11/02/17

MATH 508

Lecture 18

$f(x) = x$; $f'(x) = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$

$f(x) = 1$; $f'(x) = \lim_{x \to x_0} \frac{1 - 1}{x - x_0} = 0$

→ constant function has a derivative of zero.

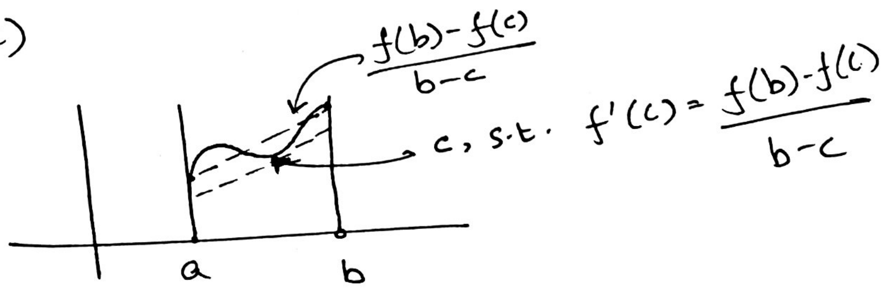
Q. If $f'(x) \equiv 0$, is $f(x)$ a constant function?

What we need is a thm that connects infinitesimal properties to local properties.

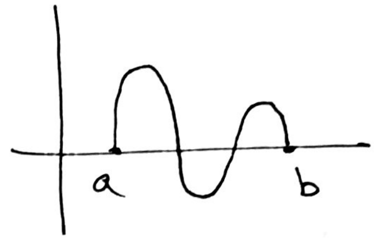
MEAN VALUE THEOREM

If f is a constant function on $[a, b]$ which is differentiable on (a, b) then $\exists\ c \in [a, b]$ s.t.

$$\frac{f(b) - f(c)}{b - c} = f'(c)$$



Suppose that $f(a) = f(b) = 0$



① $f(x) = 0 \quad \forall\, x \in (a,b)$

② $\exists\, x \in (a,b)$ s.t. $f(x) > 0$

③ $\exists\, x \in (a,b)$ s.t. $f(x) < 0$

A differentiable function is continuous.

Since $[a,b]$ is compact, in case ② there is a point $c \in [a,b]$ s.t. $f(c) = \max\{f(x) : x \in [a,b]\} > 0$

At c, $f'(c) = 0$ [proved in previous lecture]

In case ③, $\exists\, c$ s.t. $f(c) = \min\{f(x) : x \in [a,b]\} < 0$

At c, $f'(c) = 0$

ROLLE'S THEOREM

If f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b) = 0$, then $\exists\, c$ s.t.

$$f'(c) = 0 = \frac{f(b) - f(a)}{b-a}$$

Proof of MVT :

$$g(x) = f(x) - \left[f(a)\frac{(b-x)}{(b-a)} + f(b)\frac{(x-a)}{(b-a)} \right]$$

$$g(a) = g(b) = 0$$

g satisfies the hypothesis of ROLLE's Thm.

Hence $\exists\ c \in (a,b)$ s.t. $g'(c) = 0$

$$g'(x) = f'(x) - \left[\frac{-f(a)}{b-a} + \frac{f(b)}{b-a} \right]$$

$$g'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b-a} \right] = 0$$

which proves the mean value thm.

Cor : If f is differentiable on (a,b) and $f'(x) \equiv 0$, then $f(x)$ is a constant function.

Pf : use defn.

Cor: If $f'(x) \geq 0 \quad \forall x \in (a,b)$
then f is monotone increasing

Pf:
Take $a < x < y < b$

By MVT,
$$f(y) - f(x) = \underbrace{\underbrace{f'(z)}_{\geq 0} \underbrace{(y-x)}_{> 0}}_{\geq 0} \quad \text{for some } z \in (a,b)$$

$$\Rightarrow f(y) \geq f(x)$$

Note: If $f'(x) > 0 \;\; \forall x \in (a,b)$, then
f is strictly increasing

Cor: If f is differentiable in (a,b) and
$|f'(x)| \leq M \quad \forall x \in (a,b)$ then
$$|f(x) - f(y)| \leq M|x-y| \quad \forall x, y \in (a,b)$$
[Lipschitz estimate]

Pf: Left to reader.
(uses MVT).

Arithmetic properties of Derivatives

Aside : Intermediate Value Thm

If f is differentiable on $[a,b]$, $\rightarrow$ [on a "slightly" larger open interval that covers $[a,b]$]
and $f'(a) < f'(b)$, then for any
z s.t.

$$f'(a) < z < f'(b)$$

there exists a $c \in (a, b)$ s.t.

$$f'(c) = z$$

Thm : If f and g are differentiable at x_0, then

(1) $(f+g)$ is diff'l at x_0

$$(f+g)' x_0 = f'(x_0) + g'(x_0)$$

(2) $(f \cdot g)$ is diff'l at x_0

$$(f \cdot g)' x_0 = f'(x_0) \cdot g'(x_0)$$

(3) If $g'(x_0) \neq 0$, $\exists \, \delta$ s.t. $g(x) \neq 0$ if $x \in (x_0 - \delta, x_0 + \delta)$

~~and~~ f/g is diff'l at x_0,

$$\left(\frac{f}{g}\right)' x_0 = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{g'(x_0)^2}$$

Pf :

(3) Since $g(x_0) \neq 0$ if $\delta = \frac{|g(x_0)|}{2}$

then $g^{-1}(g(x_0)-\delta, g(x_0)+\delta) \supseteq (x_0-\varepsilon, x_0+\varepsilon)$

for some $\varepsilon > 0$

[since g is continuous]

$|g(x)| \geq \frac{\delta}{2}$ if $x \in (x_0-\varepsilon, x_0+\varepsilon)$

$$\left(\frac{f}{g}\right)'(x_0) = \lim_{x \to x_0} \frac{1}{x-x_0}\left[\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}\right]$$

$$= \lim_{x \to x_0} \frac{1}{x-x_0}\left[\frac{f(x)\cdot g(x_0) - f(x_0)\cdot g(x)}{g(x)\cdot g(x_0)}\right]$$

$$= \lim_{x \to x_0} \frac{1}{x-x_0}\left[\frac{f(x)\cdot g(x_0) - f(x_0)\cdot g(x_0) + f(x_0)\cdot g(x_0) - f(x_0) g(x)}{g(x)\cdot g(x_0)}\right]$$

$$= \lim_{x \to x_0} \frac{1}{x-x_0}\left[\frac{(f(x)-f(x_0))g(x_0) - f(x_0)(g(x)-g(x_0))}{g(x)\cdot g(x_0)}\right]$$

$$= \lim_{x \to x_0} \frac{g(x_0)}{g(x)\cdot g(x_0)} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{= f'(x_0)} - \lim_{x \to x_0} \frac{f(x_0)}{g(x)g(x_0)} \underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{g'(x_0)}$$

$$= \frac{f'(x_0)\cdot g(x_0) - f(x_0)\cdot g'(x_0)}{g(x_0)^2}$$

Thm :

$\forall n \in \mathbb{Z}$.

$$\frac{d}{dx} x^n = \begin{cases} n x^{n-1} & n \geq 0 \\ n x^{n-1} & n < 0, \ x \neq 0 \end{cases}$$

Pf :

$n = 1$ is proved.

Suppose it's true for some $n > 0$

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx}(x^n \cdot x)$$

$$= n x^{n-1} \cdot x + x^n = (n+1) x^n$$

This proves the formula for all $n \geq 0$

for negative numbers. Let $n > 0$

$$x^n \cdot x^{-n} = 1$$

$$\frac{d}{dx} x^n \cdot x^{-n} + \frac{d}{dx} x^{-n} \cdot x^n = 0$$

$$\Rightarrow \frac{d}{dx} x^{-n} = -n x^{-(n+1)}$$

set $m = -n$ $\Rightarrow$ $\frac{d}{dx} x^m = m x^{m-1}$ $m < 0$

Chain Rule :

Let f be diff'l at y_0 and g is a diff'l function at x_0 with $y_0 = g(x_0)$

The composition $f \circ g(x)$ is diff'l at x_0

~~ad~~ and

$$\frac{d}{dx}(f \circ g)(x_0) = f'(g(x_0)) \cdot g'(x_0) .$$

Pf :

f is diff'l at y_0

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + o(|y - y_0|)$$

g is diff'l at x_0

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \underbrace{o(|x - x_0|)}_{\substack{\downarrow \\ 0 \\ \text{as } x \to x_0}}$$

$$f(g(x)) = f(g(x_0)) + f'(g(x_0))(g(x) - g(x_0)) + \underbrace{o(|g(x) - g(x_0)|)}$$

$$= f(g(x_0)) + f'(g(x_0))\big(g'(x_0)(x - x_0) + o(|x - x_0|)\big) + o(|x - x_0|)\,]$$

equivalent asymptotically.

$$= f(g(x_0)) + f'(g(x_0))(g'(x_0)(x - x_0)) + o(|x - x_0|)$$

$$\frac{f(g(x)) - f(g(x_0))}{x - x_0} = f'(g(x_0)) \cdot g'(x_0) + \underbrace{\frac{o(|x-x_0|)}{x-x_0}}_{\substack{\downarrow \\ 0 \\ \text{as } x \to x_0}}$$

which completes the pf. ∎

Question

Is the inverse of a 1-1 differentiable function differentiable?

Thm: We'll assume that f is differentiable on (a,b), $f'(x) > 0$ on (a,b) and that $f'(x)$ is a continuous function on (a,b).

Then f^{-1} is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Pf: $f'(x) > 0$ on (a,b)

$\Rightarrow f(x) < f(y)$ if $x < y$

$\Rightarrow f$ is 1-1 and therefore is invertible

$$f^{-1} : (f(a), f(b)) \longrightarrow (a,b)$$

$$f \circ f^{-1}(x) - f \cdot f^{-1}(x_0) = x - x_0 \qquad x, x_0 \in (f(a), f(b))$$

By MVT,

$$x - x_0 = f'(c)\left(f^{-1}(x) - f^{-1}(x_0)\right) \quad \text{—— (1)}$$

where c lies b/w $f^{-1}(x)$ and $f^{-1}(x_0)$

<u>Lemma</u>: Prove that f^{-1} is continuous

<u>Pf Sketch</u>:

1) f^{-1} is monotone
2) f is continuous
3) Show that jump discontinuities cannot occur.

Since f^{-1} is continuous

$$\lim_{x \to x_0} f^{-1}(x) = f^{-1}(x_0)$$

and

$$\lim_{x \to x_0} \frac{f^{-1}(x) - f^{-1}(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{1}{f'(c)} \quad [\text{from (1)}]$$

By continuity $c \to f^{-1}(x_0)$

$$= \frac{1}{f'(f^{-1}(x_0))}$$