10/31/17

MATH 508

Lecture 17

$$\lim_{x \to x_0} f(x)$$

f is a function defined on D

x_0 is a limit point.

Consider :

f is defined on (a, b)

$$\lim_{x \to a} f(x) = L$$

If $\forall\ \varepsilon > 0\ \ \exists\ \delta$ s.t.

if $a < x < a + \delta$

$$\Rightarrow |f(x) - L| < \varepsilon$$

Notice that x approaches a from above

We can write:

$$\lim_{x \to a^+} f(x)$$

also called the right limit.

Similarly, $\lim_{x \to b^-} f(x) = L$ if

$\forall \varepsilon > 0 \quad \exists \delta$ s.t.

if $b - \delta < x < b$

$\Rightarrow |f(x) - L| < \varepsilon$

$\lim_{x \to b^-} f(x)$ is called the left limit.

It can happen that f is defined near to x_0

and $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist.

If both are equal and are equal to $f(x_0)$, then f is continuous at x_0.

If not, then f has a jump discontinuity at x_0

Two reasons for discontinuity at x_0 $\longrightarrow$ (1) jump discontinuity at x_0

(2) either the left or right limit at x_0 does not exist.

eg: $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$

$\lim_{x \to 0^+} f(x)$ does not exist

$\lim_{x \to 0^-} f(x) = 0$

Defn

A function defined on (a,b) is monotone increasing if $f(x) \le f(y)$ if $x < y$

Similar defn for monotone decreasing, strictly monotone increasing / decreasing.

Thm If $f(x)$ is monotone, then

(i) $\forall\, x_0 \in (a,b)$

$\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist, and

(ii) they differ at most countably often.

Hence a monotone ~~dis~~ function has only jump discontinuities and it has at most countably many.

Pf Let f be monotonically increasing

$$f(x) \le f(y) \quad \text{if } x < y$$

Let $\langle x_n \rangle$ be any increasing ~~convergent~~ seq. that converges to x_0

$$x_n \le x_{n+1} \le \cdots$$

$$\Rightarrow \lim_{n \to \infty} x_n = x_0$$

$$\Rightarrow f(x_n) \le f(x_{n+1}) \cdots$$

* $\lim_{n\to\infty} f(x_n)$ exists by the l.u.b. property

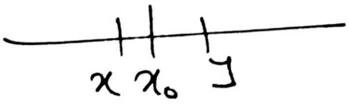
$\Rightarrow$ $\forall$ increasing seq. $\lim_{x\to x_0^-} f(x)$ exists.

A similar argument shows that $\lim_{x\to x_0^+} f(x)$ exists.

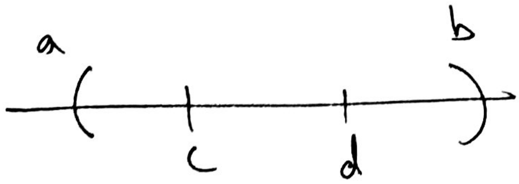
* <u>Note</u> : Need to show that considering only one increasing sequence is enough.

Now consider

$$D = \{ x_0 : \lim_{x\to x_0^-} f(x) < \lim_{x\to x_0^+} f(x) \}$$



$$f(y) - f(x) \geq f(x_0^+) - f(x_0^-)$$



$$f(d) - f(c) \geq \sum_{x_0 \in D \cap (c,d)} [f(x_0^+) - f(x_0^-)]$$

$$D_n = \{ x \in D : f(x_0^+) - f(x_0^-) > \tfrac{1}{n} \}$$

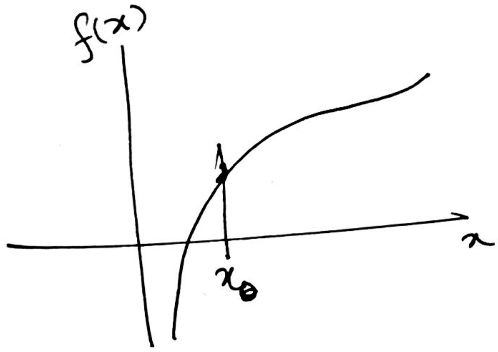
$$D = \bigcup_{n \geq 1} D_n$$

If D is uncountable, then one of the D_n's is uncountable $\longrightarrow$ leads to a contradiction.

∎

A linear function of x has the form:

$$y = \underbrace{a}_{\text{slope}} x + \underbrace{b}_{\text{intercept}}$$



want to find the best linear appx. to $f(x)$ at x_0

Represent ~~f(x) as~~ linear func. as

$$y \cancel{f(x)} = a(x - x_0) + b$$

let $y = a(x - x_0) + f(x_0)$

when is $[a(x - x_0) + f(x_0)]$ a good appx. to $f(x)$ at x_0?

want :

$$\lim_{x \to x_0} \frac{f(x) - (a(x-x_0) + f(x_0))}{x - x_0} = 0$$

— OR —

$$\boxed{\lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] = a} =: f'(x_0)$$

$f'(x_0)$ is the slope of the best linear appx. to f at x_0.

$$f(x) = f'(x_0)(x-x_0) + f(x_0) + \underbrace{o(|x-x_0|)}_{\text{error term}}$$

$$\lim_{x \to x_0} \frac{|f(x) - (f'(x_0)(x-x_0) + f(x_0))|}{|x - x_0|} = 0$$

Given $\varepsilon > 0$, $\exists\, \delta > 0$

s.t. if $|x - x_0| < \delta$ then

$$|f(x) - (f'(x_0)(x-x_0) + f(x_0))| \leq \varepsilon |x - x_0|$$

More generally,

$f(x) = O(g(x))$ at x_0

if $\displaystyle \limsup_{x \to x_0} \frac{|f(x)|}{g(x)} \leq M < \infty$

and $f(x) = o(g(x))$ at x_0

if $\displaystyle \lim_{x \to x_0} \frac{|f(x)|}{g(x)} = 0$

<u>Defn</u>: f is differentiable at x_0 iff

$$f(x) = f'(x_0)(x - x_0) + f(x_0) + o(|x - x_0|)$$

<u>Defn</u>: $\bar{h}(\delta) = \sup \{ h(x) : x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \}$

$\displaystyle \limsup_{x \to x_0} h(x) := \lim_{\delta \to 0^+} \bar{h}(\delta)$

$f(x) = O(g(x))$ at x_0 if $\exists\, c$, and a $\delta > 0$

s.t. $|f(x)| \le c\, g(x)$ if

$$0 < |x - x_0| < \delta$$

$$\text{eg: } f(x) = (x-x_0)^5 + \underbrace{(x-x_0)^{10}}_{\substack{\text{goes to}\\ \text{zero faster}\\ \text{as } x \to x_0}}$$

$$\Rightarrow f(x) = O(|x-x_0|^5)$$

Defn

If f is defined on (a,b) and differentiable at every pt. on (a,b) then we say that f is differentiable on (a,b).

$$\text{eg: } f(x) = \begin{cases} x^2 \sin \cancel{\left(\frac{1}{x^2}\right)} \left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

derivative "near" zero is unbounded but

$f'(0) = 0$.

Thm

If f is a differentiable function on (a,b) such that f is monotonically increasing near x_0 then $f'(x_0) \geq 0$.

Pf:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

If f is monotonically increasing at x_0 then $\exists\ \delta > 0$ s.t.

$$0 \leq \frac{f(x) - f(x_0)}{x - x_0} \quad \text{if} \quad 0 < |x - x_0| < \delta$$

$$\Rightarrow f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Thm: If $f'(x_0) > 0$ then f is strictly increasing near to x_0.

Pf: $\exists\ \delta > 0$ s.t. if $|x - x_0| > \delta$

then $\dfrac{f(x) - f(x_0)}{x - x_0} > \dfrac{f'(x_0)}{2}$

~~If $x \to x_0$~~

If $x > x_0$

$$f(x) - f(x_0) > \frac{f'(x_0)}{2}(x - x_0)$$

$$f(x) > \underbrace{\frac{f'(x_0)}{2}(x - x_0)}_{+ve} + f(x_0)$$

$$\Rightarrow f(x) > f(x_0)$$

$|||^{ly}$ if $x < x_0$

$$\Rightarrow f(x) < f(x_0).$$

* If f assumes max or min at x_0, then

$$f'(x_0) = 0.$$

* $f'(x_0) = 0$ does <u>not</u> imply that f has a local max or min at x_0

(eg: $f(x) = x^3$)