10/24/17

MATH 508

Lecture 15

Let $D \subseteq \mathbb{R}$,

A function defined on D is just the assignment of a unique real number $f(x)$ to each pt. $x \in D$,

$$f : D \to \mathbb{R}$$

- D (the set on which the function is defined) is called the domain of the function.

- The values assumed by the function $R = \{ f(x) : x \in D \}$ is called the range of the function.

- If $f : D \to F$, then the range of $f \subseteq F$.

- If $\forall\, y \in F$, $\exists\, x \in D$ s.t. $f(x) = y$, then f is onto (or surjective) its range.

- If $f(x_1) = f(x_2)$ iff $x_1 = x_2$ then we say that f is one-to-one or injective.

- If $\mathcal{U} \subseteq \mathbb{R}$ is any subset, then

$$f^{-1}(\mathcal{U}) = \{ x \in D : f(x) \in \mathcal{U} \}$$

then $f^{-1}(\mathcal{U})$ is called the pre-image of $\mathcal{U}$.

- If $f : D \to F$ is both 1-1 and onto then we say that f is a bijection and $f^{-1} : F \to D$ is well defined and $f^{-1} \circ f(x) = x \quad \forall\, x \in D$ and

$$f \circ f^{-1}(y) = y \quad \forall\, y \in F.$$

<u>Continuity</u>

Let f be defined on D, let $x_0 \in D$.

<u>Defn</u>: f is continuous at x_0 if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $|x - x_0| < \delta$, $x \in D$ <u>then</u>

$$|f(x) - f(x_0)| < \varepsilon$$

<u>Defn</u>: f is continuous if it is continuous at every point in its domain D.

Eg:

$$f(x) = \frac{1}{x} \qquad D = (0,1]$$

If f is continuous at $x = \frac{1}{2}$

$$f(\tfrac{1}{2}) = 2$$

Given $\varepsilon > 0$, ~~does~~ $\exists\ \delta > 0$ s.t.

$$|f(x) - 2| < \varepsilon \quad \text{if} \quad |x - \tfrac{1}{2}| < \delta$$

$$\left|\frac{1}{x} - 2\right| < \varepsilon \qquad \text{-OR-} \quad |2x-1| < 2\delta$$

$$\left|\frac{1-2x}{x}\right| < \varepsilon \quad \text{if} \quad |2x-1| < 2\delta$$

$$\cancel{\text{select} \quad \delta = \frac{\varepsilon}{2}}$$

$$\left|\frac{2x-1}{x}\right| < \varepsilon \qquad \text{if} \quad |2x-1| < 2\delta$$

— Complete.

$f(10^{-10}) = 10^{10}$

does $\exists\, \delta$ s.t. if $|x - 10^{-10}| < \delta$ then

$$\left| \frac{1}{x} - 10^{10} \right| < \varepsilon \quad \text{if } |10^{10}x - 1| < 10^{10}\delta$$

$$\frac{|10^{10}x - 1|}{|x|} < \varepsilon \Rightarrow \frac{10^{10}\delta}{|x|} < \varepsilon$$

$$\Rightarrow \delta < 10^{-10}\varepsilon |x|$$

let $|x| < \frac{10^{-10}}{2}$

$$\Rightarrow \delta < \frac{10^{-20}}{2}\varepsilon, \quad x_0 = 10^{-10}$$

Defn

Let f be a function defined on D and let x_0 be a limit point of D

i.e. $(x_0 - r, x_0 + r) \cap D \setminus \{x_0\} \neq \phi \quad \forall\, r > 0$

We say that $\lim_{x \to x_0} f(x) = L$ if

$\forall\, \varepsilon > 0, \exists\, \delta > 0$ st $|f(x) - L| < \varepsilon$ if

$x \in D$ and $0 < |x - x_0| < \delta$

Thm: Let f, g be functions defined on D and $x_0 \in D'$,

if $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M$ then:

(i) $\lim_{x \to x_0} (f(x) + g(x)) = L + M$

(ii) $\lim_{x \to x_0} (f(x) \cdot g(x)) = L \cdot M$

(iii) if $M \neq 0$, $\exists \delta > 0$ s.t. $g(x) \neq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$$

Defn:

Let f be defined on D, then f is continuous at $x_0 \in D$ if

$$\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$$

Note: (1) Well defined because $\lim_{x \to x_0} f(x)$ has nothing to do with $f(x_0)$

(2) Nothing is required for continuity if x_0 is not a limit point.

- The two definitions for being continuous at a point $x_0 \in D$ are equivalent.

Sequential limits

Suppose that for any sequence $\langle x_n \rangle$ such that

① $x_n \neq x_0 \ \forall n$

② $\lim_{n\to\infty} x_n = x_0$

the sequence $\langle f(x_n) \rangle$ has a limit, then the limit is the same for every sequence and equals $\lim_{x\to x_0} f(x)$ and conversely.

Pf. ($\Rightarrow$) Suppose $\langle x_n \rangle$ $\langle x_n' \rangle$ are sequences that satisfy ① and ②, then $\langle f(x_n) \rangle$, $\langle f(x_n') \rangle$ converge

Let $\left.\begin{array}{l} \tilde{x}_{2n} = x_n \\ \tilde{x}_{2n-1} = x'_n \end{array}\right\}$ — $x_1, x_1', x_2, x_2', \ldots$ converges to x_0 (exercise)

then $\langle f(\tilde{x}_n) \rangle$ also converges

If $\lim_{x\to x_0} f(x) \neq L$ ~~or does not exist~~, ~~but $\lim_{n\to\infty} f(x_n)$~~ $= L$

$\exists\ \varepsilon > 0$ s.t. $\forall n$, $x_n \neq x_0$

$$|x_n - x_0| < \frac{1}{n}, \quad |f(x_n) - L| \geq \varepsilon$$

But $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n) = L$, a contradiction.

($\Leftarrow$) follows from definition. ■

ε-δ defn

Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $|x - x_0| < \delta$ then

$|f(x) - f(x_0)| < \varepsilon$.

A neighborhood of x_0, $(x_0 - \delta, x_0 + \delta)$ is mapped to a neighborhood of $f(x_0)$, $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

$$f^{-1}\left(\left(f(x_0) - \varepsilon, f(x_0) + \varepsilon\right)\right) \supseteq (x_0 - \delta, x_0 + \delta)$$

-OR- equivalently, we can say that if f is continuous then $f^{-1}(\mathcal{U})$ is open whenever $\mathcal{U}$ is open.

Thm Let $D \subseteq \mathbb{R}$ be open and let $f : D \to \mathbb{R}$. Then f is continuous iff $f^{-1}(\mathcal{U})$ is an open set for any $\mathcal{U} \subseteq \mathbb{R}$, an open set.

i.e. "The inverse image of any open set is an open set."

Pf :

($\Rightarrow$) f is ε-δ continuous and let $\mathcal{U} \subseteq \mathbb{R}$ be an open set.

want to show that $f^{-1}(\mathcal{U})$ is open.

① $f^{-1}(\mathcal{U}) = \phi$ which is open

② $f^{-1}(\mathcal{U}) \neq \phi$

$\Rightarrow \exists\, x_0 \in f^{-1}(\mathcal{U})$

need to show $\exists\ \delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(\mathcal{U})$

If $x_0 \in f^{-1}(\mathcal{U})$ then $f(x_0) \in \mathcal{U}$.

Since $\mathcal{U}$ is open, $\exists\ \varepsilon > 0$ s.t.

$(f(x_0) - \varepsilon,\ f(x_0 + \varepsilon)) \subseteq \mathcal{U}$

ε-δ continuity $\Rightarrow$ $\exists\, \delta > 0$ s.t. if $x \in (x_0 - \delta, x_0 + \delta)$

then $f(x) \in (f(x_0) - \varepsilon,\ f(x_0) + \varepsilon) \subseteq \mathcal{U}$

and thus $f^{-1}(\mathcal{U}) \supseteq (x_0 - \delta, x_0 + \delta)$

$\Rightarrow f^{-1}(\mathcal{U})$ is open.

($\Leftarrow$) Suppose $f^{-1}(\mathcal{U})$ is open for all open sets $\mathcal{U}$.

Let $x_0 \in D$

let $\mathcal{U} = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ for some $\varepsilon > 0$

$f^{-1}(\mathcal{U})$ is open and $x_0 \in f^{-1}(\mathcal{U})$

$$\Rightarrow \exists \delta > 0 \text{ s.t. } (x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(\mathcal{U})$$

$$\Updownarrow$$

$$\forall x \in (x_0 - \delta, x_0 + \delta) \quad f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$$

implying continuity. ∎

Thm

Let $f : D \to E$ and $g : E \to F$ be continuous maps, where D, E are open sets, then $g \circ f : D \to F$ is also continuous.

Pf: Let $\mathcal{U} \subseteq F$ be an open set.

$$(g \circ f)^{-1}(\mathcal{U}) = \underbrace{f^{-1}(g^{-1}(\mathcal{U}))}_{\text{is open}}$$

$$\Rightarrow g \circ f \text{ is continuous}$$

∎