mathbb{R}$ is an open set then

$$\mathcal{U} = \coprod_{j=1}^{\infty} I_{r_j} \qquad [\text{disjoint union}]$$

where $\{I_{r_j}\}$ are open intervals and

$I_{r_j} \cap I_{r_k} = \phi$ if $j \neq k$.

<u>Axiom of Choice</u>

If $A_\alpha \neq \phi$, $\alpha \in \mathcal{A}$ is a collection of non-empty sets then we can choose an element $a_\alpha \in A_\alpha$ for each $\alpha \in \mathcal{A}$.

<u>Defn</u> : Given a set A, define <u>interior</u> of A

$$\text{int } A = \bigcup_{\substack{\mathcal{U} \text{ open} \\ \mathcal{U} \subseteq A}} \mathcal{U}$$

int A is the largest open interval in A.

Defn

If A is not closed, then the closure of A

$$\overline{A} = \bigcap_{\substack{F \text{ is closed} \\ F \supseteq A}} F$$

It is the smallest closed set containing A.

Thm : For any set A, let A' be the set of limit points of A, then $\overline{A} = A \cup A'$.