10/17/17

MATH 508

Lecture 14

RECAP :

- A is open if $\forall x \in A$, $\exists r > 0$ s.t. $(x-r, x+r) \subseteq A$

- A set F is closed iff F contains all of its limit points

- F is closed $\longleftrightarrow$ F^c is open

- $A \subseteq \mathbb{R}$ is any arbitrary set.

$$\text{int } A = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$$, int A is the largest open set $\subseteq A$.

$$\overline{A} = \bigcap_{\substack{F \text{ is closed} \\ F \supseteq A}} F$$ The closure is the smallest closed set containing A.

$$\text{int } A \subseteq A \subseteq \overline{A}$$

Thm : $\overline{A} = A \cup A'$, where A' is the set of limit points of A

Pf : If $F \supseteq A$ and F is closed, then

$F \supseteq A'$

$\overline{A} \supseteq A \cup A'$ —— (1)

Let $x \in \overline{A} \setminus A$.

Claim : $x \in A'$

Pf: If not, then $x \notin A'$

and $\exists\ r > 0$ s.t. $(x-r, x+r) \cap A = \emptyset$

But if $F \supseteq A$ and F is closed, then

closed set containing A → $F \cap (x-r, x+r)^c \supseteq A$

$\Rightarrow x \notin \overline{A}$, $\times$

$\Rightarrow \overline{A} \subseteq A \cup A'$ —— (2)

Thus $\overline{A} = A \cup A'$ (from (1), (2))

$\langle x_n \rangle = 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$

The limit points of $\langle x_n \rangle = \mathbb{N}$

$\overline{\{x_n\}} = \{x_n\}$, $\{x_n\}' = \phi$

[This set has no limit points]

If A is set, then x is a limit point of A

iff $\exists \langle x_j \rangle \subseteq A$ s.t. $x_j \neq x_k$ if $j \neq k$

and $\lim\limits_{j \to \infty} x_j = x$.

- If $B \subseteq A$ and $A \subseteq \overline{B}$, then we say that B is <u>dense</u> in A.

$[0,1] \cap \mathbb{Q} = B$

$\overline{B} = [0,1]$

The rationals are dense in $[0,1]$

The Cantor Set

$C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$

$C_2 = C_1 \setminus (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$

$\vdots$

$C_n = C_{n-1} \setminus$ (middle-third of each remaining interval)

$C_n \supseteq C_{n+1} \supseteq \dots$

each is closed

$\mathbb{C} = \bigcap_{n=1}^{\infty} C_n$ is called the Middle-thirds Cantor set.

$|C_n| = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \longrightarrow 0$ as $n \to \infty$

$\boxed{\text{int } C = \phi}$

Claim : C is uncountable

Pf :

Ternary Expansion

$$x = \sum_{j=1}^{\infty} a_j \cdot 3^{-j} \quad ; \quad a_j \in \{0,1,2\}$$

Points in the Cantor set

→ At each level i we remove all numbers that have $a_i = 1$.

→ If $x = \sum_{j=1}^{\infty} a_j \cdot 3^{-j}$ ~~then~~ $\in C$, then $a_j \in \{0,2\}$

→ let $b_j = \frac{a_j}{2}$; $b_j \in \{0,1\}$

this puts every $x \in C$ in a bijection with a number $y \in [0,1]$

→ C is uncountable as $[0,1]$ is uncountable.

∎

Thm : Every point of C is a limit point.

Pf : (maybe) on Pset.

- A set $A \subseteq \mathbb{R}$ with the property $A = A'$ is called a perfect set.

Thm : If F is closed then $F = P \cup C$ where P is perfect and C is countable

Bolzano - Weierstrass Thm :

If $\langle x_n \rangle$ is a bounded sequence, then it has a convergent subsequence

Let $K \subseteq \mathbb{R}$

When is it possible to assert that every sequence $\langle x_n \rangle \subseteq K$ has a limit point also in K?

(1) K must be bounded. If not, $\exists \langle x_n \rangle \subseteq K$ such that $|x_n| > n$.

(2) K must be closed. If not $\exists x \in K' \setminus K$, $\exists \langle x_n \rangle \subseteq K$ s.t. $\lim_{n\to\infty} x_n = x \notin K$.

<u>Defn</u> : If $K \subseteq \mathbb{R}$ has the property that every sequence $\langle x_n \rangle \subseteq K$ has a limit point in K, then we say that K is <u>compact</u>.

<u>Thm</u> : K is compact iff K is closed and bounded.

<u>Pf</u> :

($\Rightarrow$) K is compact $\longrightarrow$ K is closed and bounded (proved earlier)

($\Leftarrow$) K is closed and bounded

Consider $\langle x_n \rangle \subseteq K$

By B-W, $\exists \langle x_{n_j} \rangle$ s.t.

$$\lim_{j \to \infty} x_{n_j} = x$$

If $x_{n_j} \neq x$,

$$\forall \, l \in \mathbb{N}, \ |\{ j : 0 < |x - x_j| \leq \tfrac{1}{l} \}| = \infty$$

then $x \in K' \subseteq K$ (K is closed)

Cannot be finite, ~~...~~ If it was then seq would not converge

The other possibility, $\forall \, l \in \mathbb{N}, |\{ j : 0 < |x - x_j| < \tfrac{1}{l} \}| < \infty$

$$\Rightarrow x_{n_j} = x \ \ \forall j \geq N \Rightarrow x \in K.$$

Thm : Suppose that $K_n \supseteq K_{n+1} \supseteq \ldots$ are compa[cts]
sets. Then $\bigcap_{n=1}^{\infty} K_n \neq \phi$

Pf :

$$[0, \infty) \supseteq [1, \infty) \ldots \supseteq [n, \infty)$$

$$\bigcap_{n=1}^{\infty} [n, \infty) = \phi$$

$$\mathcal{U}_n = \bigcup_{j=n}^{\infty} \left(\frac{1}{j}, \frac{1}{j+1} \right)$$

$$\bigcap_{n=1}^{\infty} \mathcal{U}_n = \phi$$

Let $x_n \in K_n$ [Axiom of Choice]

$$\langle x_n \rangle$$

$$\langle x_n \rangle \in K_1$$

$\exists \langle x_{n_j} \rangle$ s.t.

$\lim_{j \to \infty} x_{n_j} = x^* \in K_1$ (from compactness)

$\langle x_{n_j} \rangle_{j=k}^{\infty} \subseteq K_k$, $x_{n_j} \in K_{n_j} \subseteq K_k$

$x^* = \lim_{j \to \infty} x_{n_j} \in K_k$ (this is true for all k)

$$\Rightarrow x^* \in \bigcap_{n=1}^{\infty} K_n$$

■

- Compactness is not neccessary

Consider : $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

$\cancel{\bigcap_{n=1}^{\infty} \dots}$ $\bigcap_{n=1}^{\infty} U_n = \{0\}$